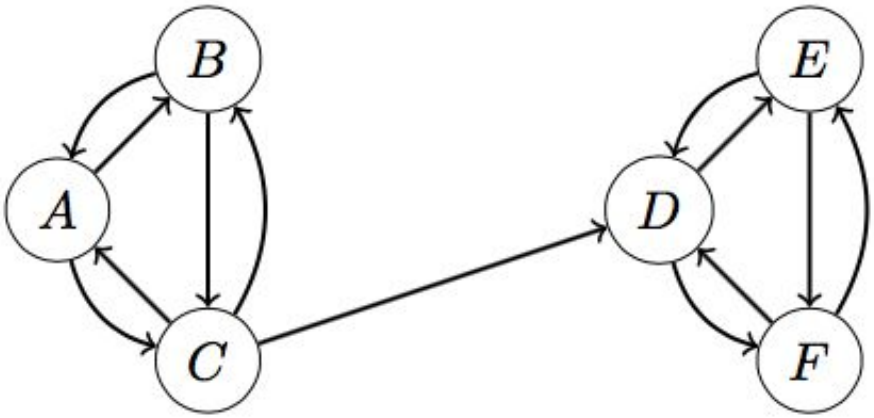
Thought experiment

- Run DFS starting at D:



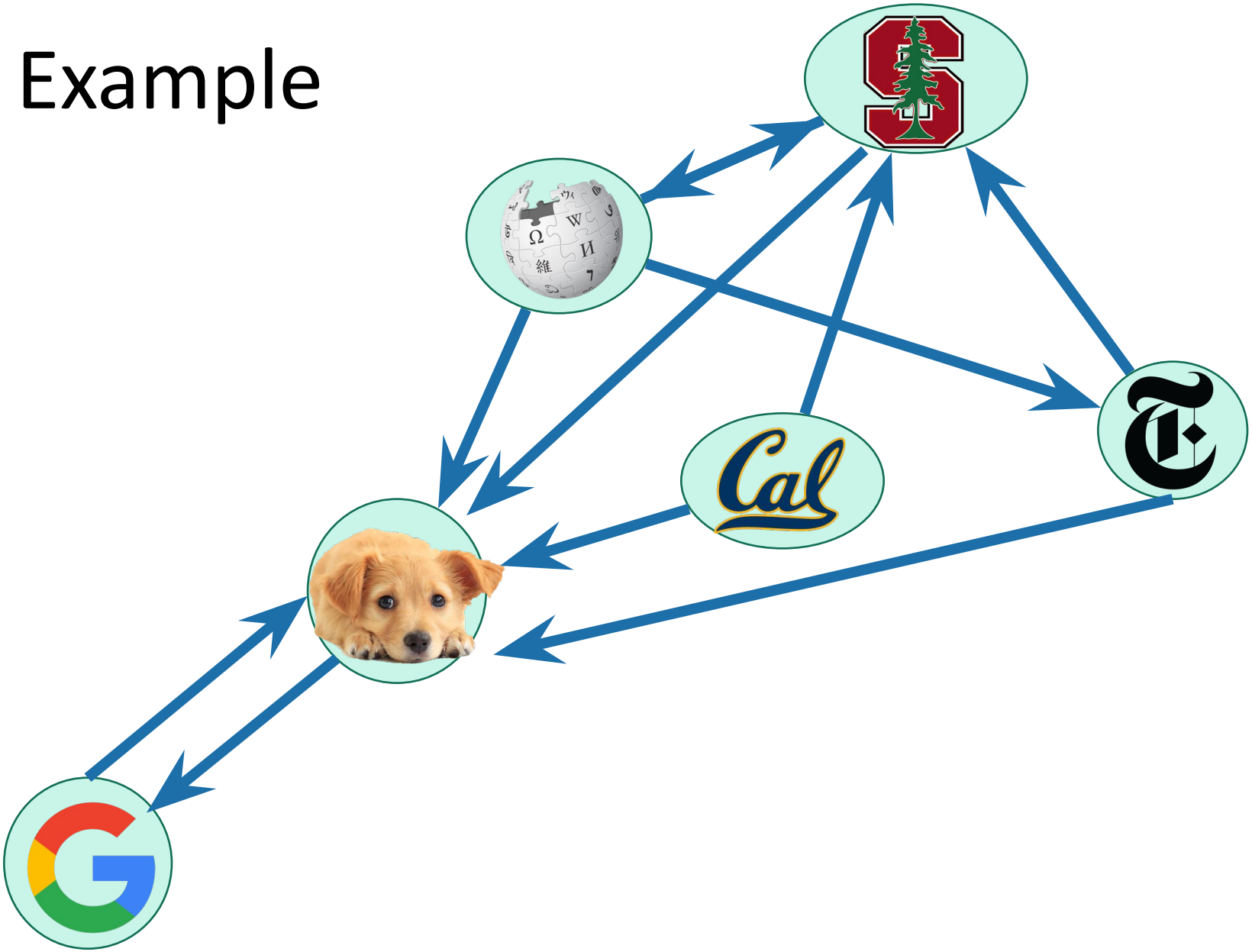
- That will identify SCCs…
- Issues:
  - How do we know where to start DFS?
  - It wouldn't have found the SCCs if we started from A.

Algorithm

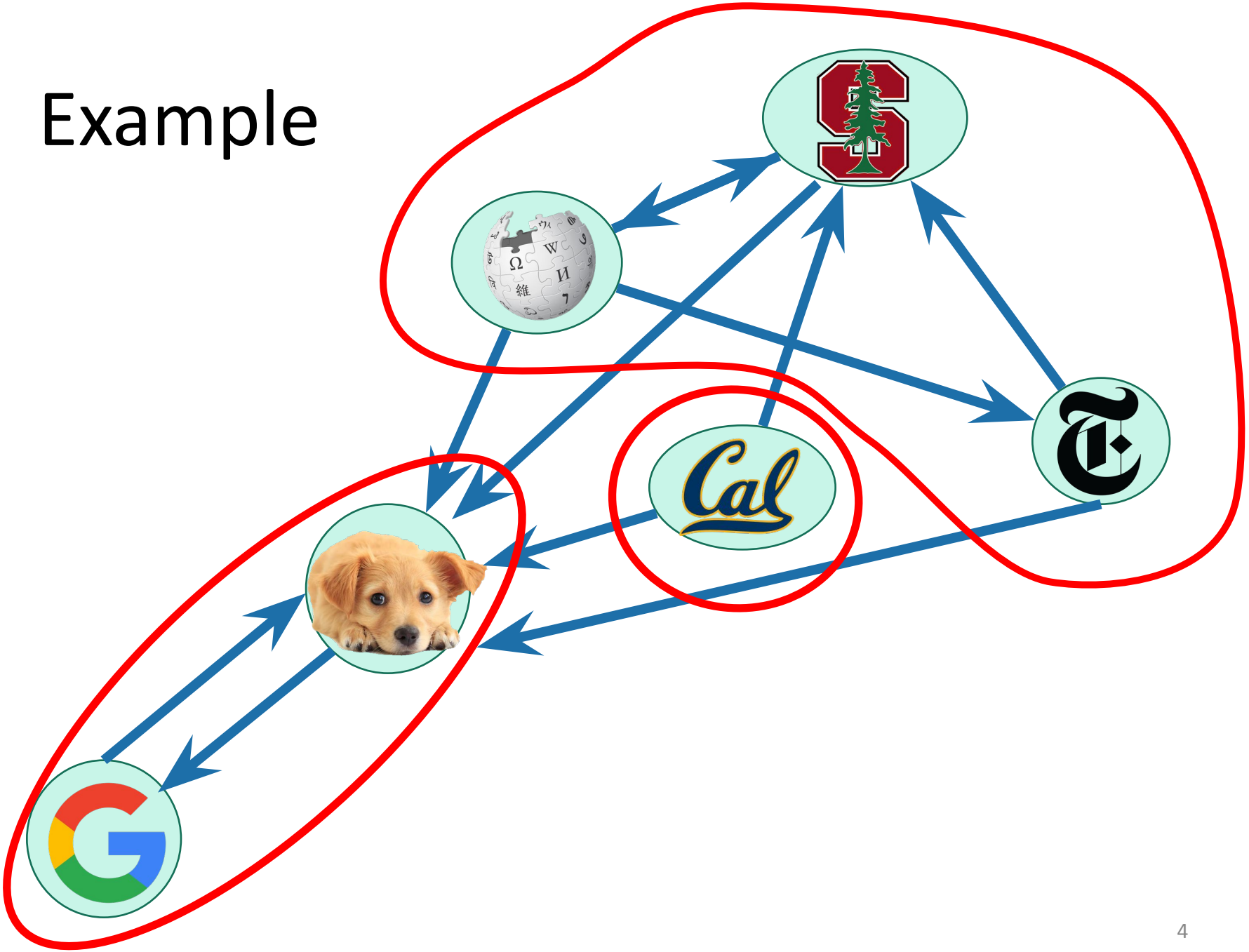
Running time: O(n + m)

- Do DFS to create a DFS forest.
  - Choose starting vertices in any order.
  - Keep track of finishing times.
- Reverse all the edges in the graph.
- Do DFS again to create **another DFS forest**.
  - This time, order the nodes in the reverse order of the finishing times that they had from the first DFS run.
- The SCCs are the different trees in the **second DFS forest.**

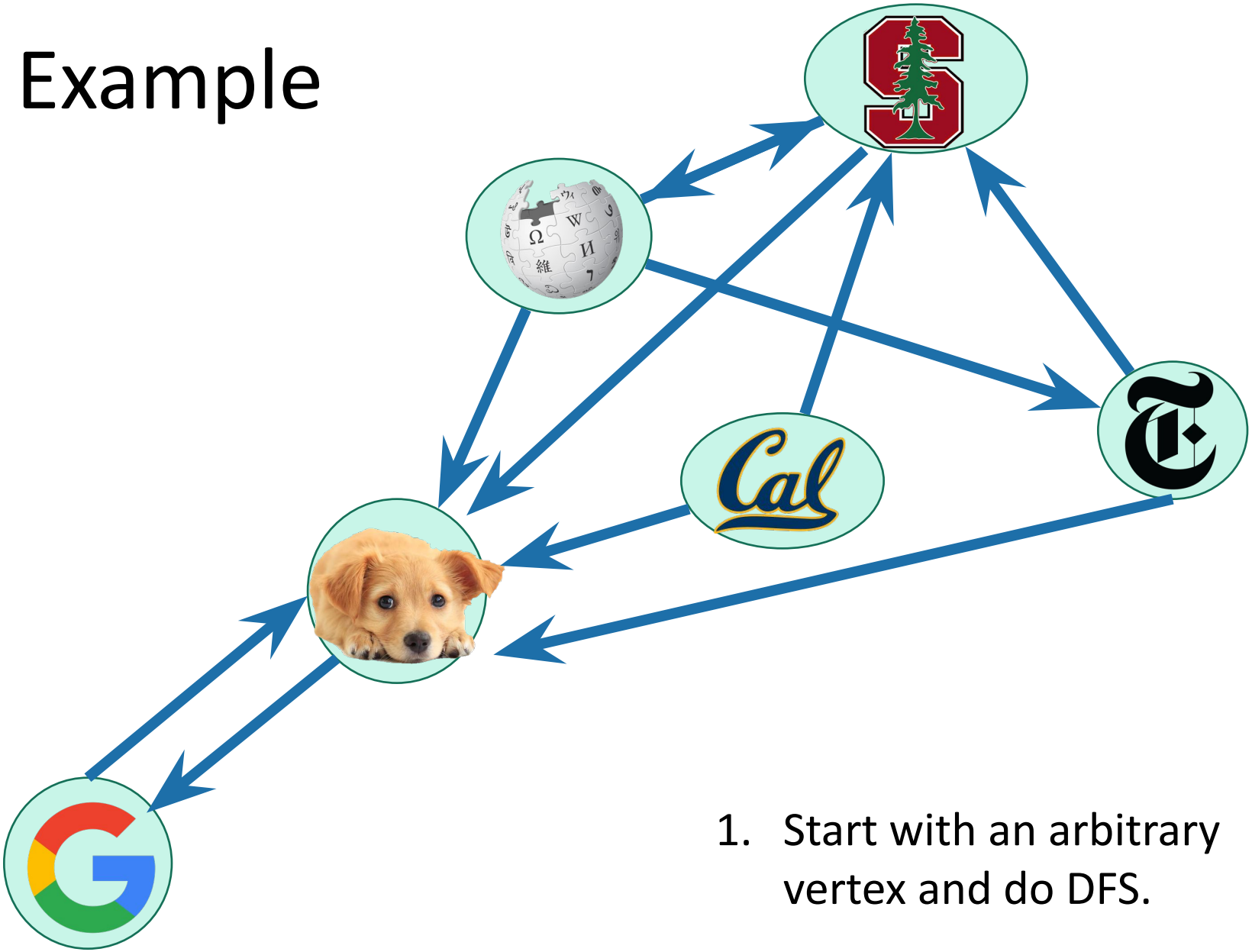
Example



Example

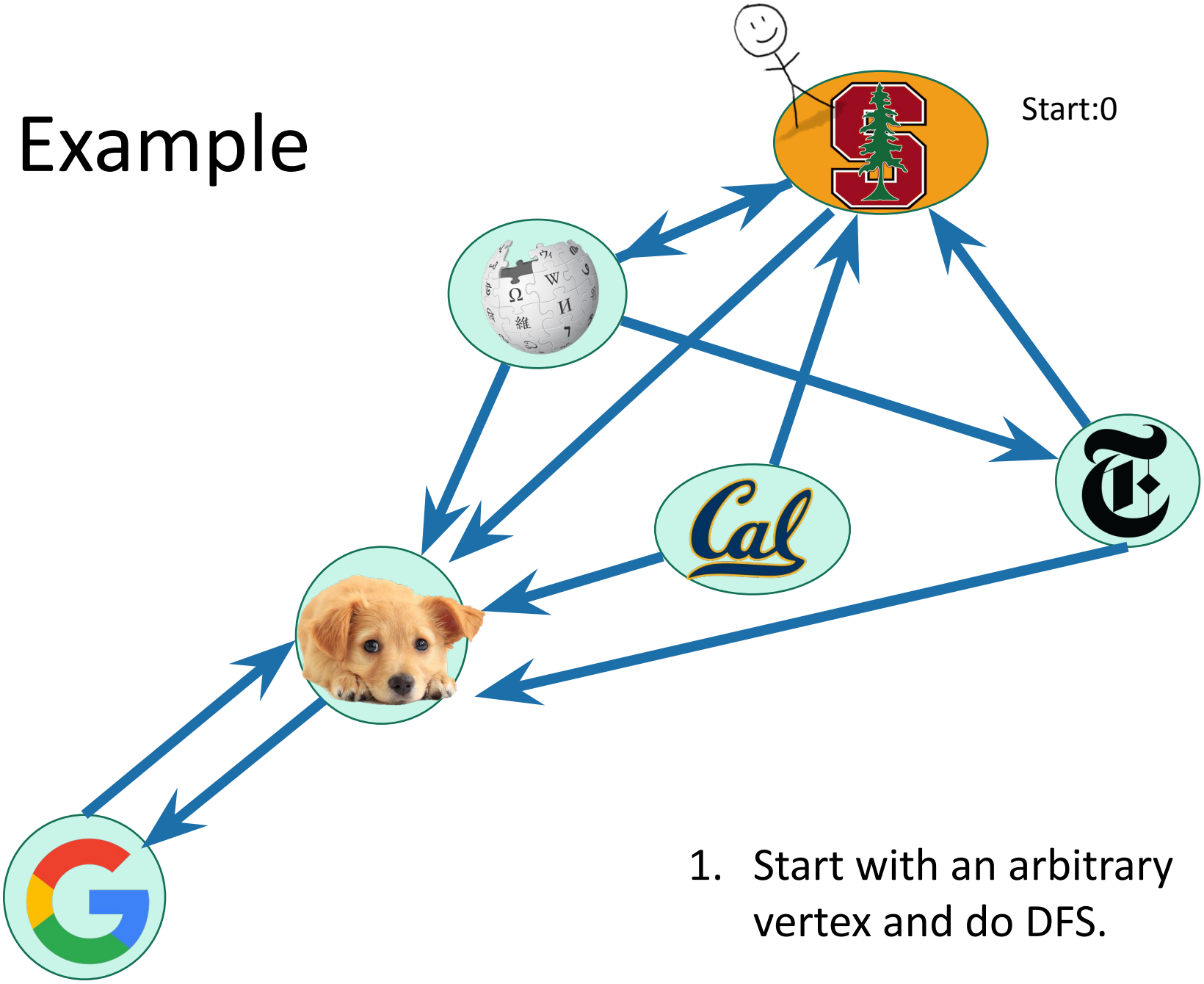


Example



1. Start with an arbitrary vertex and do DFS.

Example



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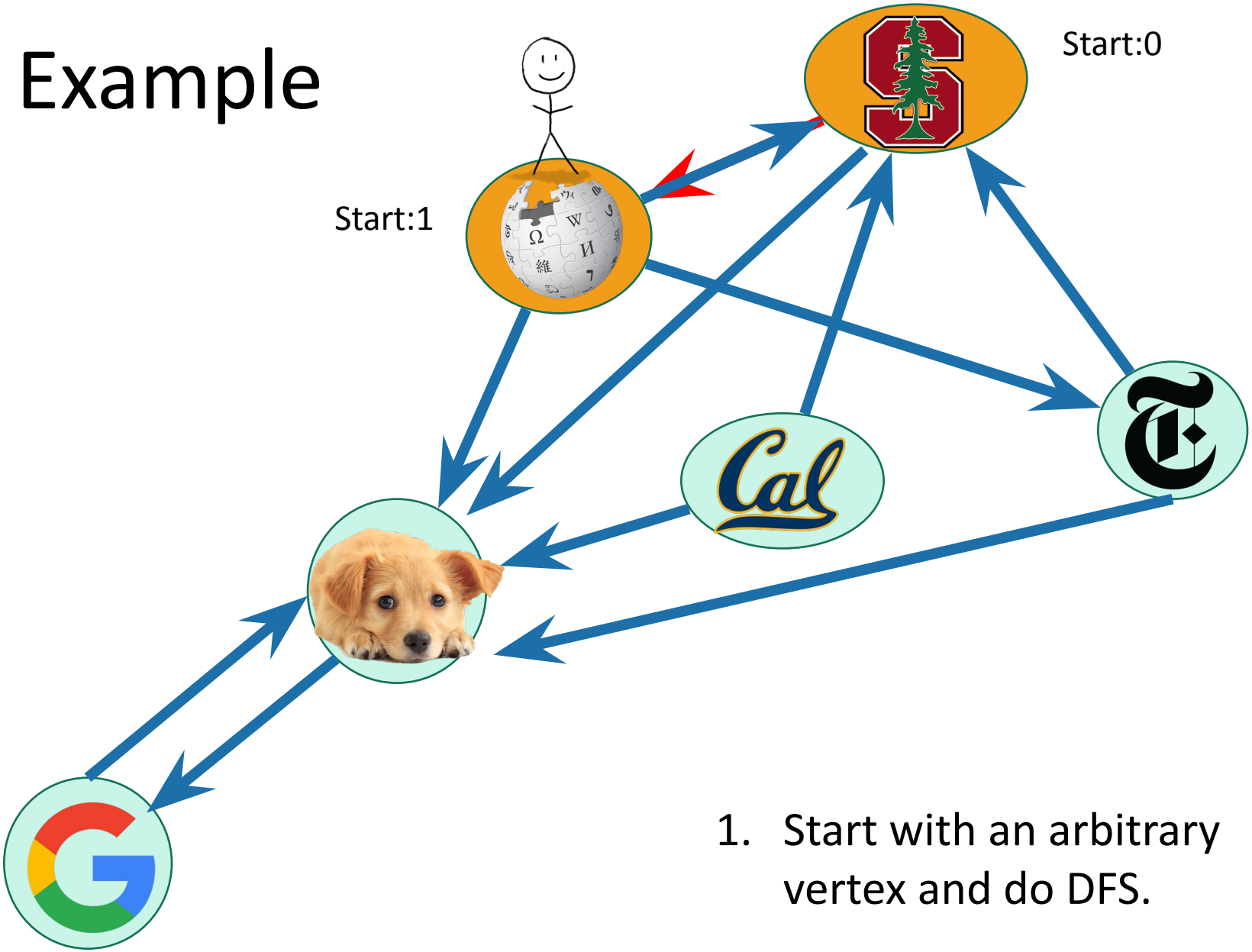

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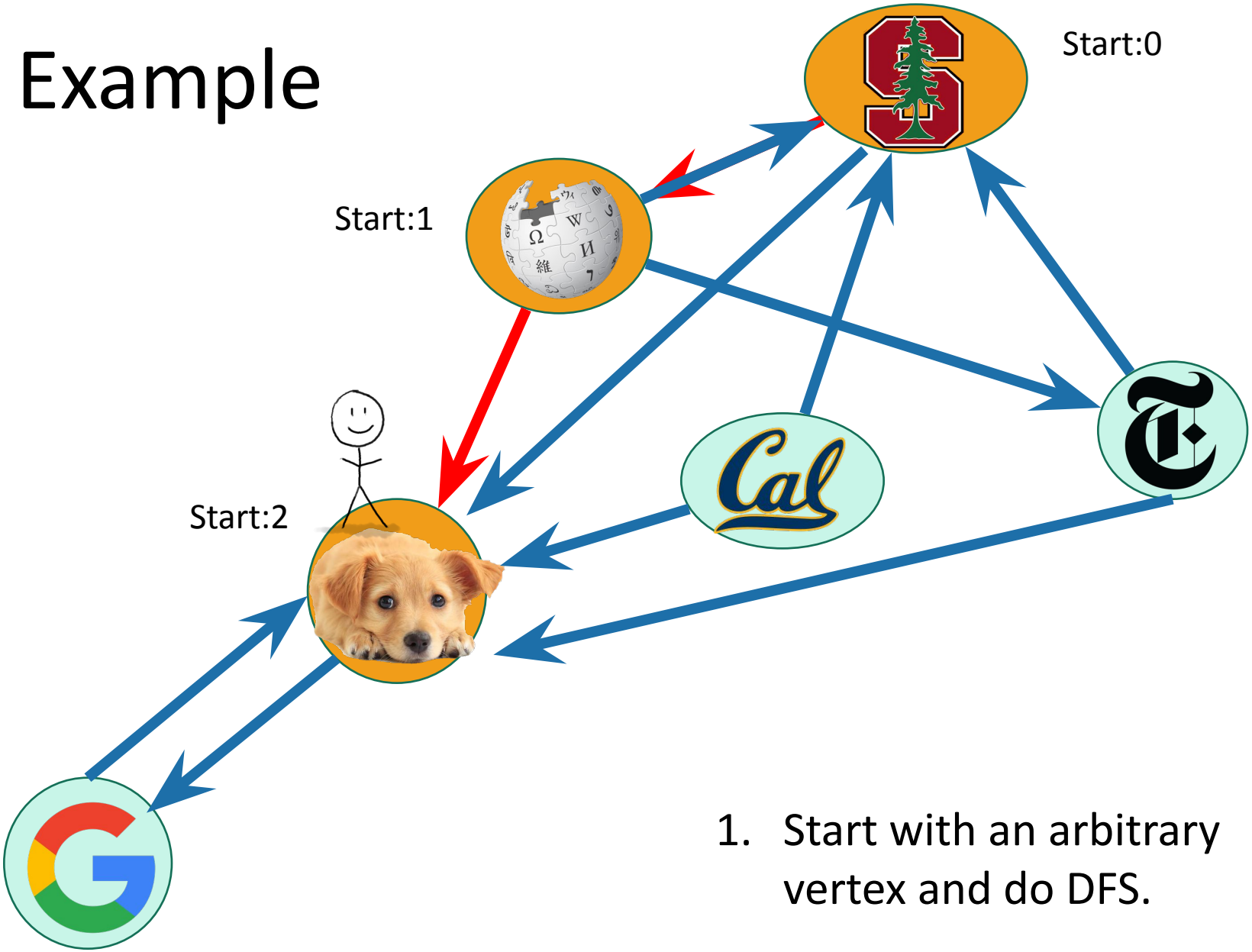

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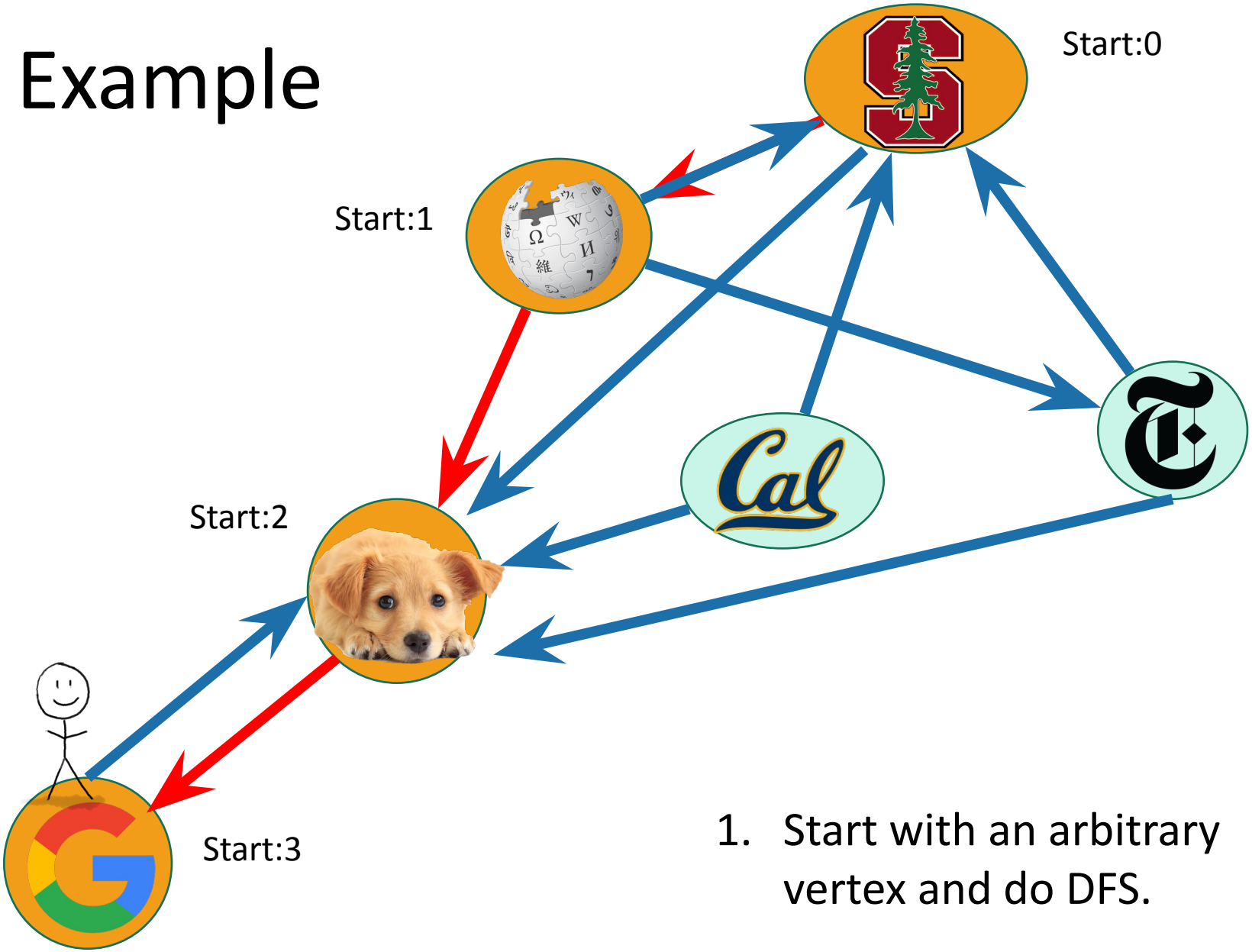
Example



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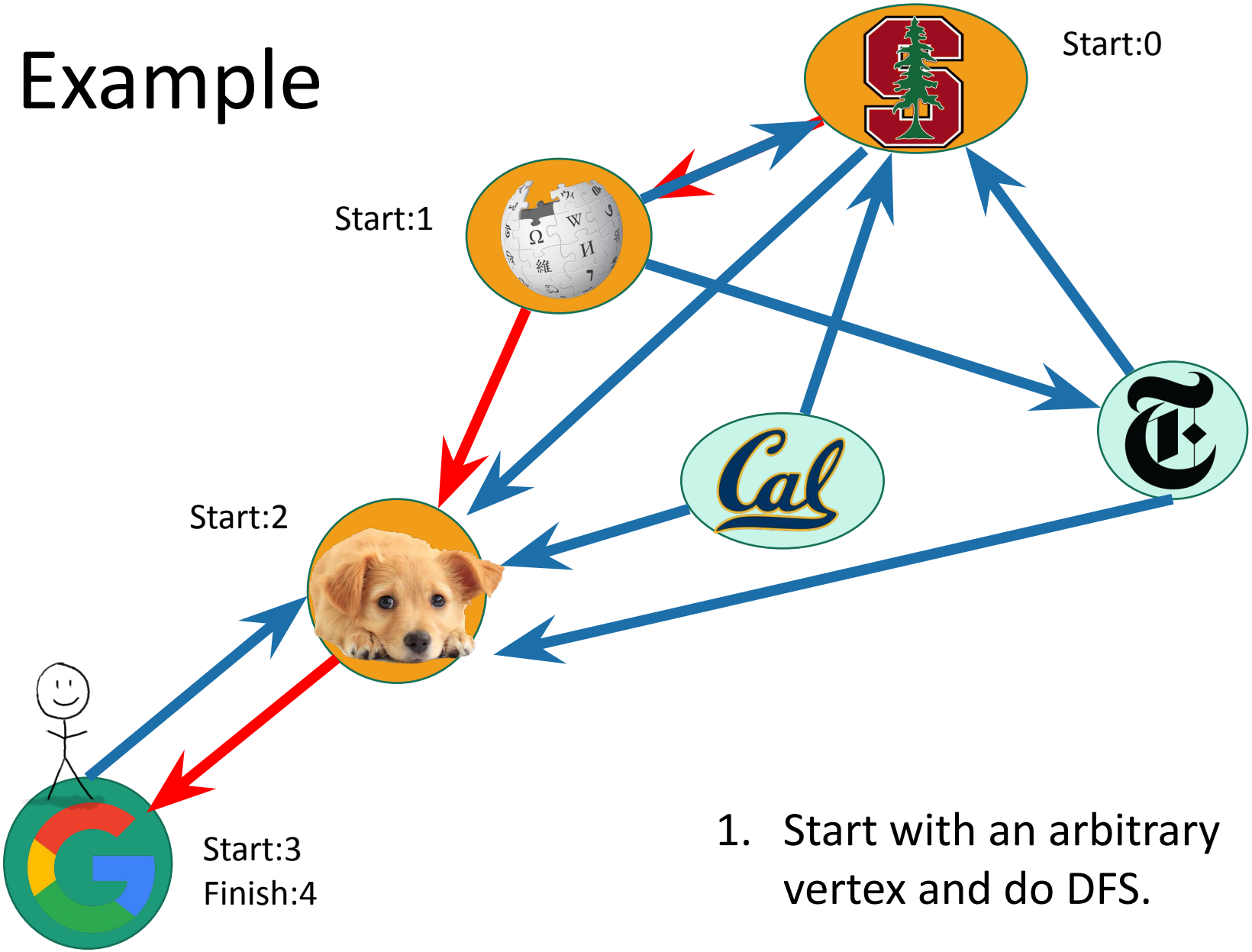

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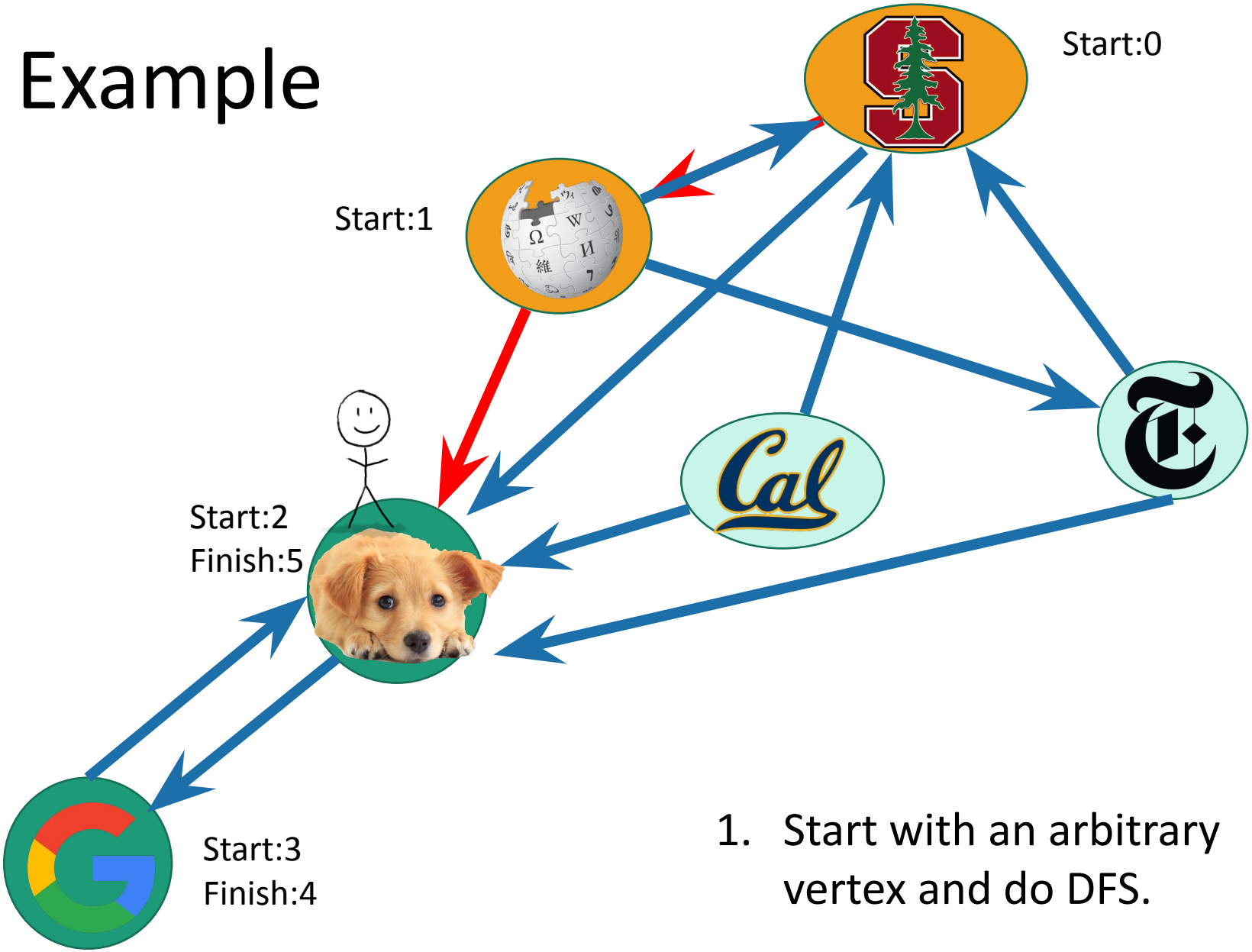
Example



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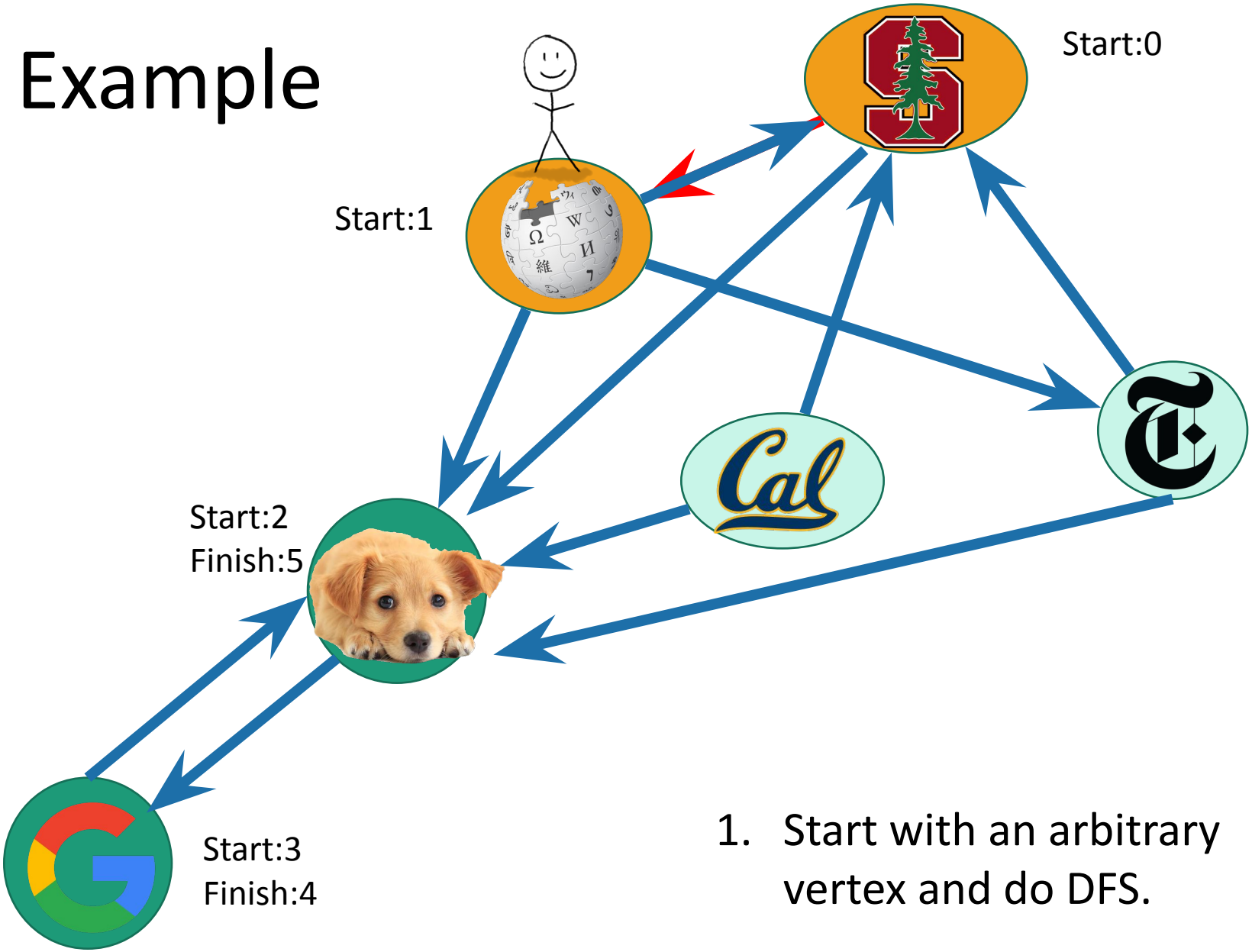
Example



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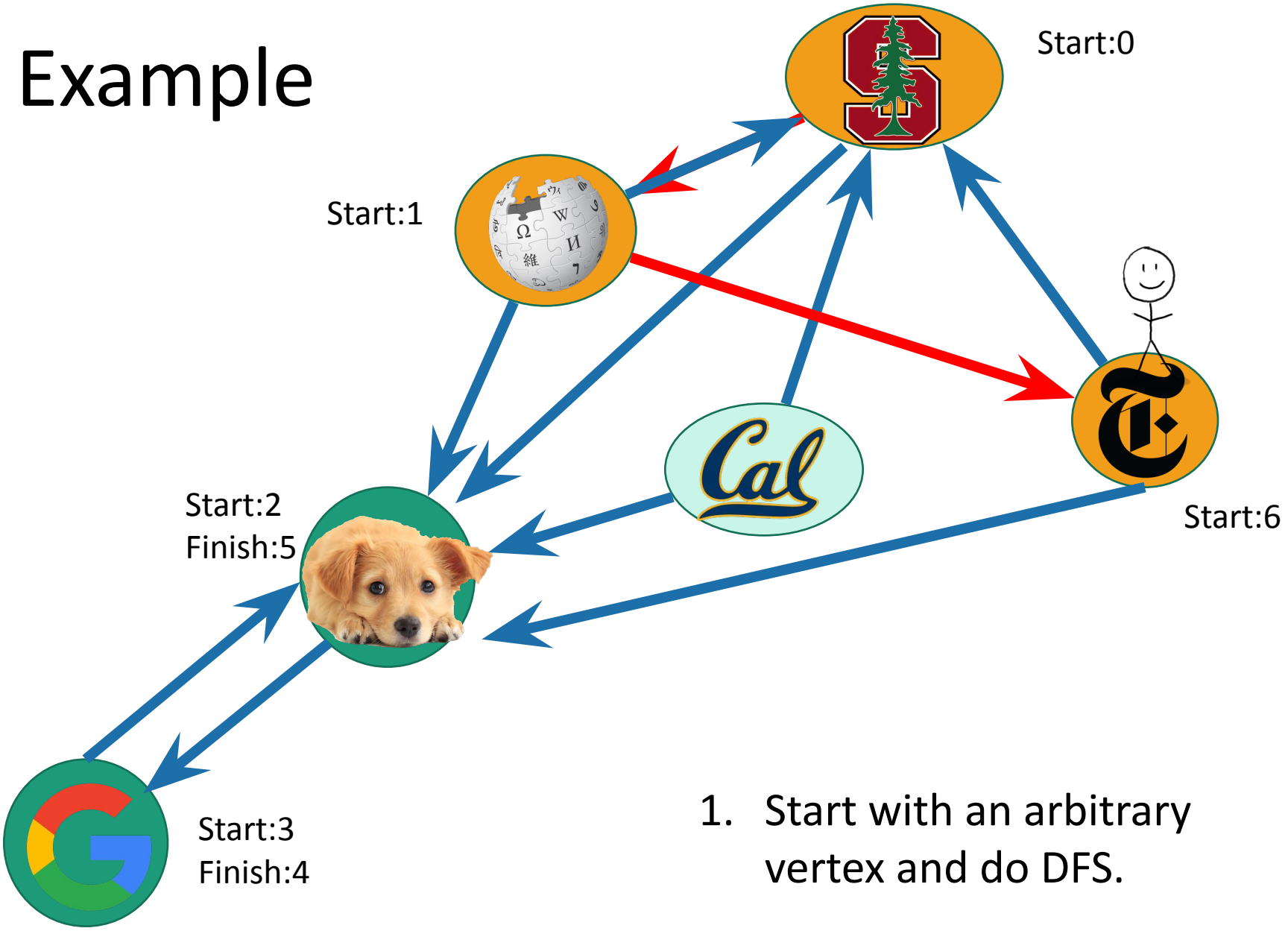
Example



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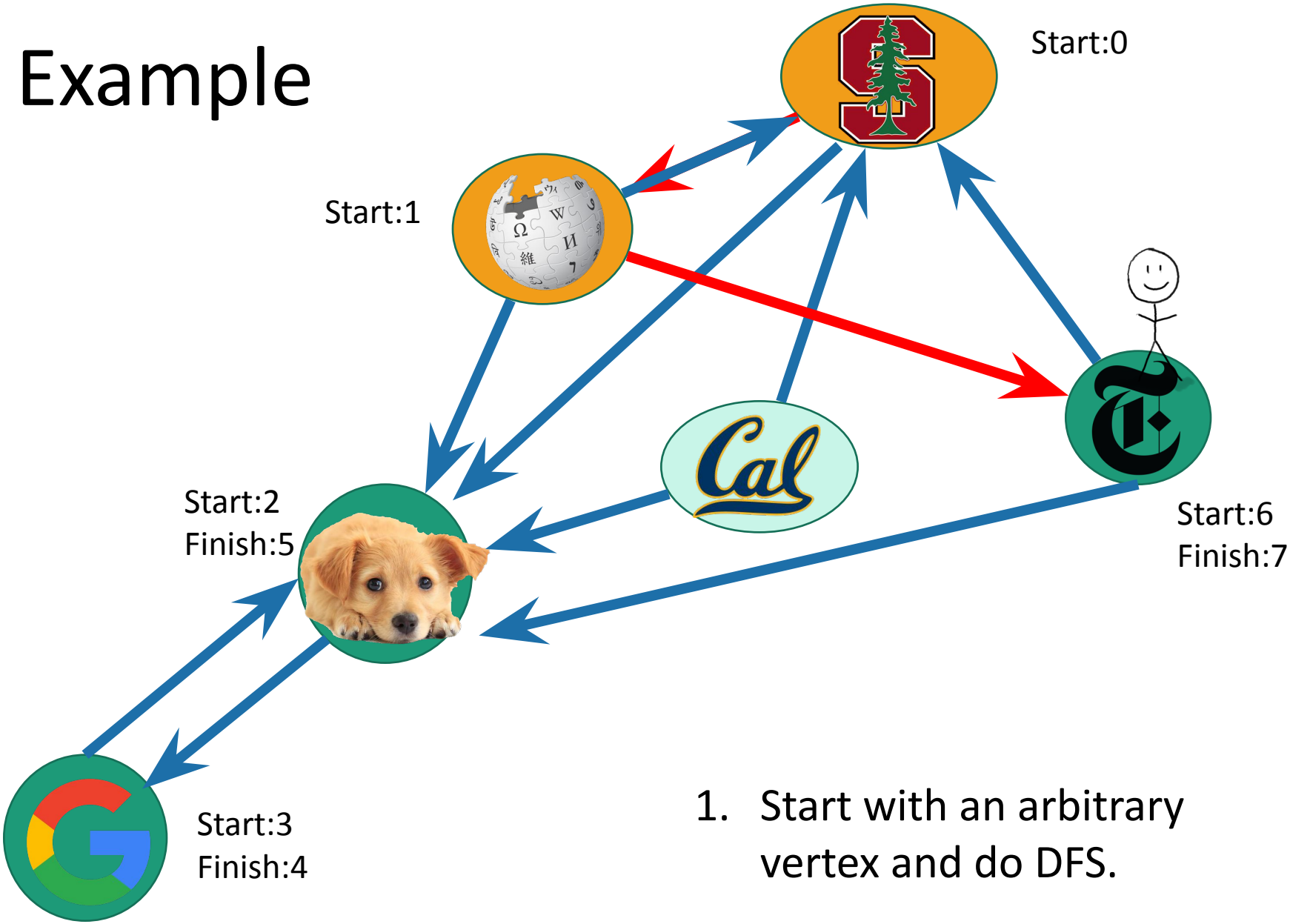

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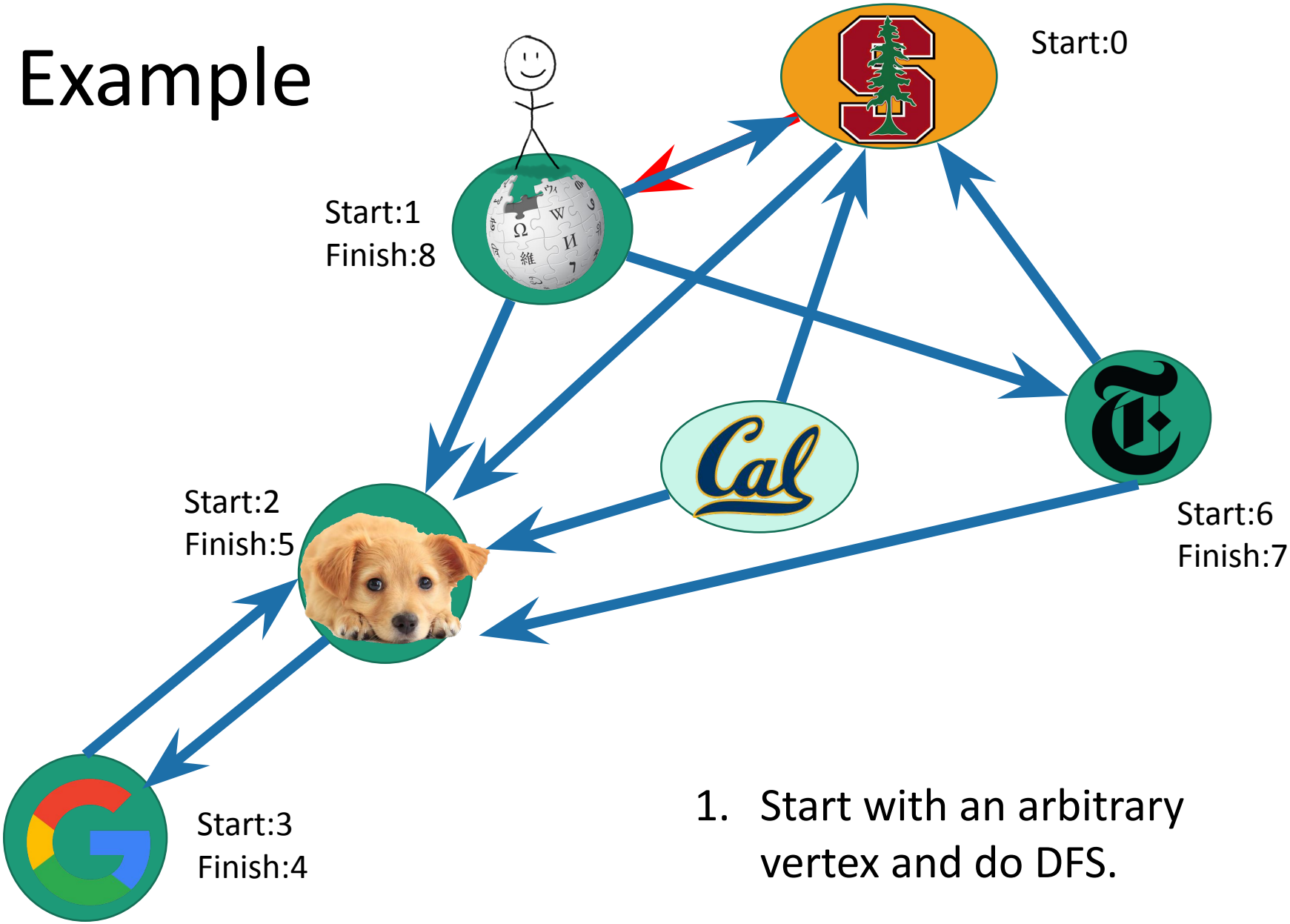

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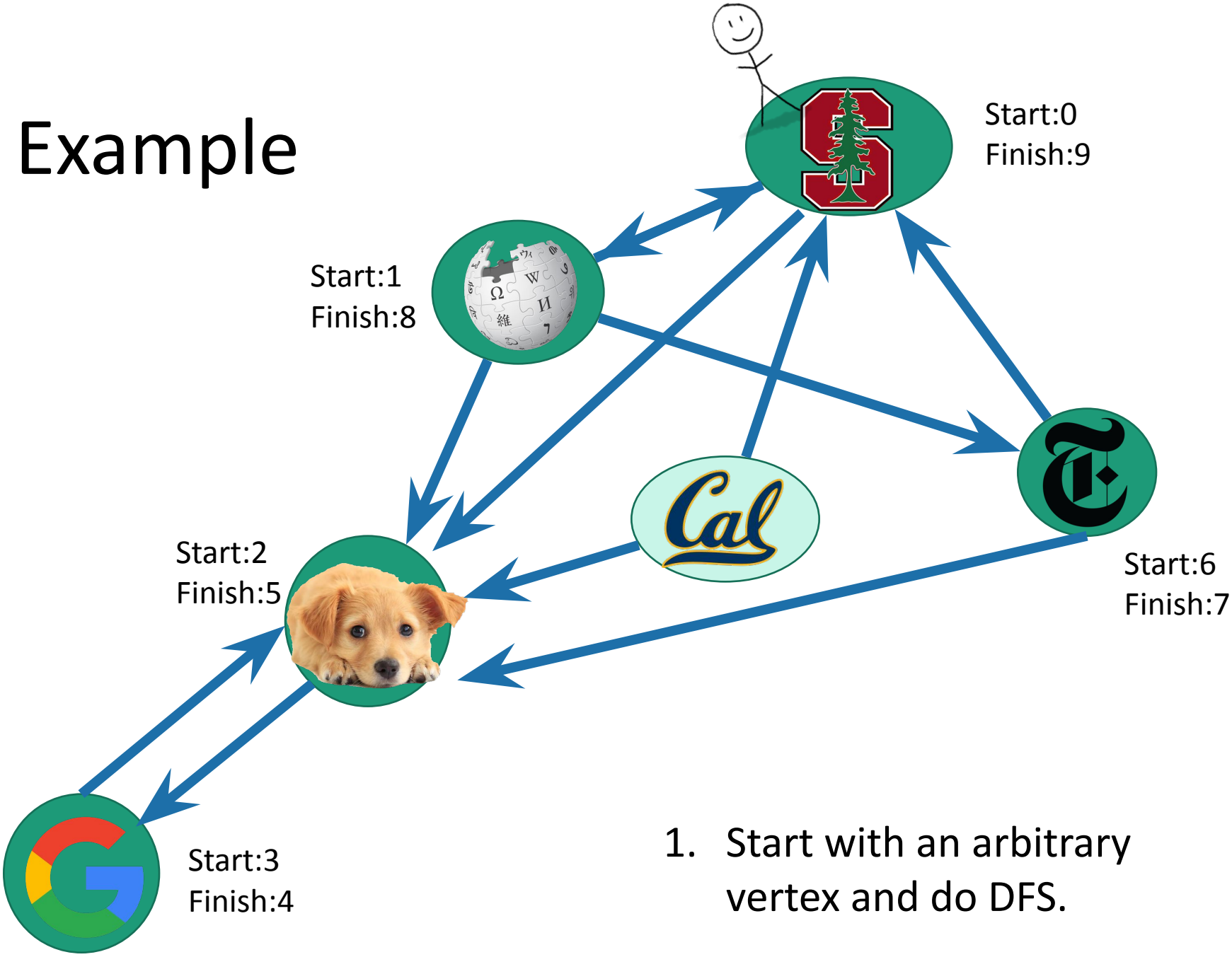

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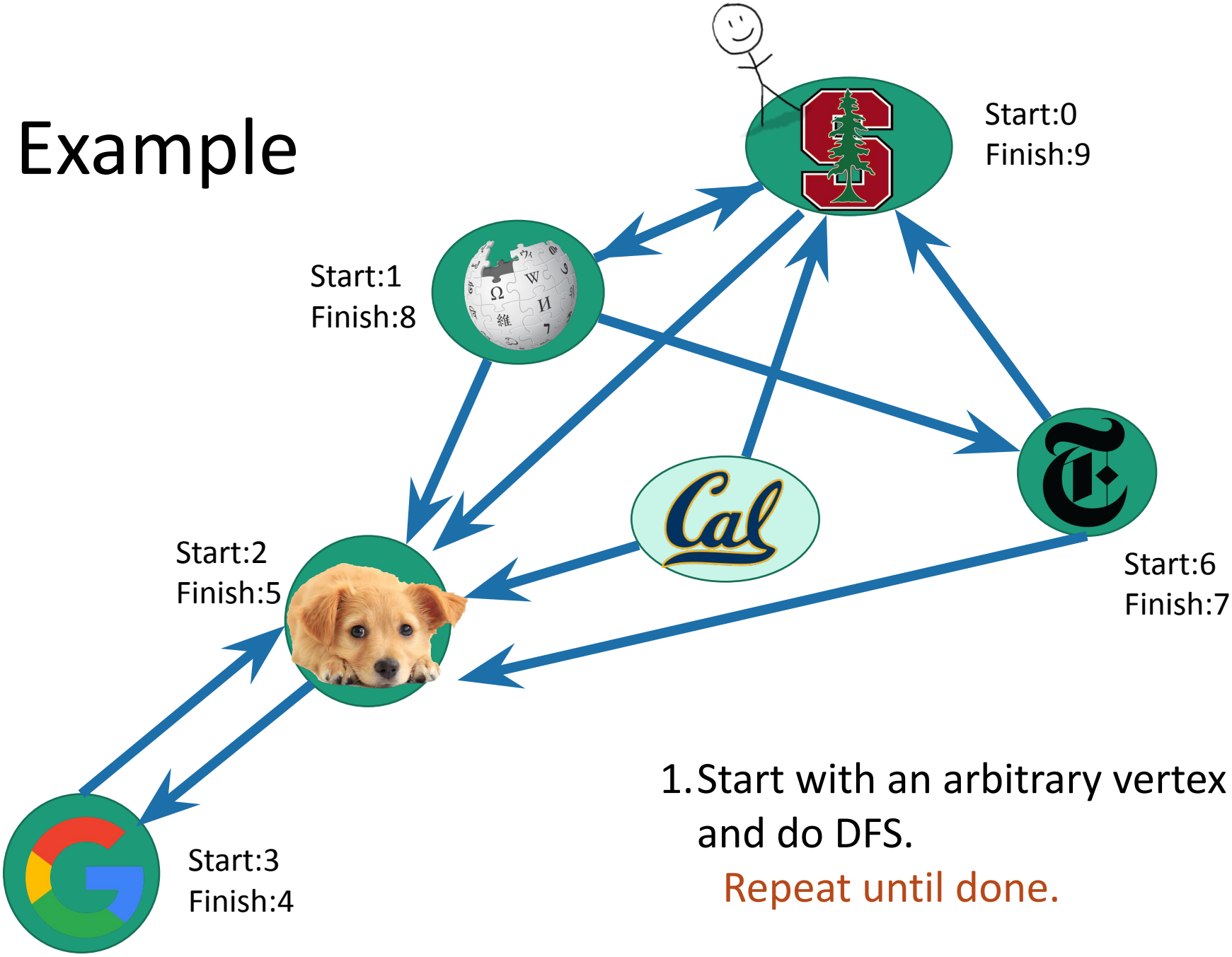
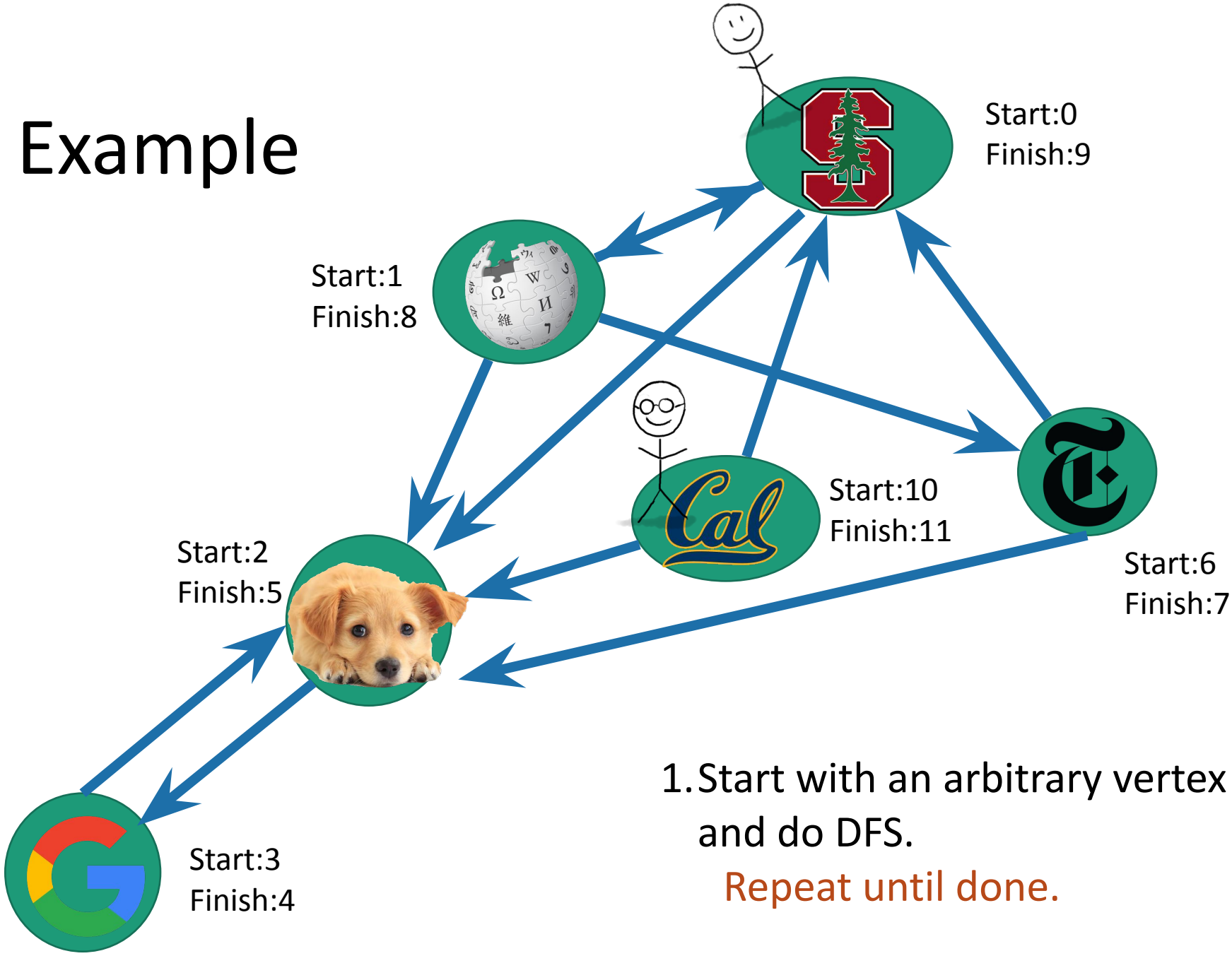
Example



1. Start with an arbitrary vertex and do DFS.
   Repeat until done.

Example



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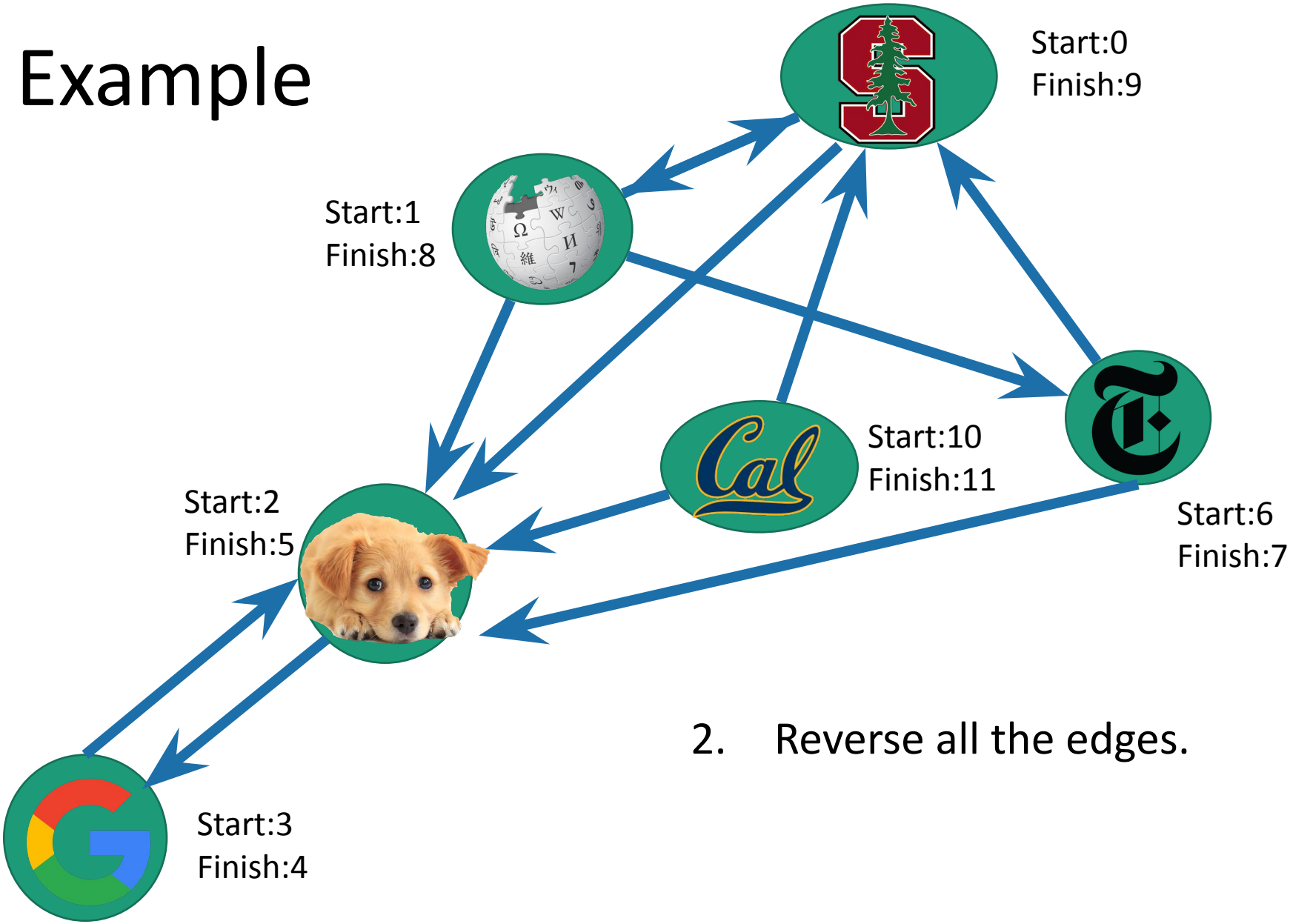
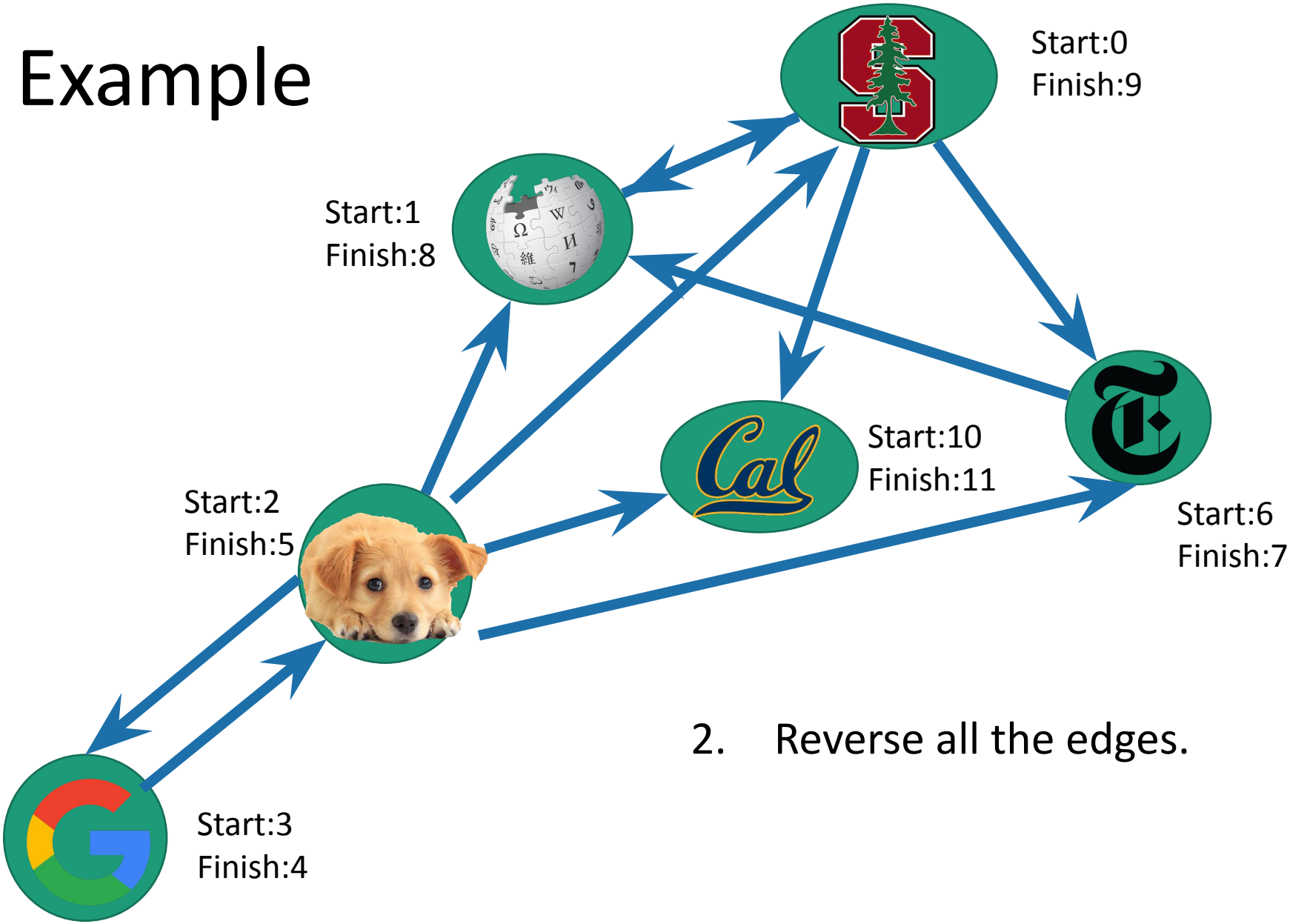


2. Reverse all the edges.

Example



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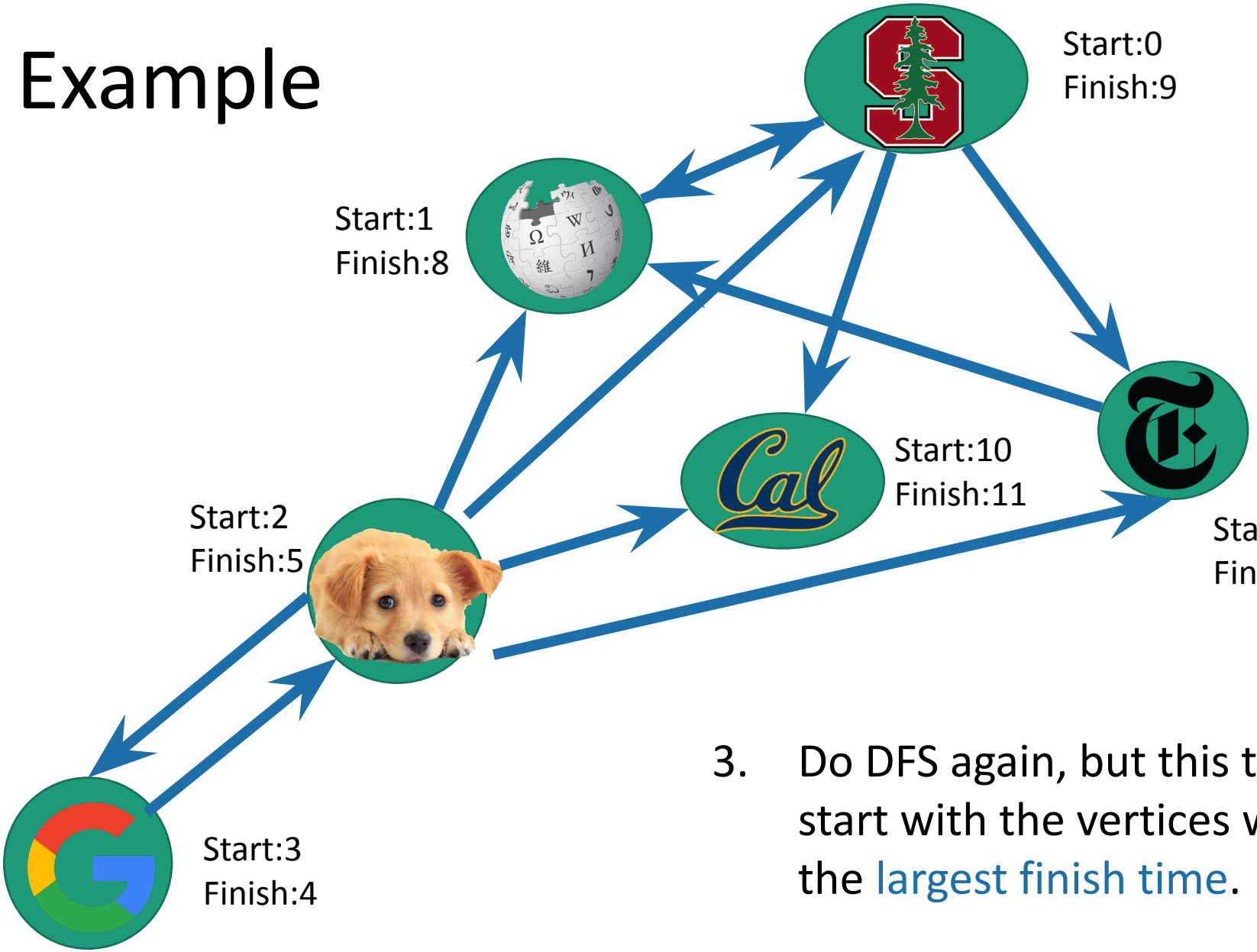
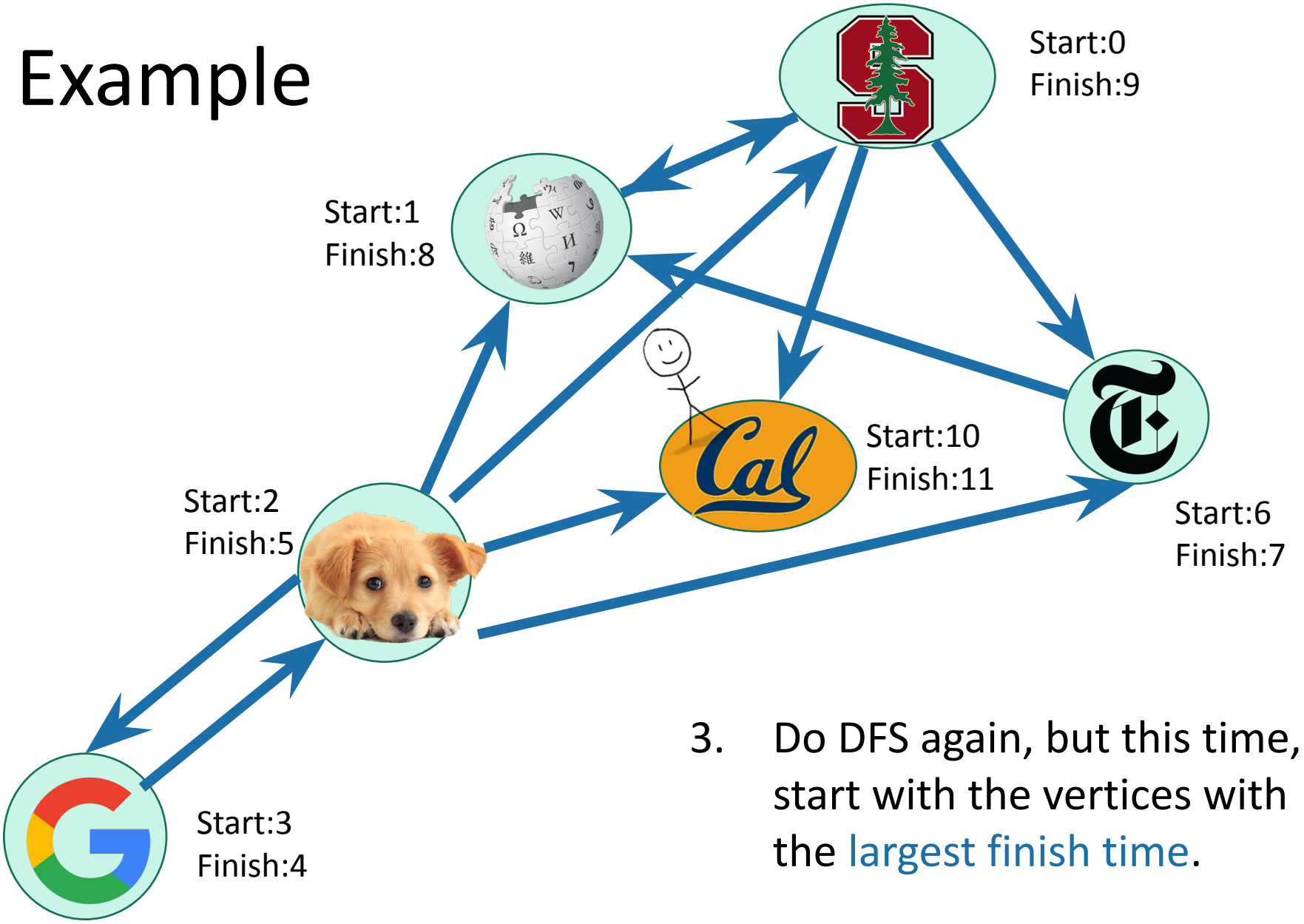


3. Do DFS again, but this time, start with the vertices with the largest finish time.

Example



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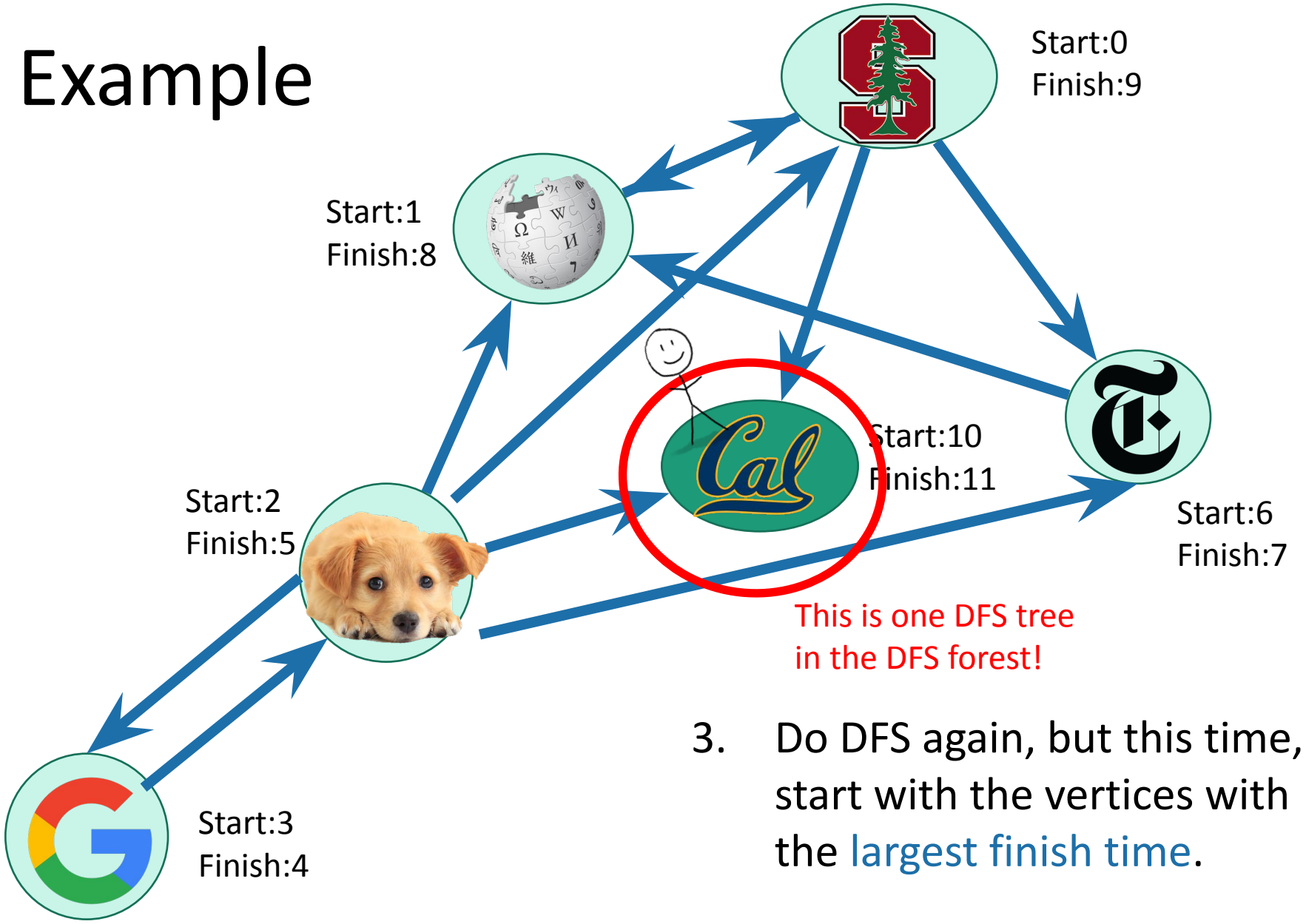

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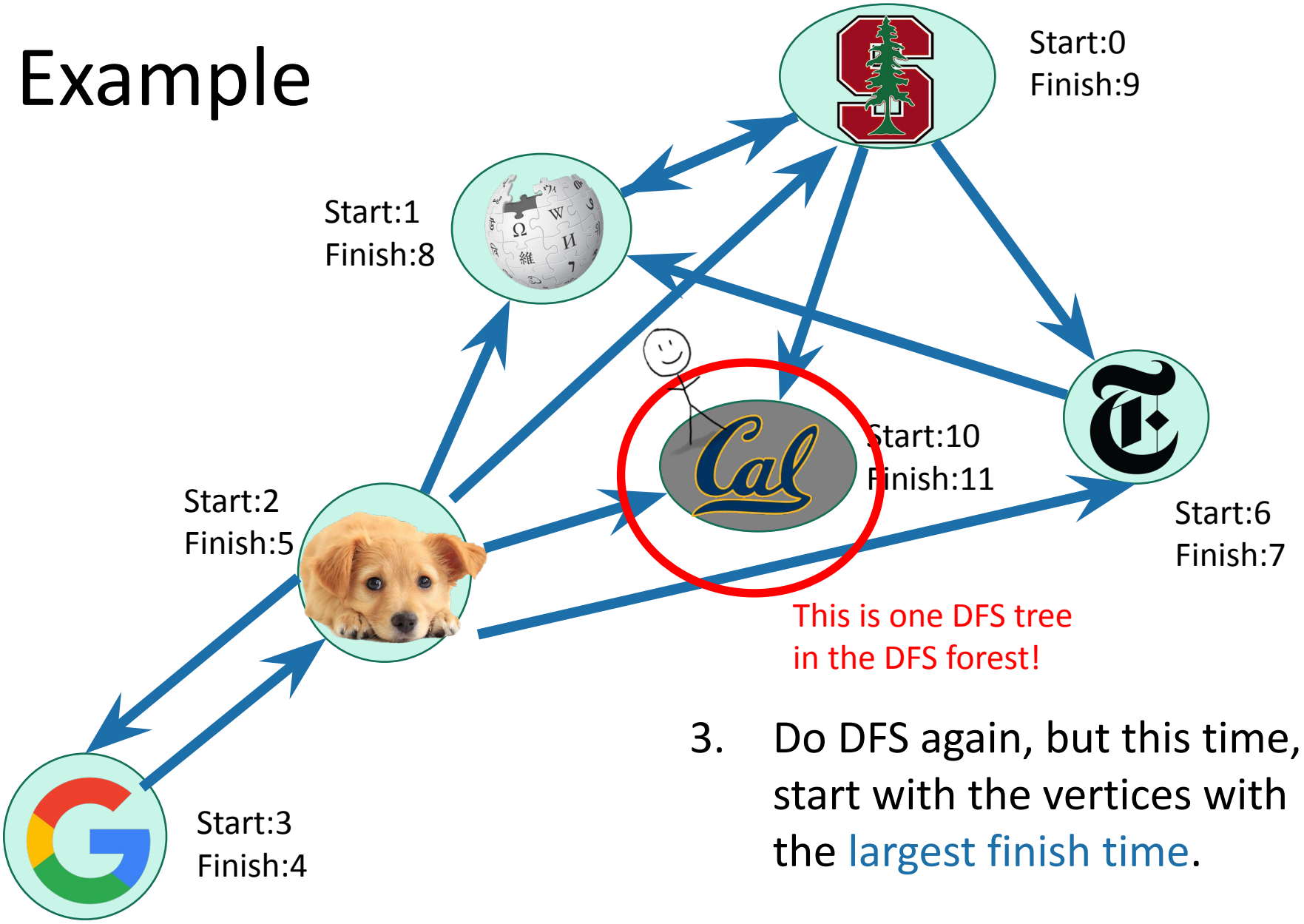

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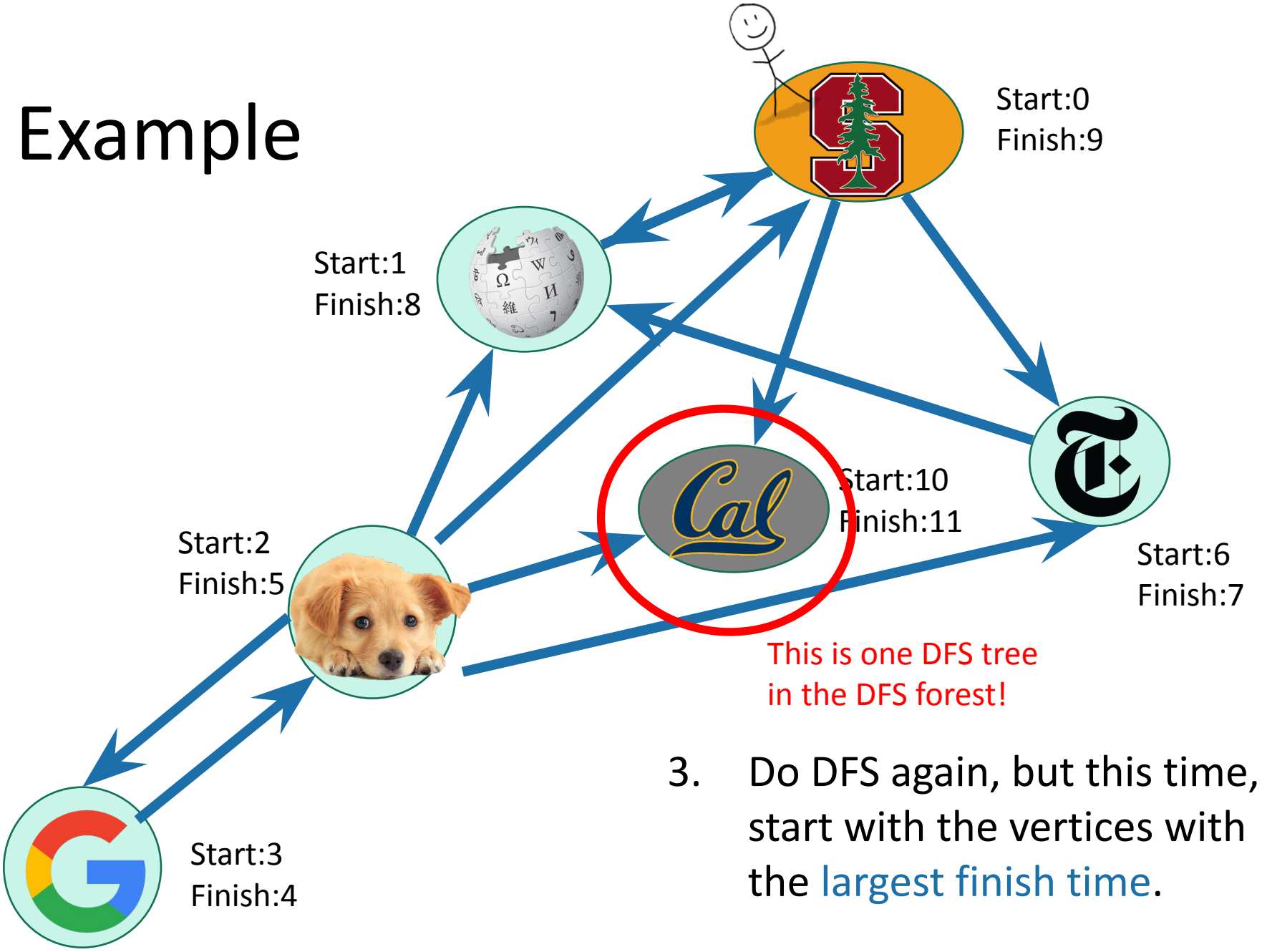

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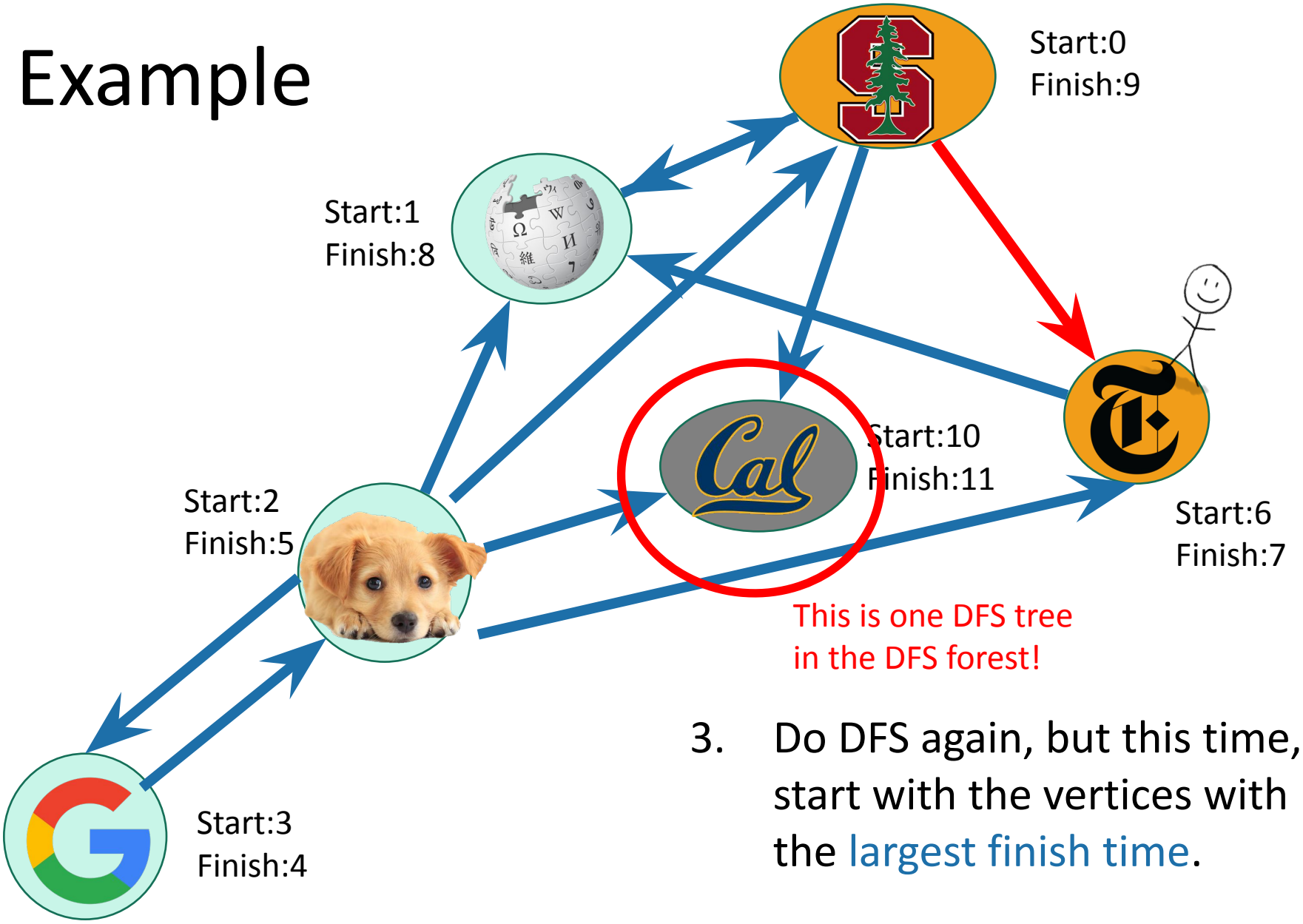

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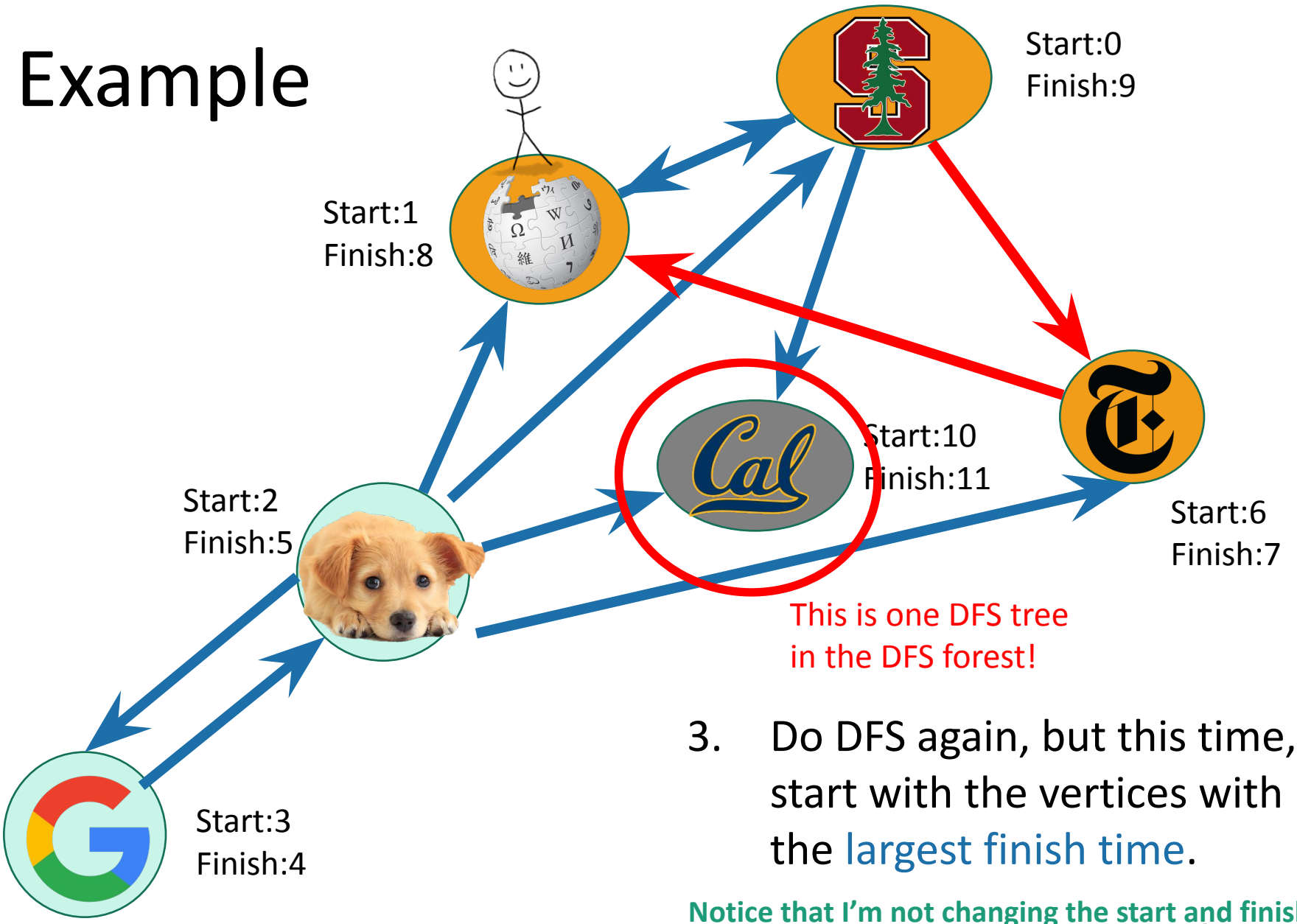

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Notice that I'm not changing the start and finish times – I'm keeping them from the first run.

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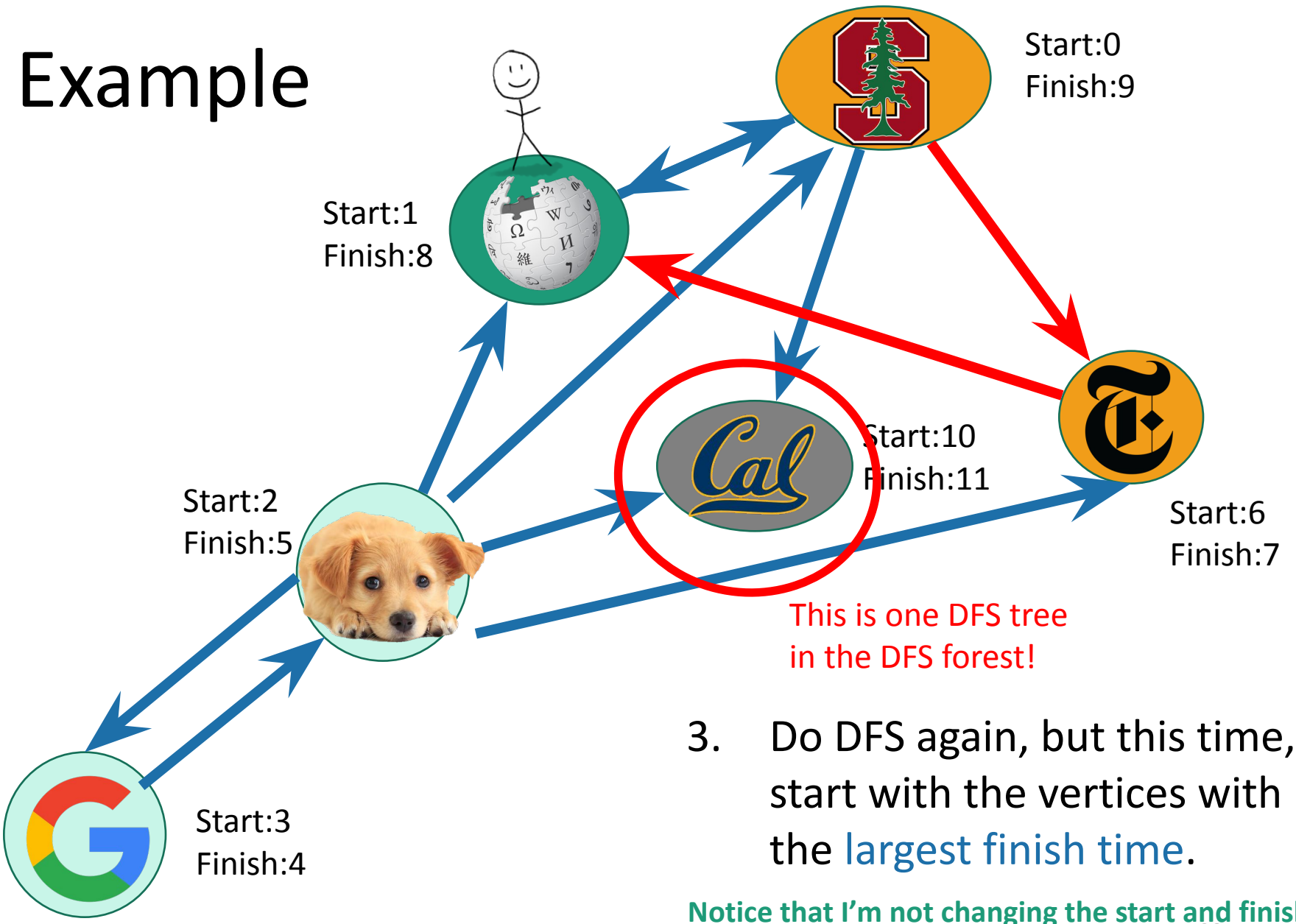

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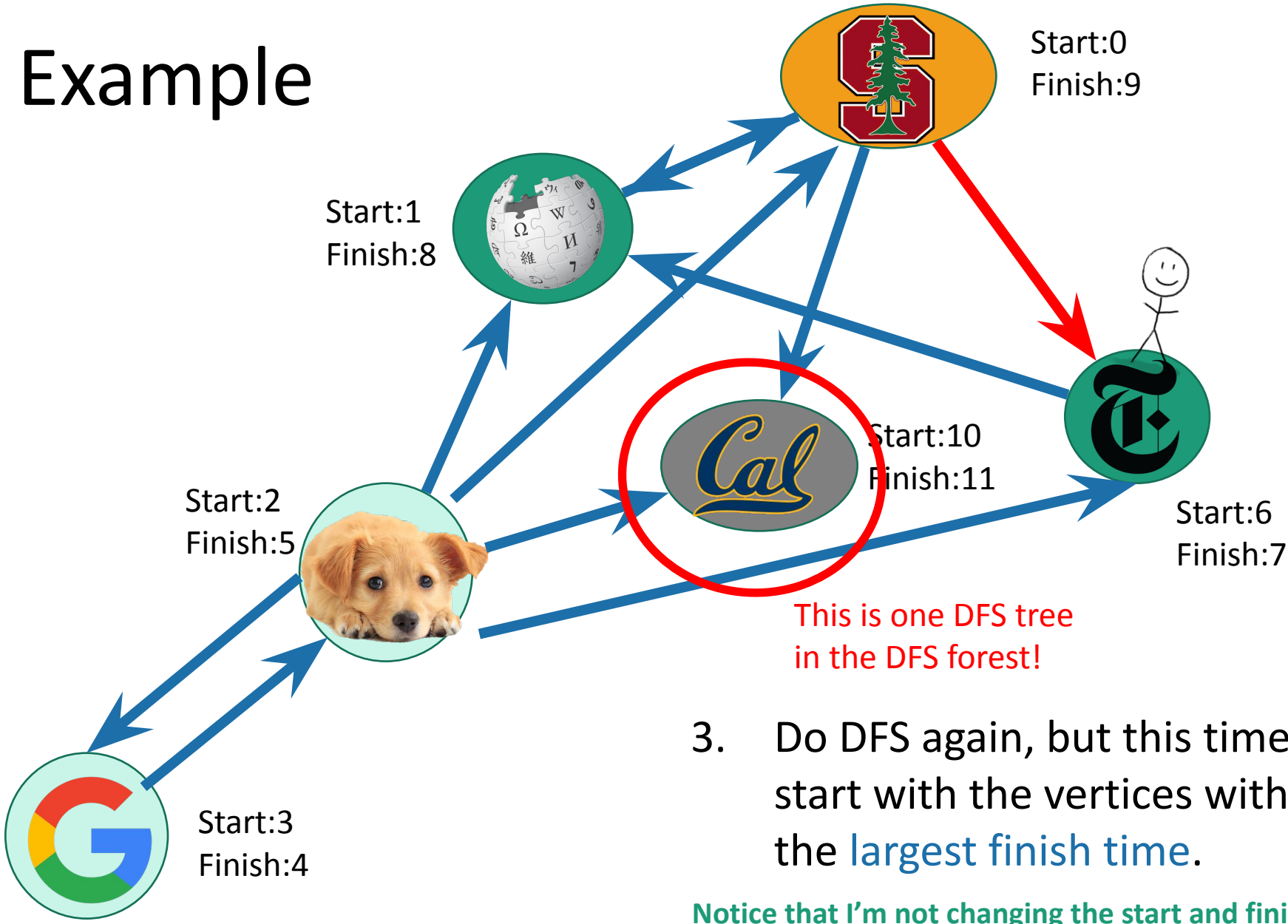

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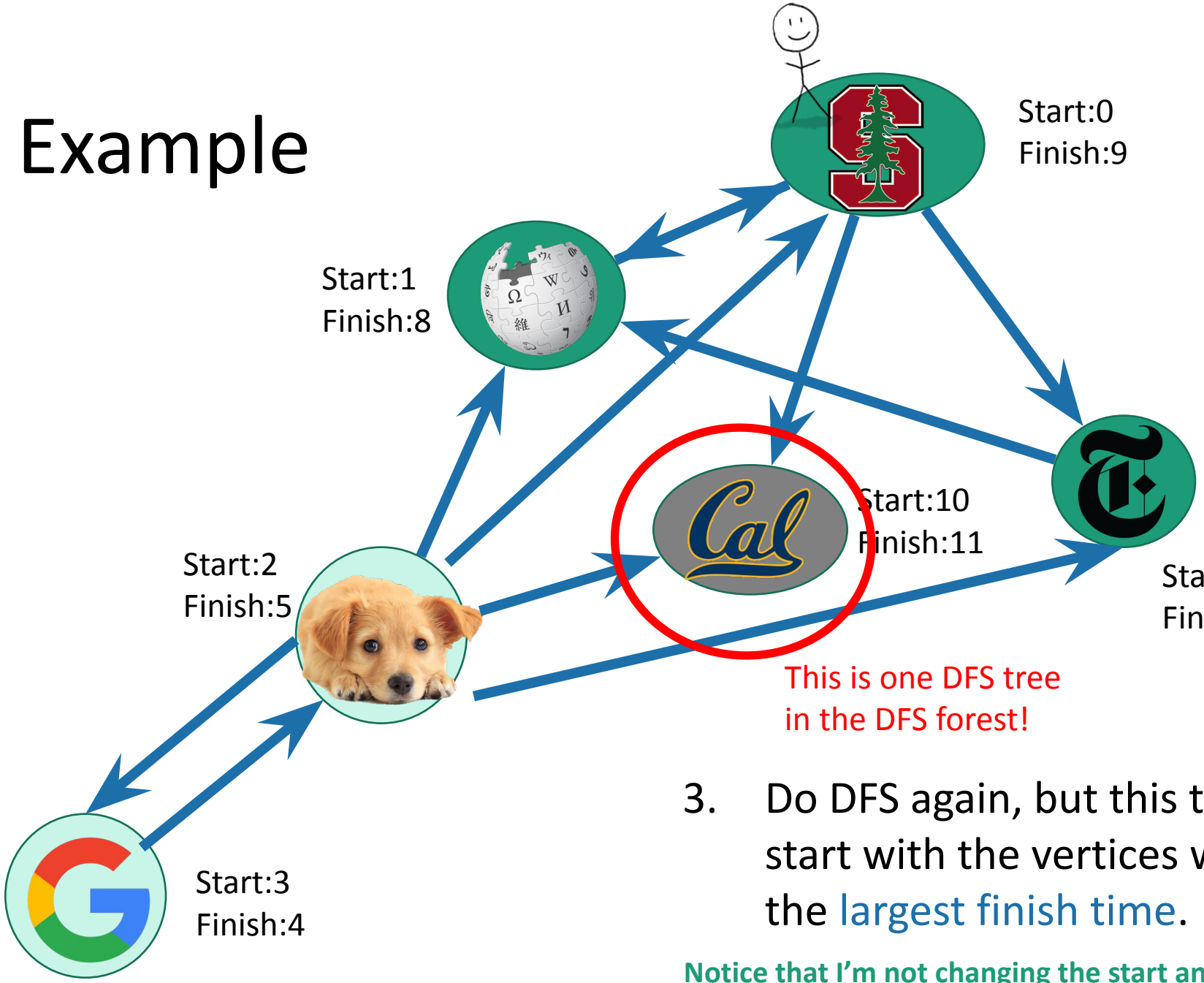

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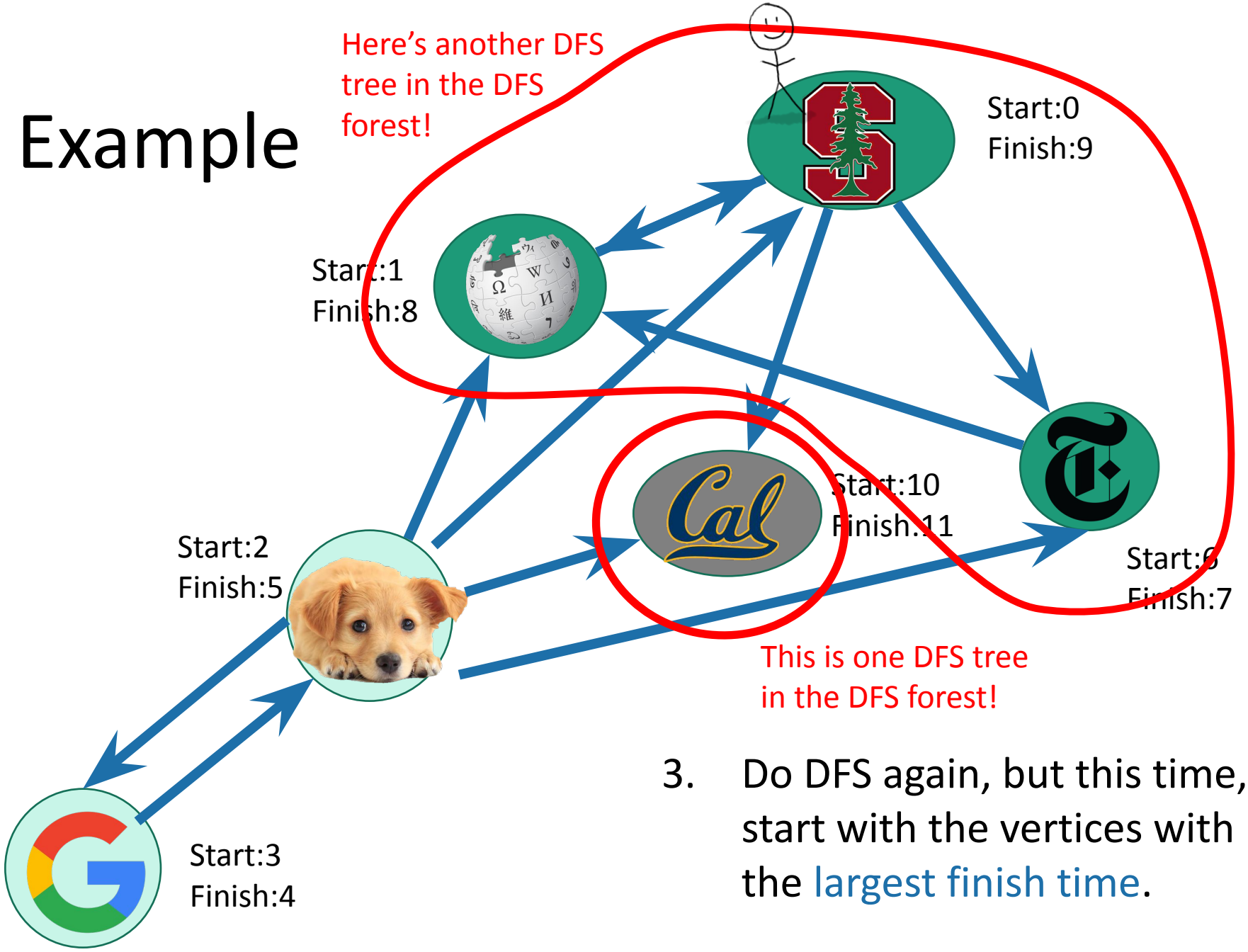

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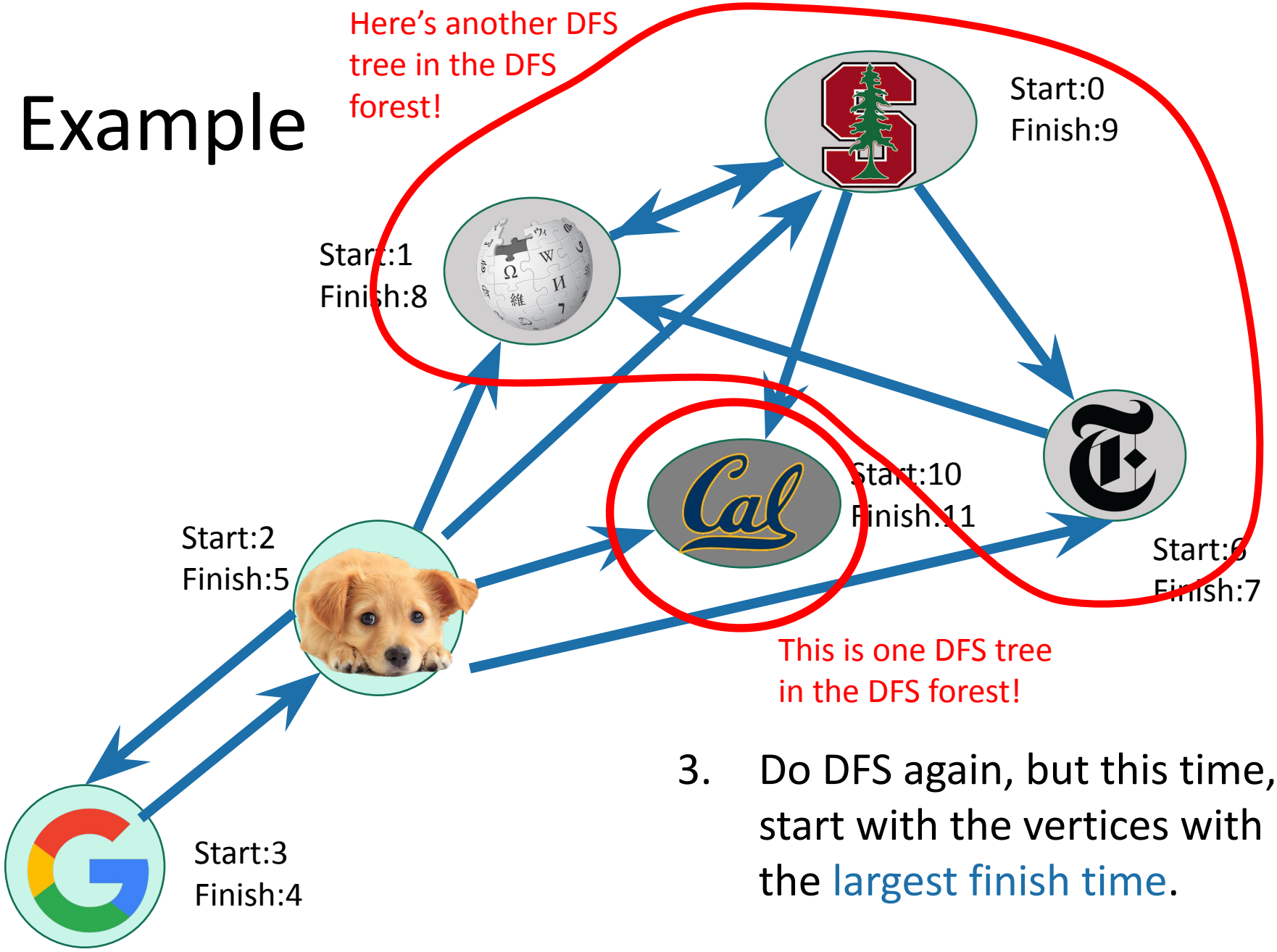

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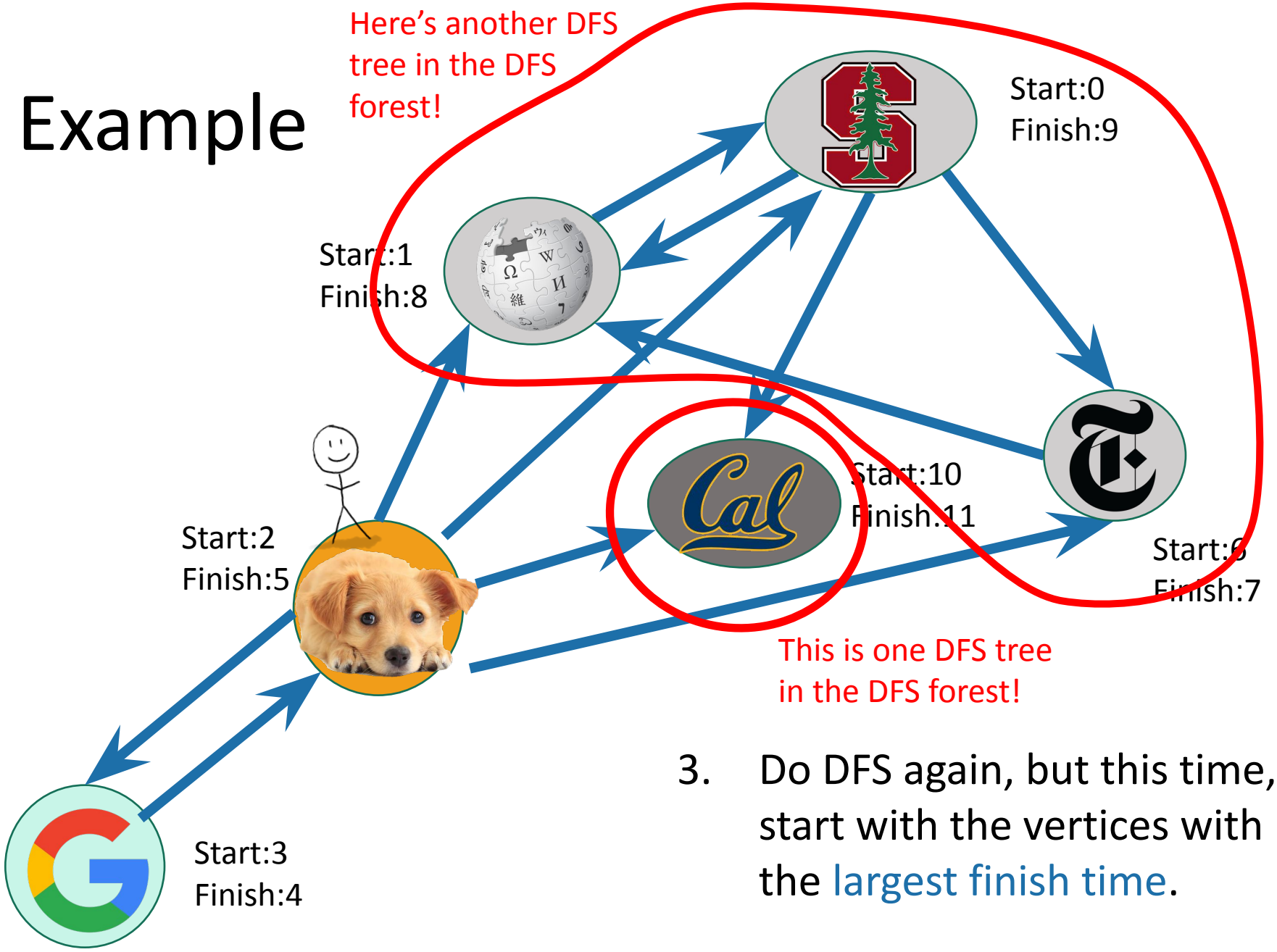

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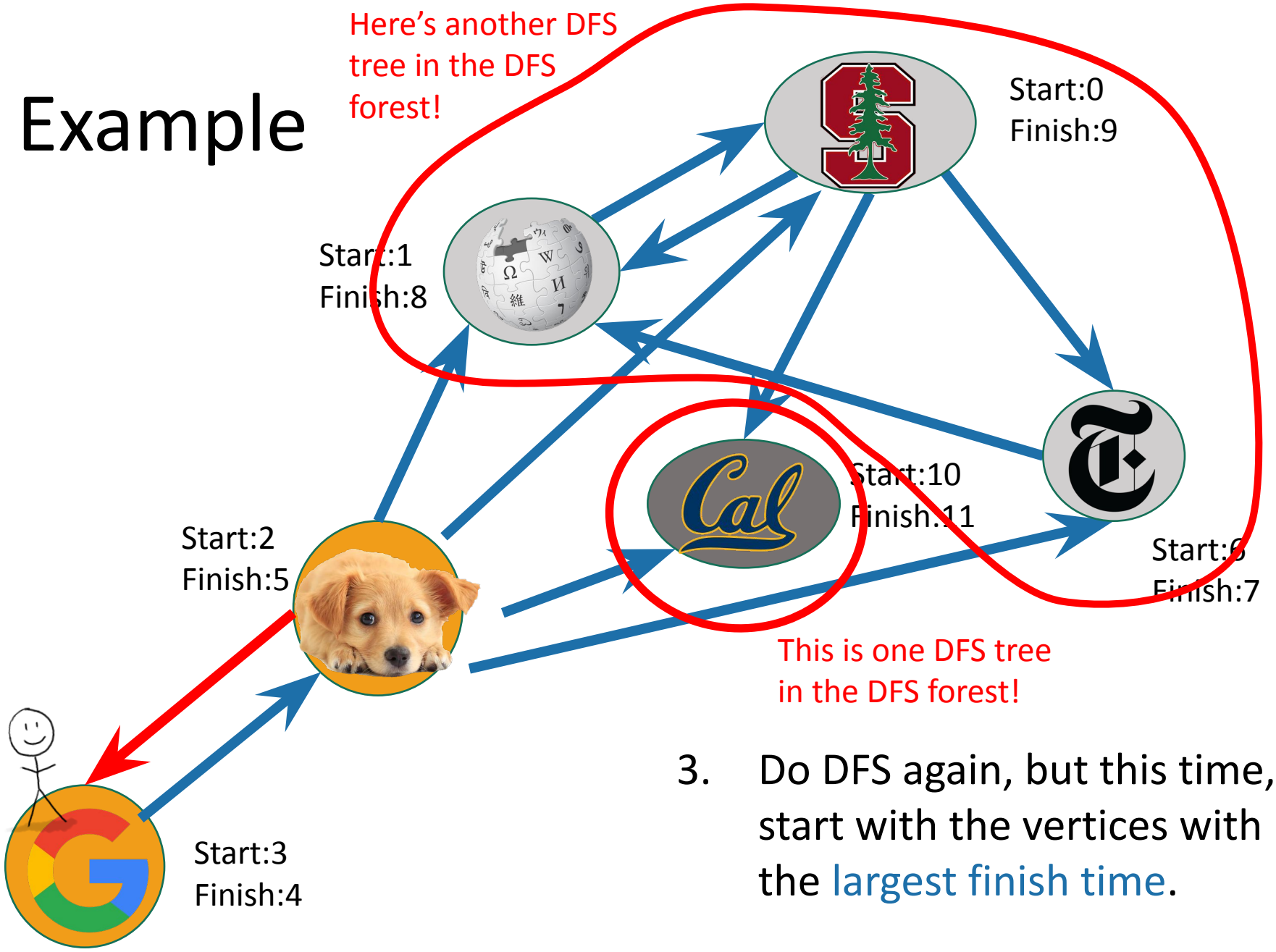
Example



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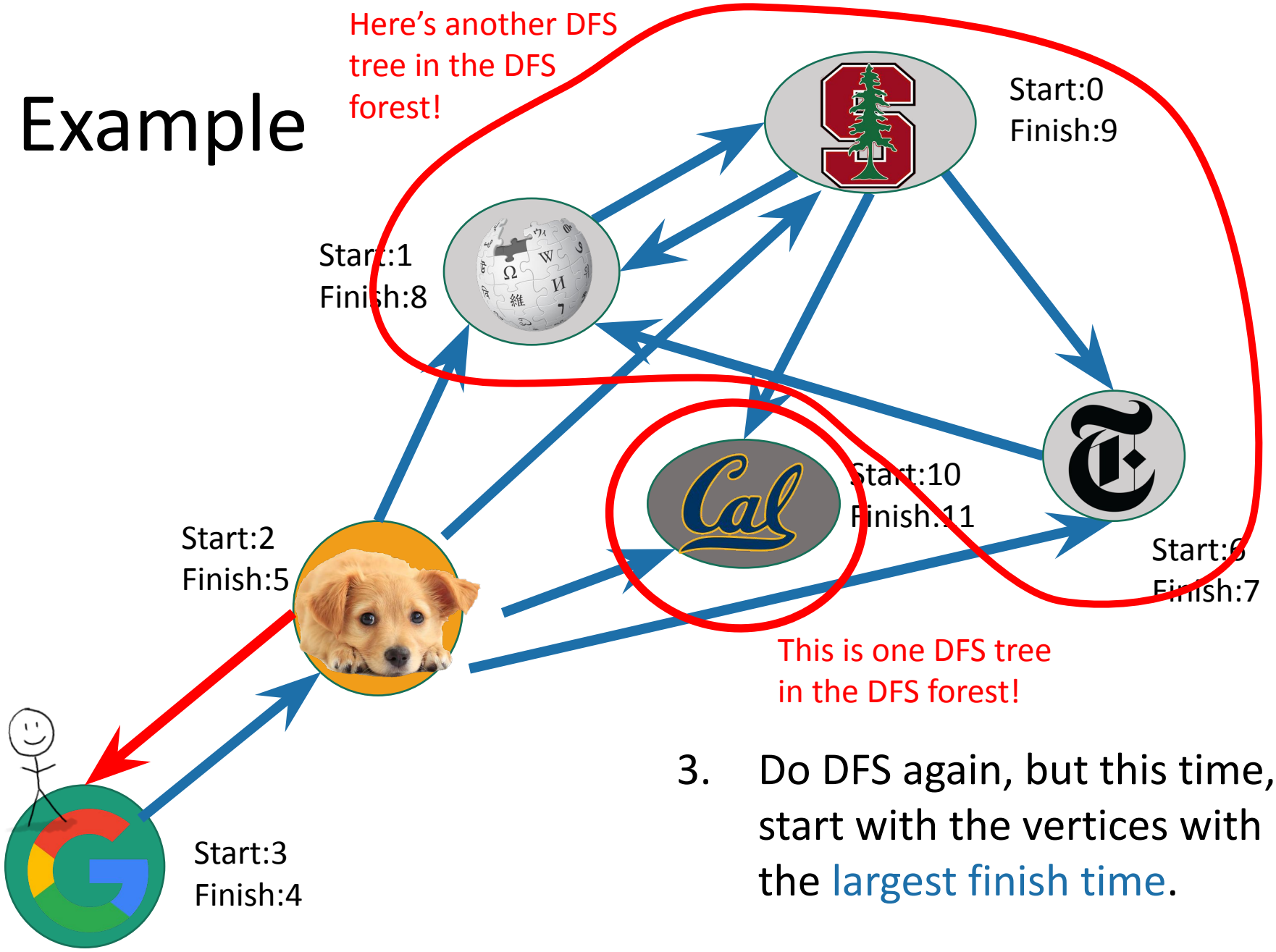

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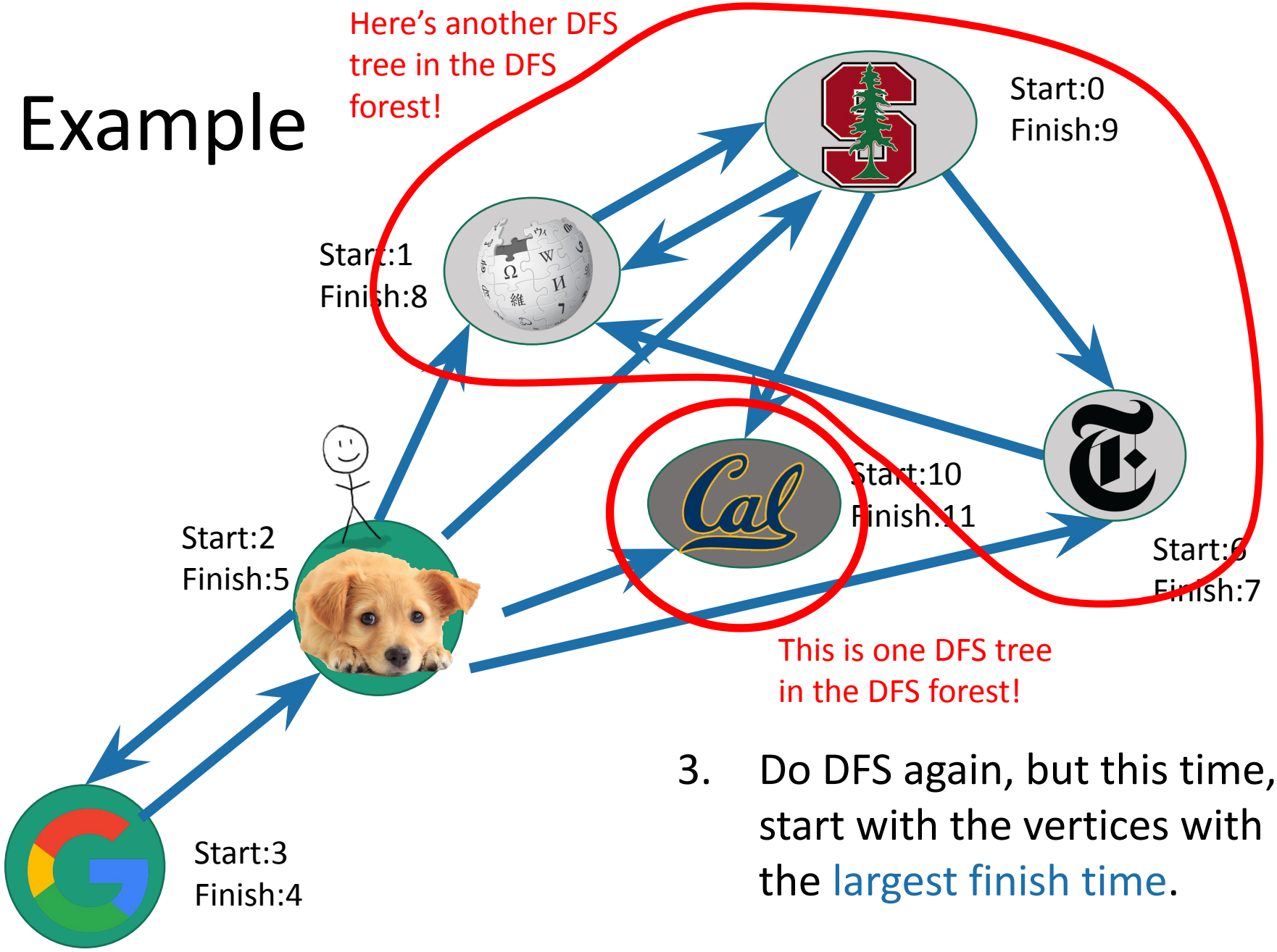

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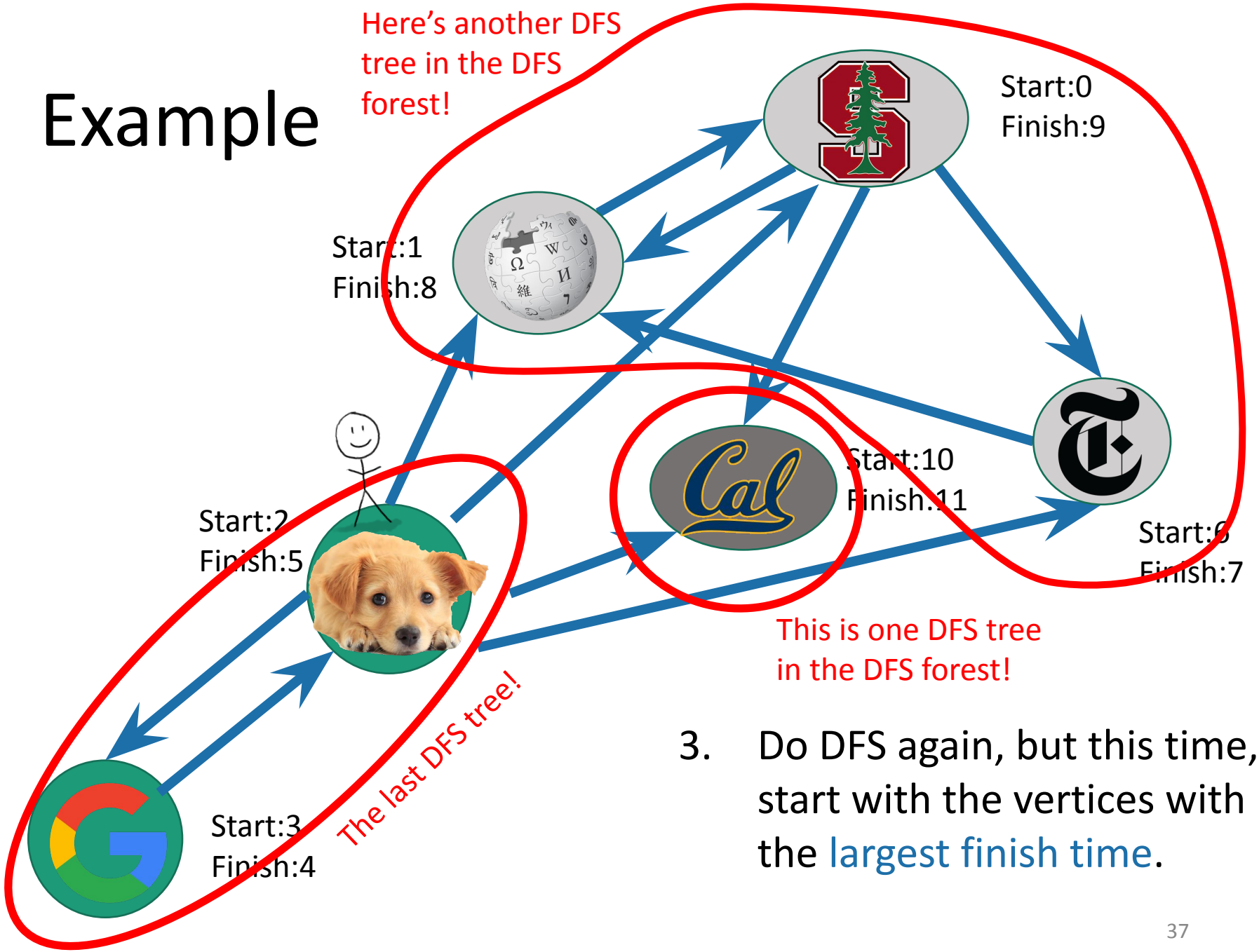

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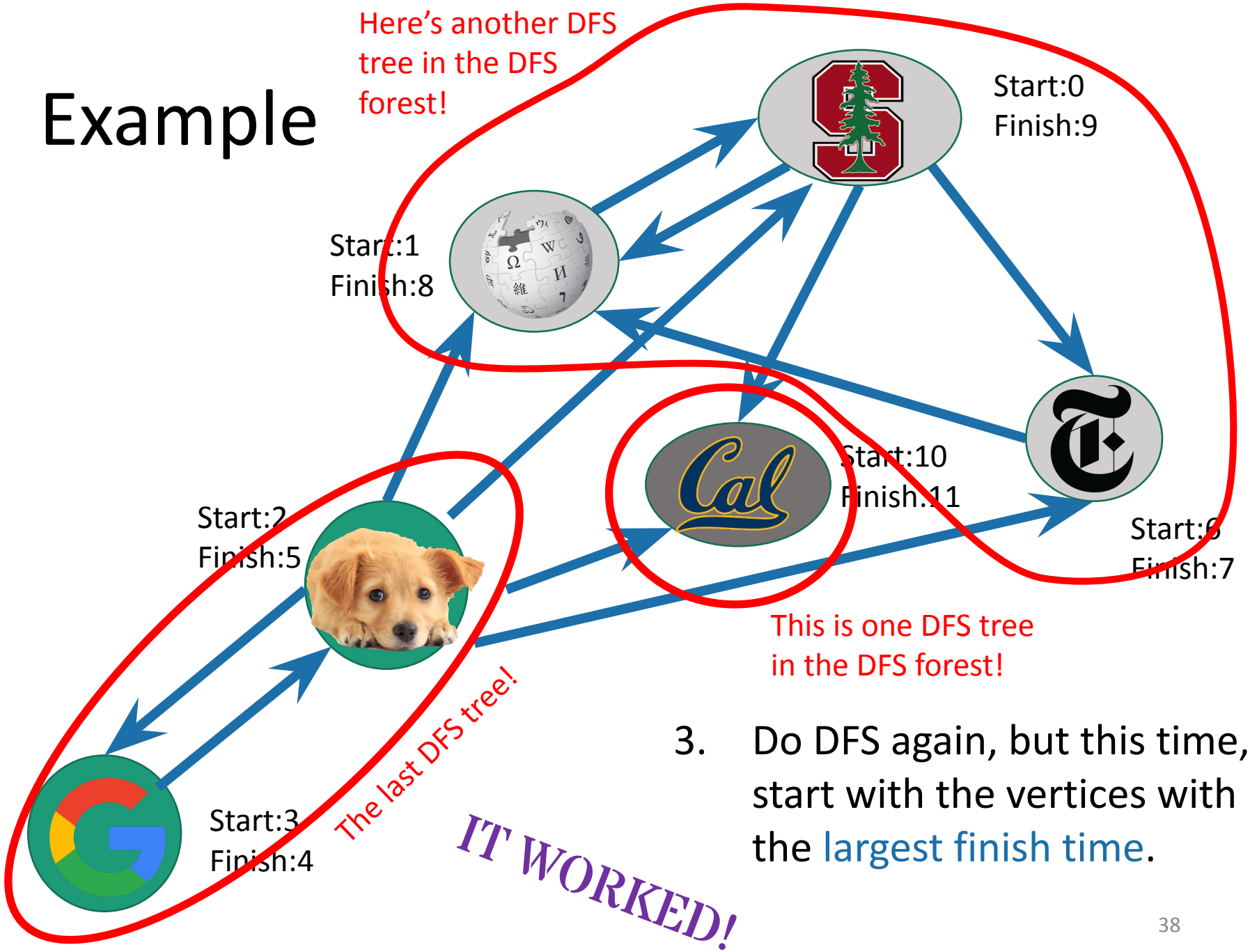


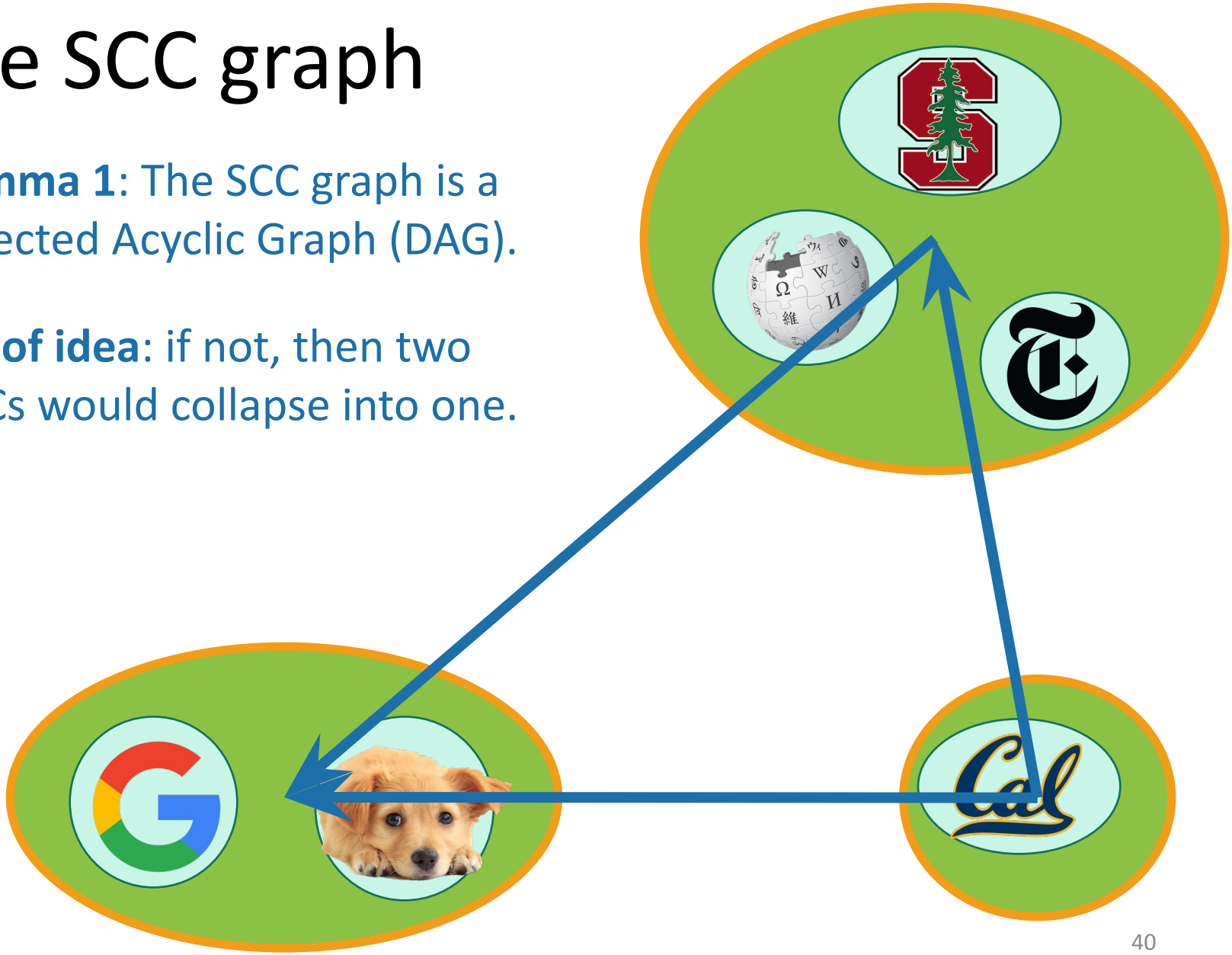
3. Do DFS again, but this time, start with the vertices with the largest finish time.

One question

The SCC graph

Lemma 1: The SCC graph is a Directed Acyclic Graph (DAG).

Proof idea: if not, then two SCCs would collapse into one.

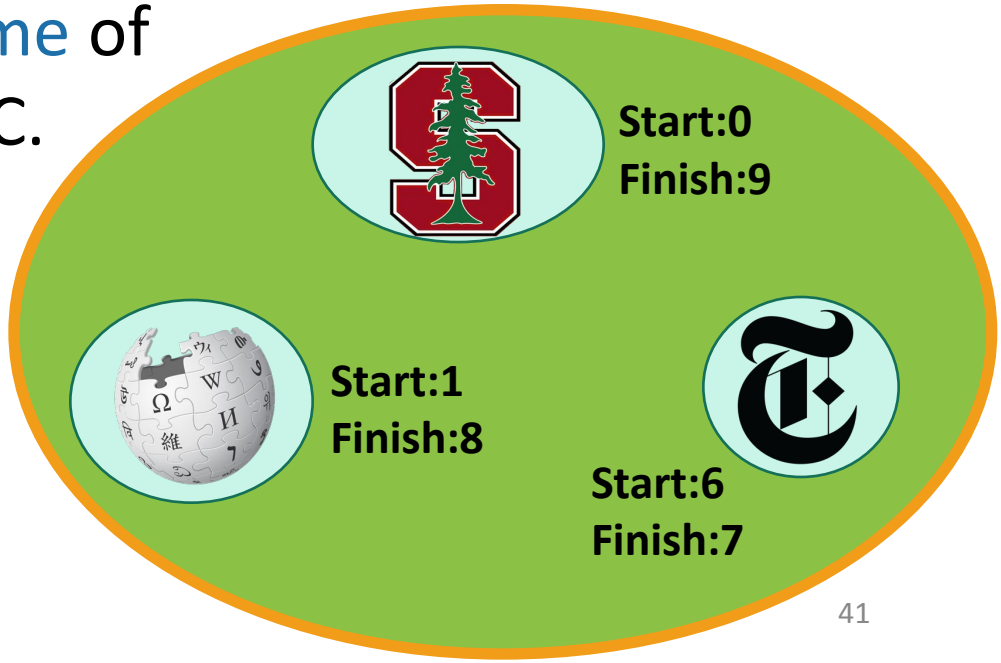


all times are with respect to the **first** DFS run

Starting and finishing times in a SCC

Definitions:

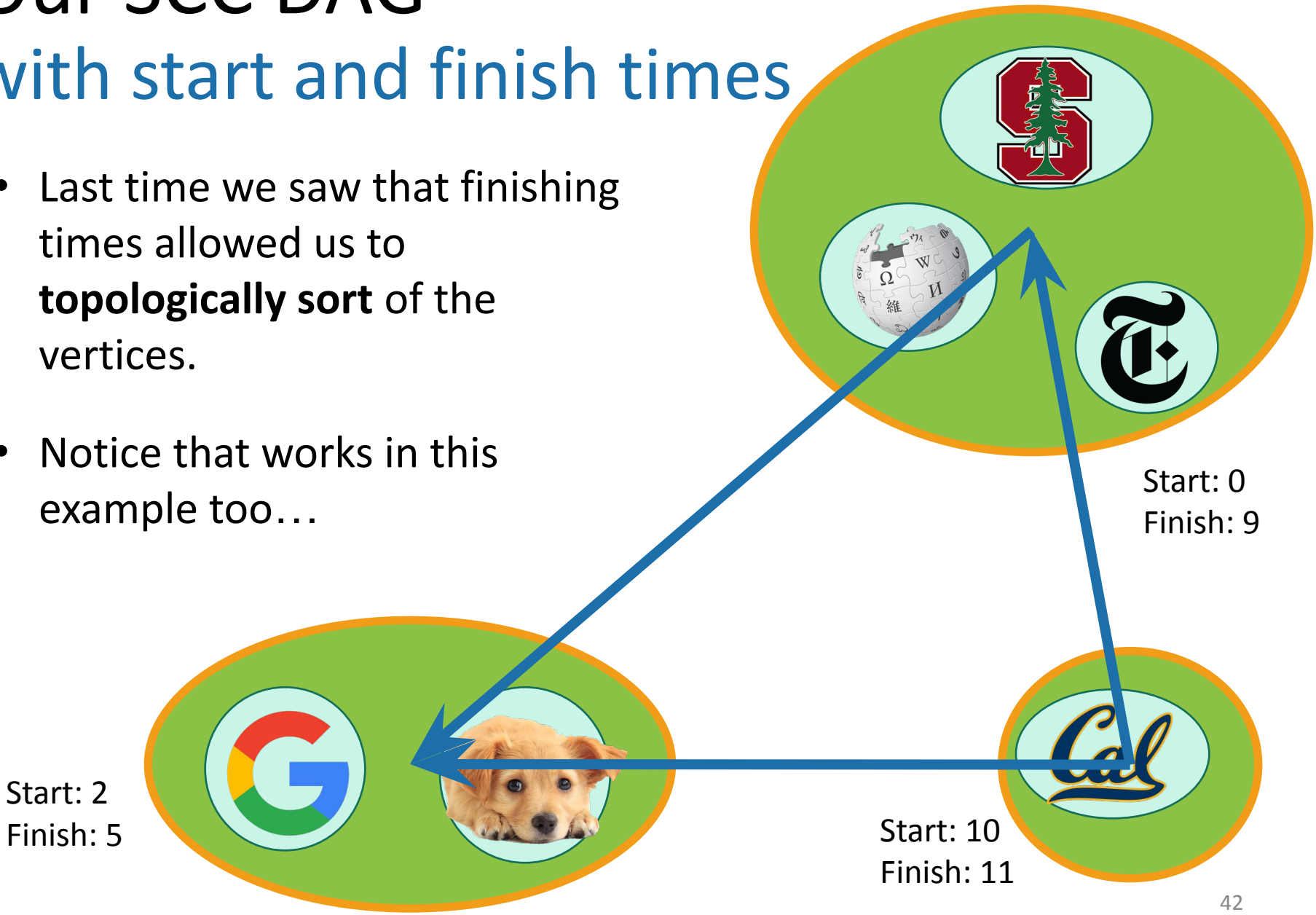
- The **finishing time** of a SCC is the largest finishing time of any element of that SCC.
- The **starting time** of a SCC is the smallest starting time of any element of that SCC.



Start: 0
Finish: 9

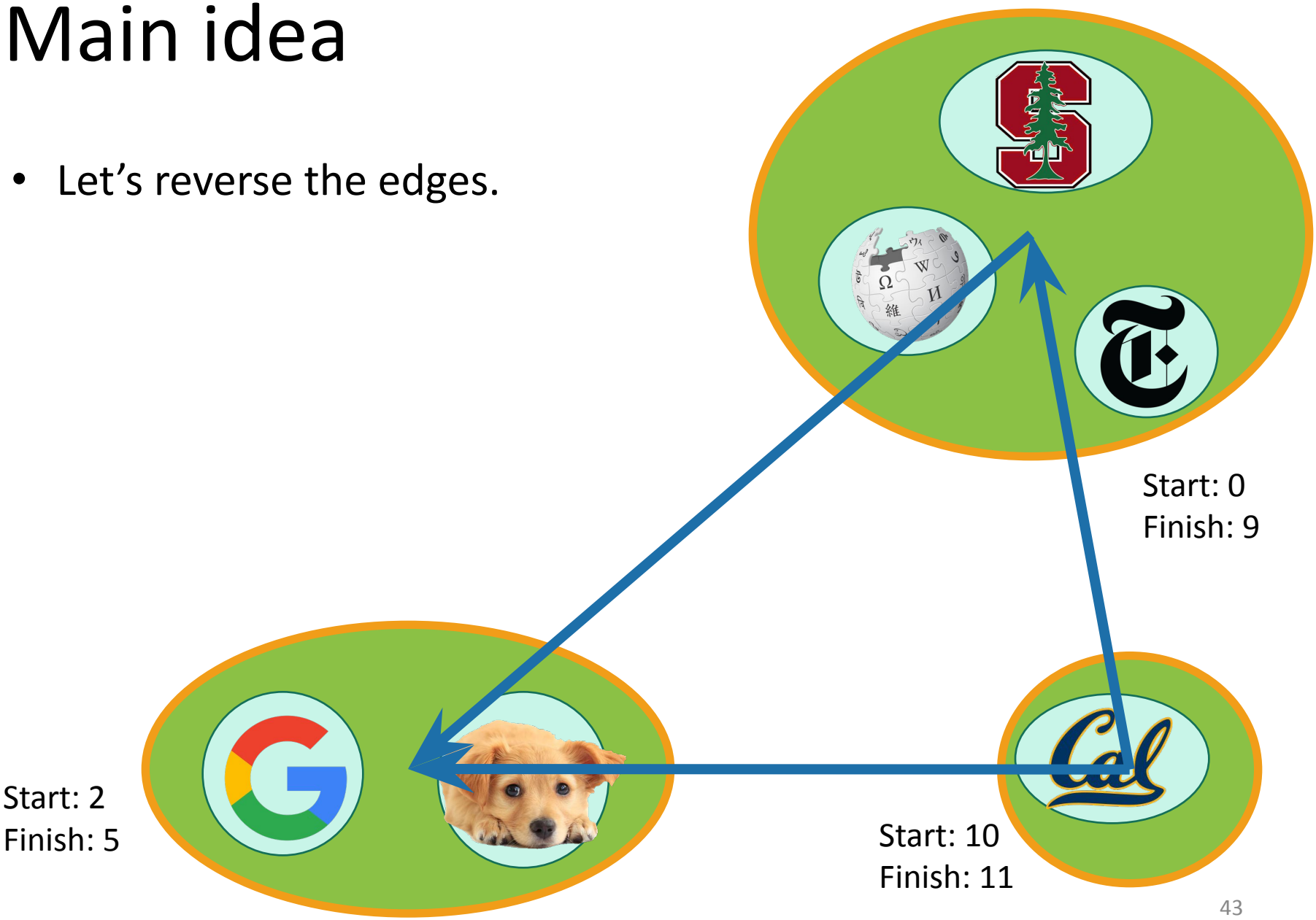
Our SCC DAG
with start and finish times

- Last time we saw that finishing times allowed us to **topologically sort** of the vertices.
- Notice that works in this example too…



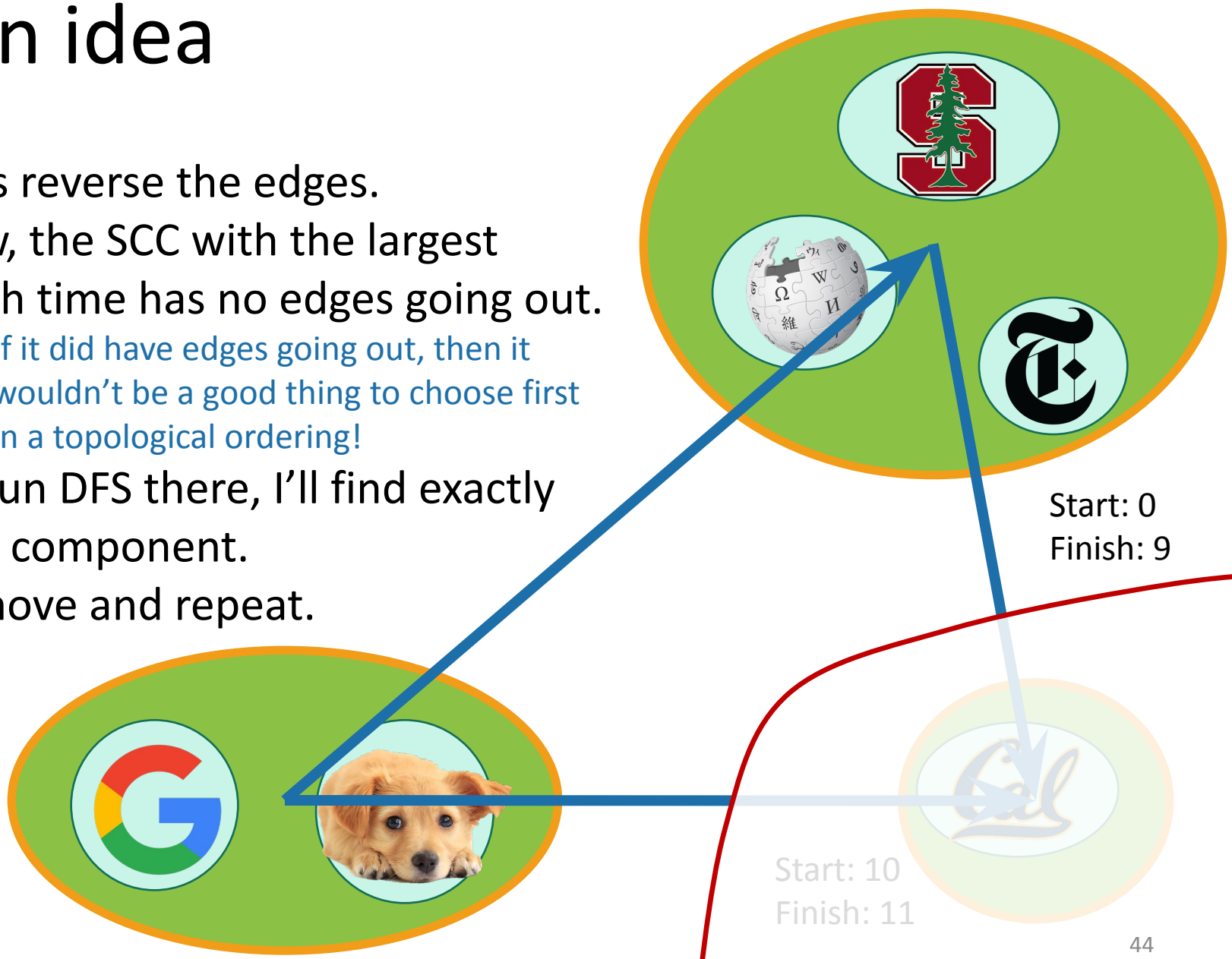
Main idea

- Let's reverse the edges.



Main idea

- Let's reverse the edges.
- Now, the SCC with the largest finish time has no edges going out.
  - If it did have edges going out, then it wouldn't be a good thing to choose first in a topological ordering!
- If I run DFS there, I'll find exactly that component.
- Remove and repeat.



Let’s make this idea formal.

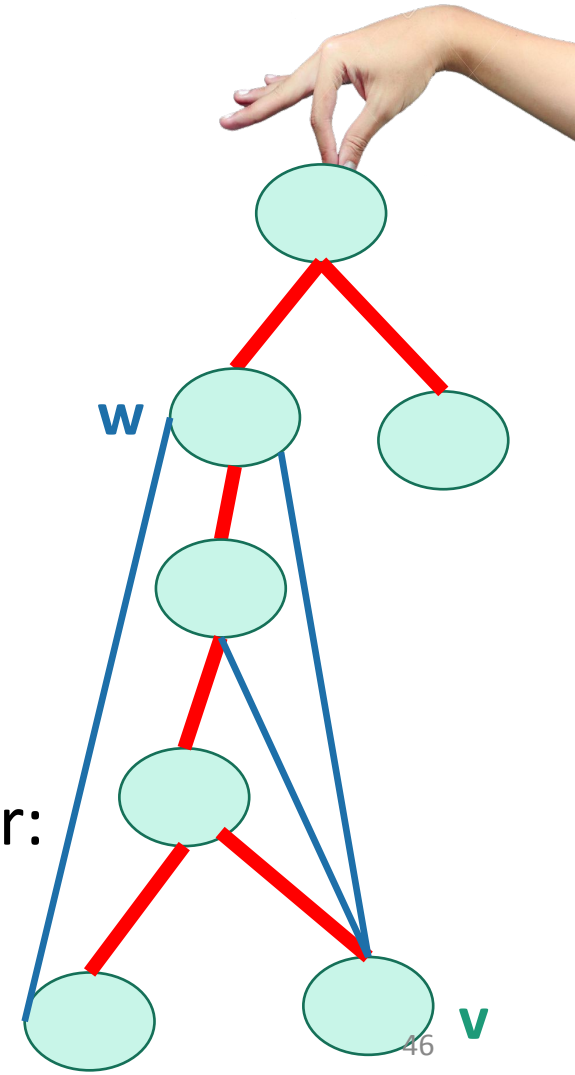
Recall

- If v is a descendant of w in this tree:

- If w is a descendant of v in this tree:

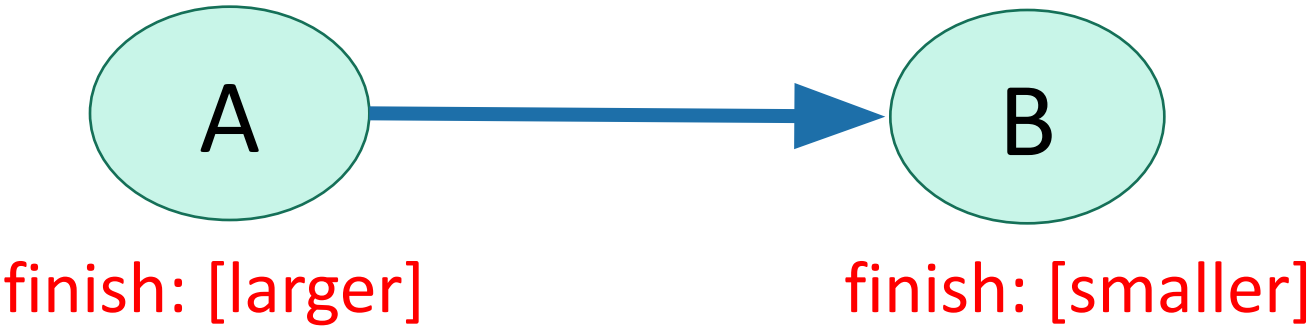
- If neither are descendants of each other:

(or the other way around)



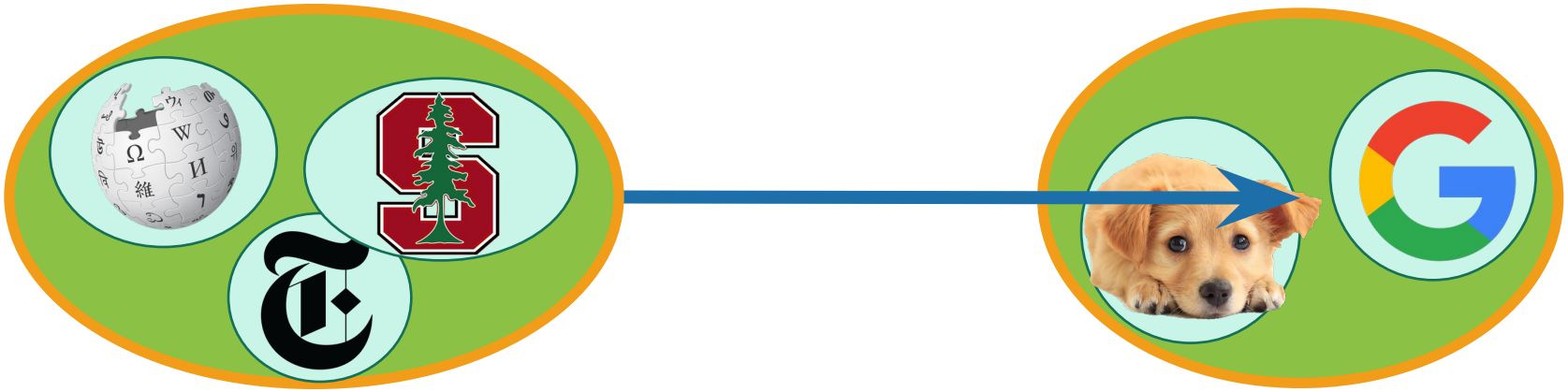
As we mentioned earlier:

Claim: In a DAG, we'll always have:



Same thing, in the SCC DAG.

- Claim: we'll always have



finish: [larger]

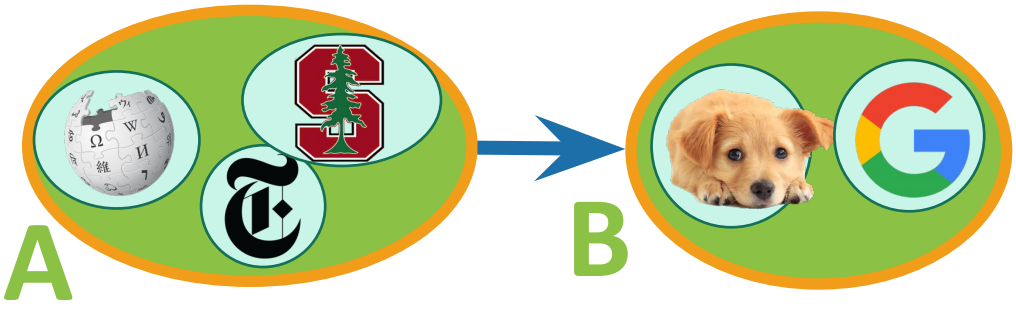
finish: [smaller]

Let's call it Lemma 2

- If there is an edge like this:

- Then A.finish > B.finish.

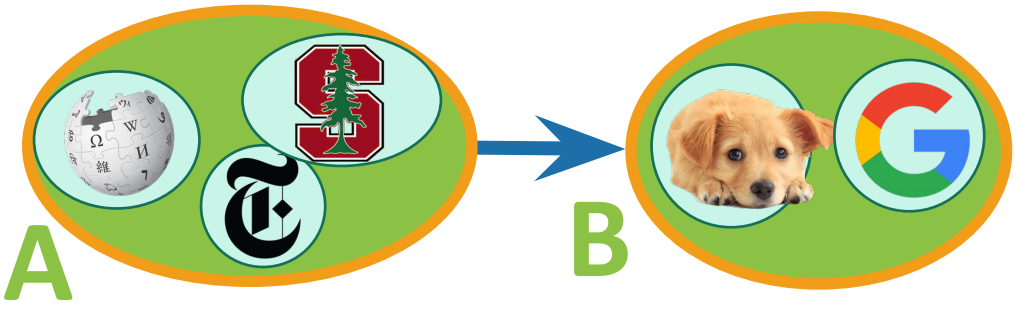
Proof idea



Want to show A.finish > B.finish.

- **Two cases**:
  - We reached **A** before **B** in our first DFS.
  - We reached **B** before **A** in our first DFS.

Proof idea



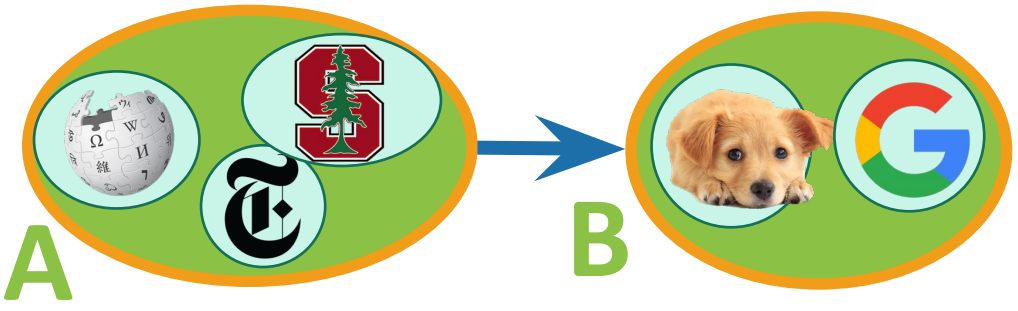
Want to show A.finish > B.finish.

- **Case 1**: We reached **A** before **B** in our first DFS.
- Say that:
  - **y** has the largest finish in **B**; B.finish = y.finish
  - **z** was discovered first in **A**; A.finish >= z.finish
- Then:
  - Reach **A** before **B**
  - => we will discover **y** via **z**
  - => **y** is a descendant of **z** in the DFS forest.

- Then

aka,
A.finish > B.finish

Proof idea



Want to show A.finish > B.finish.

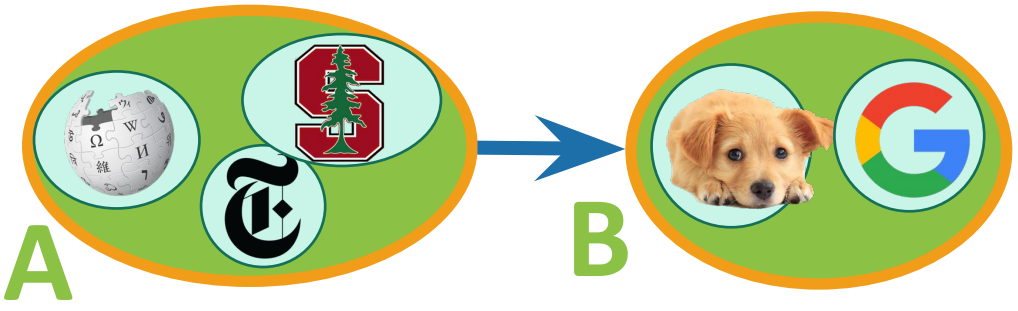
- **Case 2**: We reached **B** before **A** in our first DFS.

- There are no paths from B to A
  - because the SCC graph has no cycles

- So we completely finish exploring B and never reach A.
- A is explored later after we restart DFS.

aka,
A.finish > B.finish

Proof idea



Want to show A.finish > B.finish.

- **Two cases**:
  - We reached **A** before **B** in our first DFS.
  - We reached **B** before **A** in our first DFS.

- In either case:

A.finish > B.finish

which is what we wanted to show.

This establishes:

Lemma 2

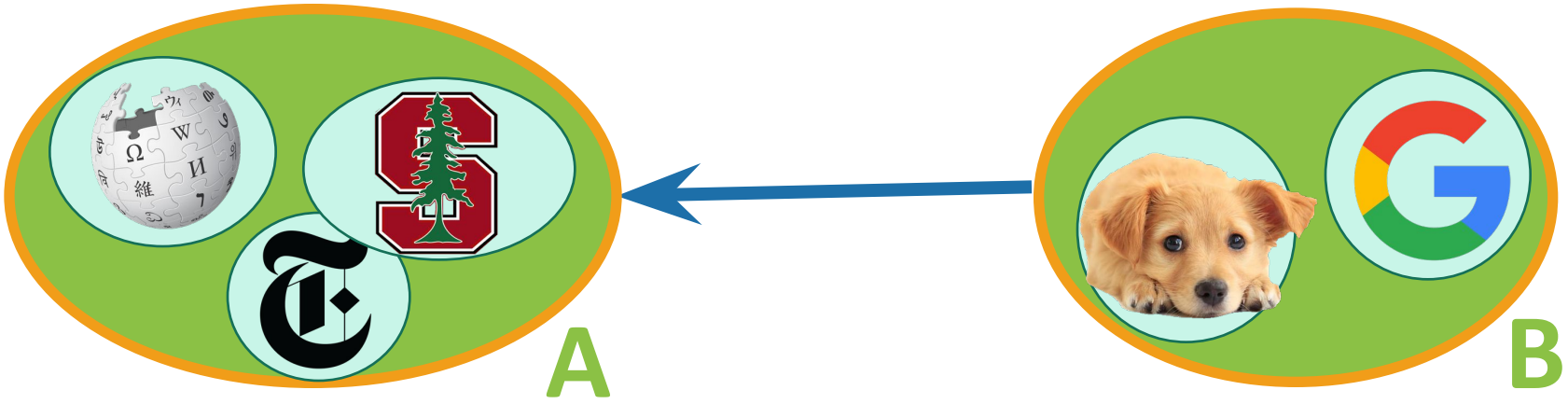
- If there is an edge like this:

- Then A.finish > B.finish.

This establishes:

Corollary 1

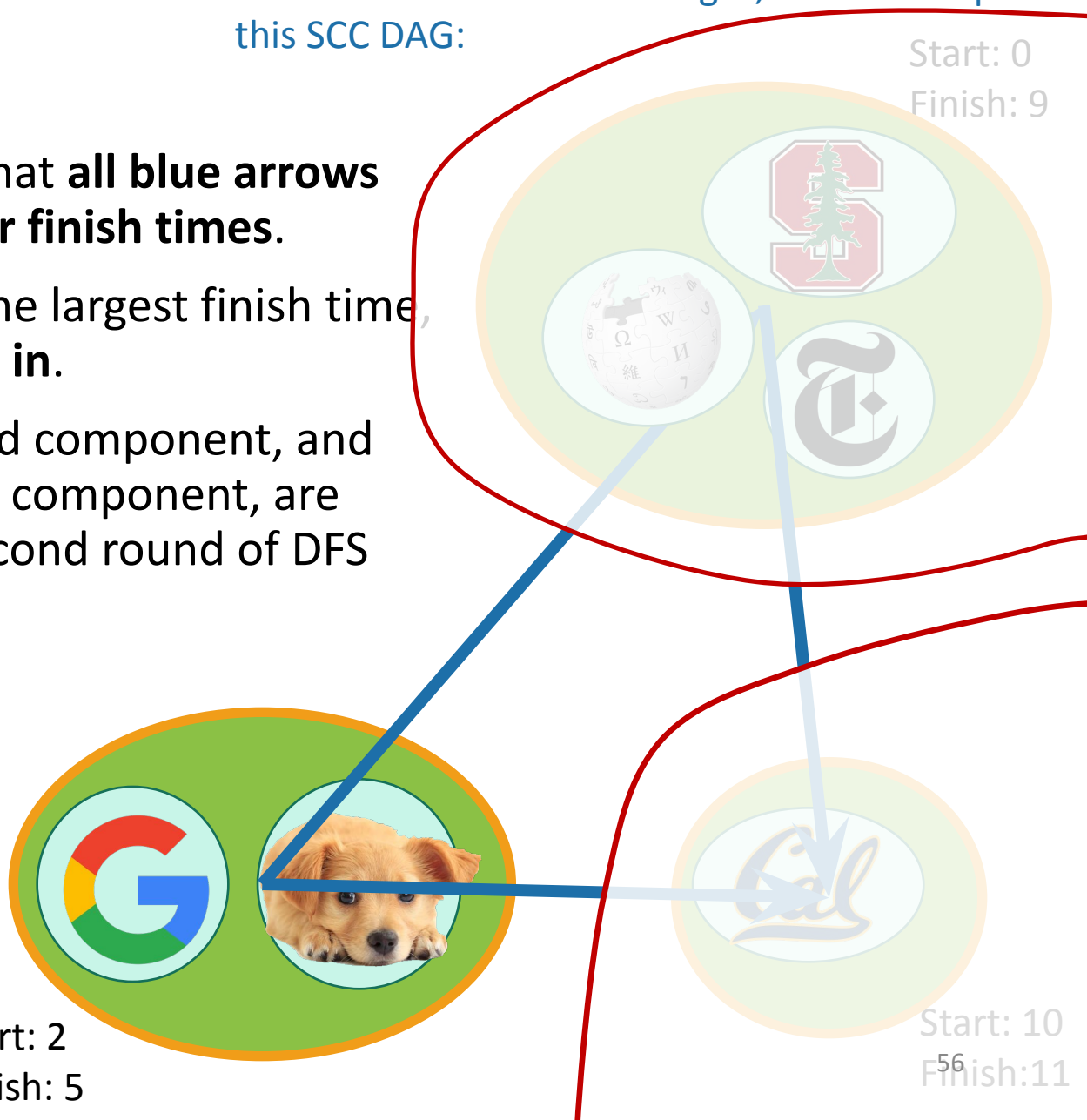
- If there is an edge like this in the **reversed graph**:



- Then A.finish > B.finish.

Now we see why this finds SCCs.

Remember that after the first round of DFS, and after we reversed all the edges, we ended up with this SCC DAG:



- The Corollary says that **all blue arrows point towards larger finish times**.
- So if we start with the largest finish time, **all blue arrows lead in**.
- Thus, that connected component, and only that connected component, are reachable by the second round of DFS

- Now, we've deleted that first component.
- The next one has the **next biggest finishing time.**
- So **all remaining blue arrows lead in.**
- Repeat.

Formally, we prove it by induction

- **Theorem**: The algorithm we saw before will correctly identify strongly connected components.

- **Inductive hypothesis**:
  - The first t trees found in the second (reversed) DFS forest are the t SCCs with the largest finish times.

- **Base case**: (t=0)
  - The first 0 trees found in the reversed DFS forest are the 0 SCCs with the largest finish times. **(TRUE)**

Inductive step

- **Assume by induction that the first t trees are the last-finishing SCCs.**
- Consider the $(t+1)^{st}$ tree produced, suppose the root is **x**.
- Suppose that **x** lives in the SCC **A**.
- Then **A**.finish > **B**.finish for all remaining SCCs **B**.
  - This is because we chose **x** to have the largest finish time.
- Then there are no edges leaving **A** in the remaining SCC DAG.
  - This follows from the Corollary.
- Then DFS started at **x** recovers exactly **A**.
  - It doesn't recover any more since nothing else is reachable.
  - It doesn't recover any less since A is strongly connected.
  - (Notice that we are using that A is still strongly connected when we reverse all the edges).
- **So the $(t+1)^{st}$ tree is the SCC with the $(t+1)^{st}$ biggest finish time.**

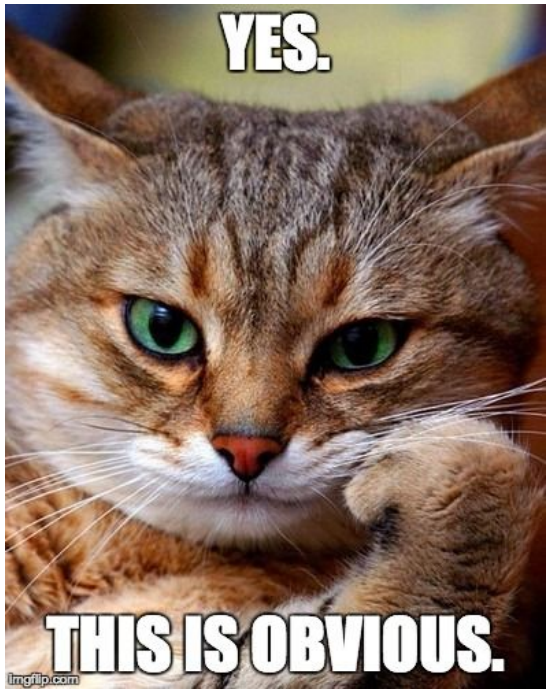
Formally, we prove it by induction

- **Theorem**: The algorithm we saw before will correctly identify strongly connected components.

- **Inductive hypothesis**:
  - The first t trees found in the second (reversed) DFS forest are the t SCCs with the largest finish times.

- **Base case**: *[done]*
- **Inductive step**: *[done]*
- **Conclusion:** The second (reversed) DFS forest contains all the SCCs as its trees!
  - (This is the **IH** when t = #SCCs)

Punchline:
we can find SCCs in time O(n + m)



`Algorithm:`

- Do DFS to create a **DFS forest**.
  - Choose starting vertices in any order.
  - Keep track of finishing times.
- Reverse all the edges in the graph.
- Do DFS again to create **another DFS forest**.
  - This time, order the nodes in the reverse order of the finishing times that they had from the first DFS run.
- The SCCs are the different trees in the **second DFS forest.**

(Clearly it wasn't obvious since it took so many slides! But hopefully it is less mysterious now.)