Lecture 13

More dynamic programming!

Longest Common Subsequences, Knapsack, and (if time) independent sets in trees.

Announcements

- Exam 3 is live! Please do not ask or discuss exam-related stuff in chat/Q&A during lecture.

- If you find any errors, typos, or omissions during the first 24 hours (EOD today Pacific Time), you can PRIVATELY bring them to the course staff attention (private Ed post would do). We will publicly clarify these in a pinned Ed thread.

- We will not answer general clarification questions. If something is not clear, state your assumptions.

Last time

Dynamic Programming!

- Not coding in an action movie.

Last time

Dynamic Programming!

- Dynamic programming is an algorithm design paradigm.
- Basic idea:
  - Identify **optimal sub-structure**
    - Optimum to the big problem is built out of optima of small sub-problems
  - Take advantage of **overlapping sub-problems**
    - Only solve each sub-problem once, then use it again and again
  - Keep track of the solutions to sub-problems in a table as you build to the final solution.

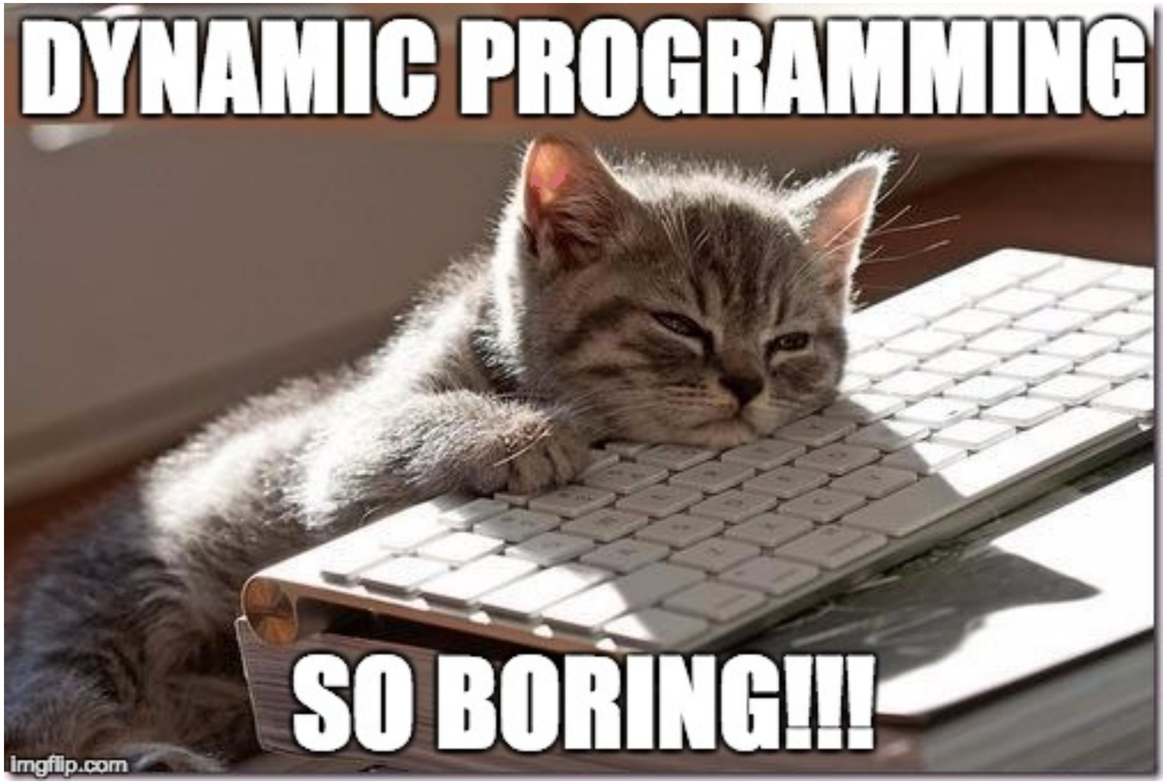
Today

- Examples of dynamic programming:
  1. Longest common subsequence
  2. Knapsack problem
     - Two versions!
  3. Independent sets in trees
     - If we have time…
     - (If not the slides will be there as a reference)

- Yet more examples of DP in CLRS!
  - Optimal order of matrix multiplications
  - Optimal binary search trees
  - Longest paths in DAGs, …

The goal of this lecture

- For you to get really bored of dynamic programming



Longest Common Subsequence

- How similar are these two species?

DNA:
AGCCCTAAGGGCTACCTAGCTT

DNA:
GACAGCCTACAAGCGTTAGCTTG

Longest Common Subsequence

- How similar are these two species?

DNA:
AGCCCTAAGGGCTACCTAGCTT

DNA:
GACAGCCTACAAGCGTTAGCTTG

- Pretty similar, their DNA has a long common subsequence:

AGCCTAAGCTTAGCTT

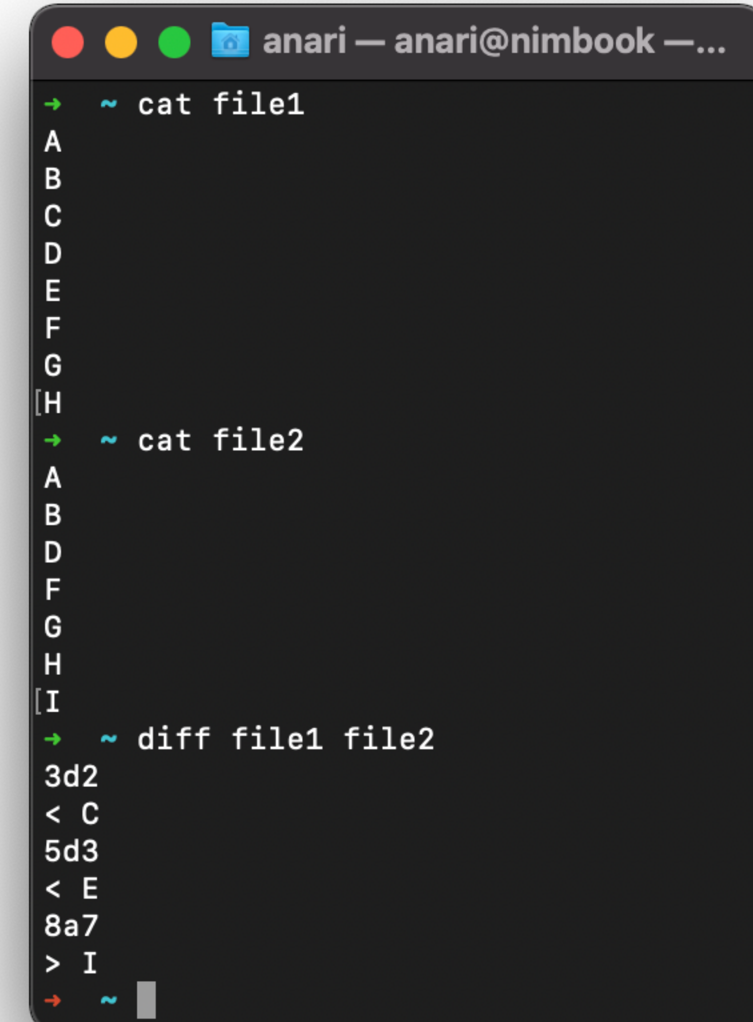
Longest Common Subsequence

- Subsequence:
  - BDFH is a **subsequence** of ABCDEFGH
- If X and Y are sequences, a **common subsequence** is a sequence which is a subsequence of both.
  - BDFH is a **common subsequence** of ABCDEFGH and of ABDFGHI
- A **longest common subsequence**…
  - …is a common subsequence that is longest.
  - The **longest common subsequence** of ABCDEFGH and ABDFGHI is ABDFGH.

We sometimes want to find these

- Applications in bioinformatics

- The unix command `diff`
- Merging in version control
  - svn, git, etc…

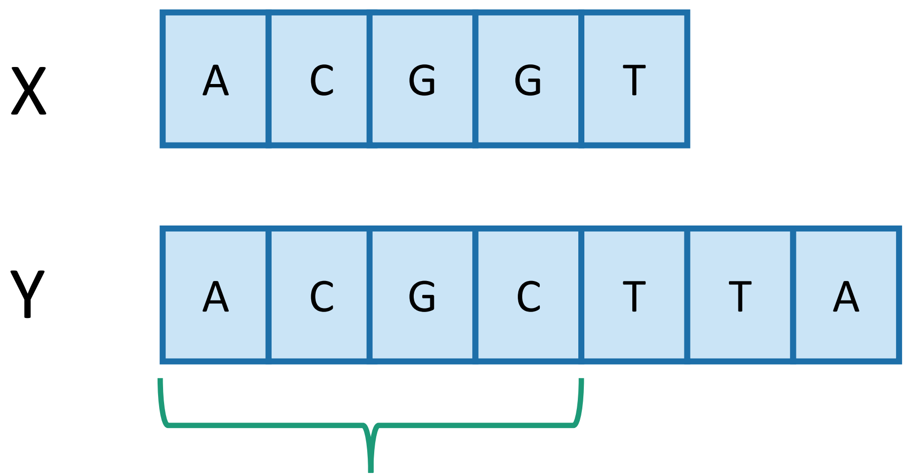


Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the length of the longest common subsequence.
- **Step 3:** Use dynamic programming to find the length of the longest common subsequence.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual LCS.
- **Step 5:** If needed, code this up like a reasonable person.

Step 1: Optimal substructure

Prefixes:



Notation: denote this prefix **ACGC** by Y_4

- Our sub-problems will be finding LCS's of prefixes to X and Y.
- Let C[i,j] = length_of_LCS(X_i, Y_j)

Examples: C[2,3] = 2
C[4,4] = 3

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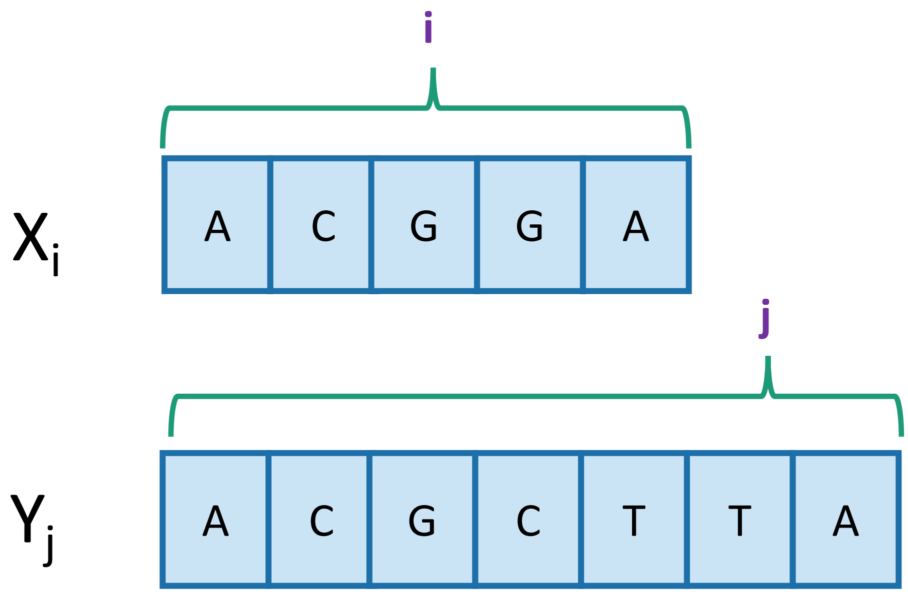
Goal

- Write C[i,j] in terms of the solutions to smaller sub-problems

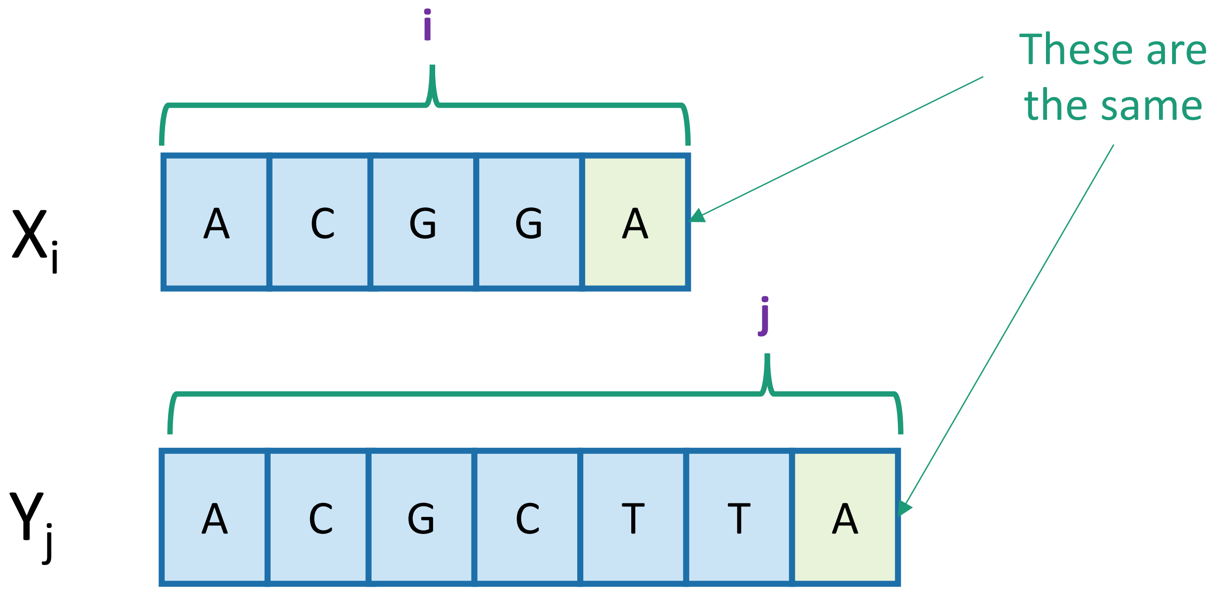


$C[i,j] = \text{length\_of\_LCS}(X_i, Y_j)$

Two cases

Case 1: X[i] = Y[j]

- Our sub-problems will be finding LCS's of prefixes to X and Y.
- Let C[i,j] = length_of_LCS(X_i, Y_j)

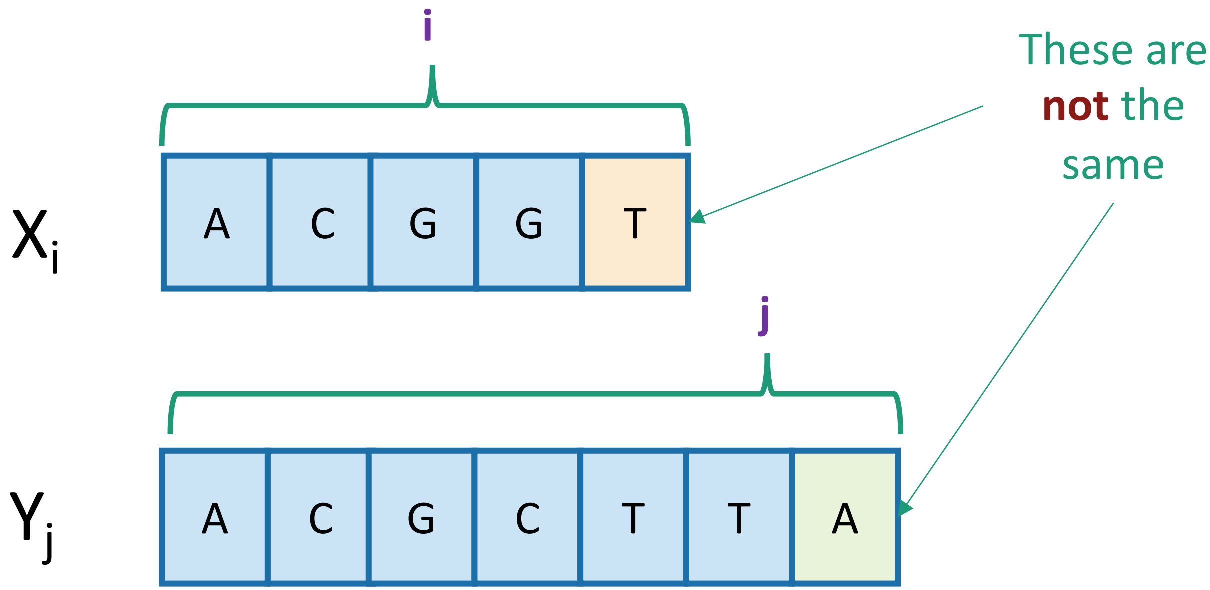


- Then C[i,j] = 1 + C[i-1,j-1].
  - because LCS(X_i,Y_j) = LCS(X_{i-1},Y_{j-1}) followed by $\boxed{A}$

Two cases

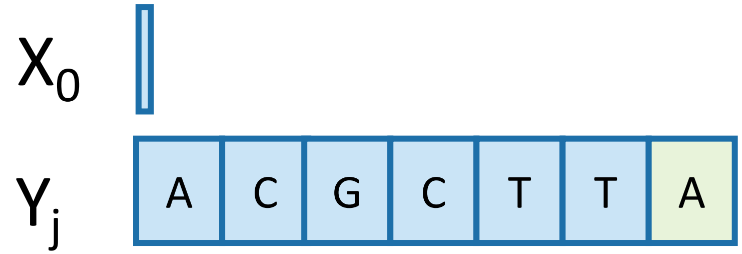
Case 2: X[i] != Y[j]

- Our sub-problems will be finding LCS's of prefixes to X and Y.
- Let C[i,j] = length_of_LCS(X_i, Y_j)



- Then C[i,j] = max{ C[i-1,j], C[i,j-1] }.
  - either LCS(X_i,Y_j) = LCS(X_{i-1},Y_j) and T is not involved,
  - or LCS(X_i,Y_j) = LCS(X_i,Y_{j-1}) and A is not involved,
  - (maybe both are not involved, that's covered by the "or").

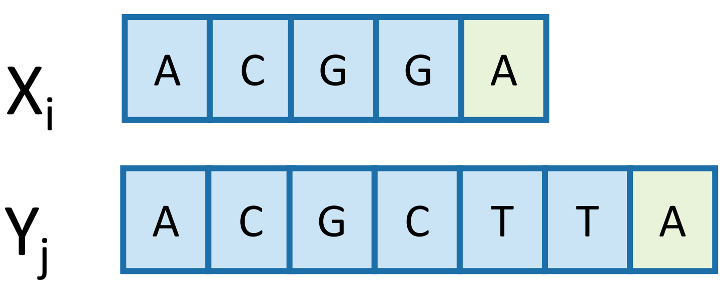
Recursive formulation
of the optimal solution



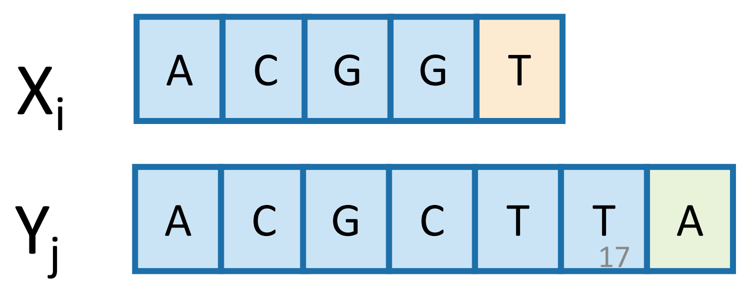
Case 0

- $C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{\, C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$

Case 1



Case 2



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LCS DP

- **LCS**(X, Y):
  - C[i,0] = C[0,j] = 0 for all i = 0,…,m, j=0,…n.
  - **For** i = 1,…,m and j = 1,…,n:
    - **If** X[i] = Y[j]:
      - C[i,j] = C[i-1,j-1] + 1
    - **Else:**
      - C[i,j] = max{ C[i,j-1], C[i-1,j] }
  - Return C[m,n]

Running time: O(nm)

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1,j-1]+1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{ C[i,j-1], C[i-1,j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X: A C G G A

Y: A C T G

X \ Y		A	C	T	G
	0	0	0	0	0
A	0				
C	0				
G	0				
G	0				
A	0				

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{\, C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X	A	C	G	G	A
Y	A	C	T	G	

		Y: A	C	T	G
X	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

So the LCM of X and Y has length 3.

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{\, C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

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Example

X	A	C	G	G	A

Y	A	C	T	G

Y

X		A	C	T	G
	0	0	0	0	0
A	0				
C	0				
G	0				
G	0				
A	0				

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X	A	C	G	G	A

Y	A	C	T	G

Y

X		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
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$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{\, C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X	A	C	G	G	A
Y	A	C	T	G	

Y

X			A	C	T	G
		0	0	0	0	0
	A	0	1	1	1	1
	C	0	1	2	2	2
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	3

- Once we've filled this in, we can work backwards.

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{\, C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X	A	C	G	G	A
Y	A	C	T	G	

Y

X		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.

That 3 must have come from the 3 above it.

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1,j-1]+1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{ C[i,j-1], C[i-1,j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X	A	C	G	G	A
Y	A	C	T	G	

Y

X		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

This 3 came from that 2 – we found a match!

G

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X	A	C	G	G	A

Y	A	C	T	G

Y

X		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

That 2 may as well have come from this other 2.

G

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X	A	C	G	G	A

Y	A	C	T	G

Y

			A	C	T	G
		0	0	0	0	0
X	A	0	1	1	1	1
	C	0	1	2	2	2
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	3

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

G

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{\, C[i, j-1], C[i-1, j] \} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X: A C G G A

Y: A C T G

		Y: A	C	T	G
	0	0	0	0	0
X: A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

C G

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{ C[i, j-1], C[i-1, j] \} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X: A C G G A

Y: A C T G

		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

A C G

This is the LCS!

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{ C[i, j-1], C[i-1, j] \} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Finding an LCS

- Good exercise to write out pseudocode for what we just saw!
  - Or you can find it in lecture notes.
- Takes time O(mn) to fill the table
- Takes time O(n + m) on top of that to recover the LCS
  - We walk up and left in an n-by-m array
  - We can only do that for n + m steps.
- Altogether, we can find LCS(X,Y) in time O(mn).

Recipe for applying Dynamic Programming

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Our approach actually isn't so bad

- If we are only interested in the length of the LCS we can do a bit better on space:
  - Since we go across the table one-row-at-a-time, we can only keep two rows if we want.
- If we want to recover the LCS, we need to keep the whole table.

- Can we do better than O(mn) time?
  - A bit better.
    - By a log factor or so.
  - But doing much better (polynomially better) is an open problem!

What have we learned?

- We can find LCS(X,Y) in time O(nm)
  - if |Y|=n, |X|=m

- We went through the steps of coming up with a dynamic programming algorithm.
  - We kept a 2-dimensional table, breaking down the problem by decrementing the length of X and Y.

Example 2: Knapsack Problem

- We have n items with weights and values:

Item:					
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- And we have a knapsack:
  - it can only carry so much weight:

Capacity: 10

Capacity: 10

Item:	turtle	lightbulb	watermelon	taco	fire truck
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- Unbounded Knapsack:
  - Suppose I have infinite copies of all items.
  - What's the most valuable way to fill the knapsack?

Total weight: 10
Total value: 42

- 0/1 Knapsack:
  - Suppose I have only one copy of each item.
  - What's the most valuable way to fill the knapsack?

Total weight: 9
Total value: 35

Some notation

Item:					
Weight:	w_1	w_2	w_3	…	w_n
Value:	v_1	v_2	v_3		v_n

Capacity: W

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Optimal substructure

- Sub-problems:
  - Unbounded Knapsack with a smaller knapsack.
  - K[x] = value you can fit in a knapsack of capacity x

First solve the problem for small knapsacks

Then larger knapsacks

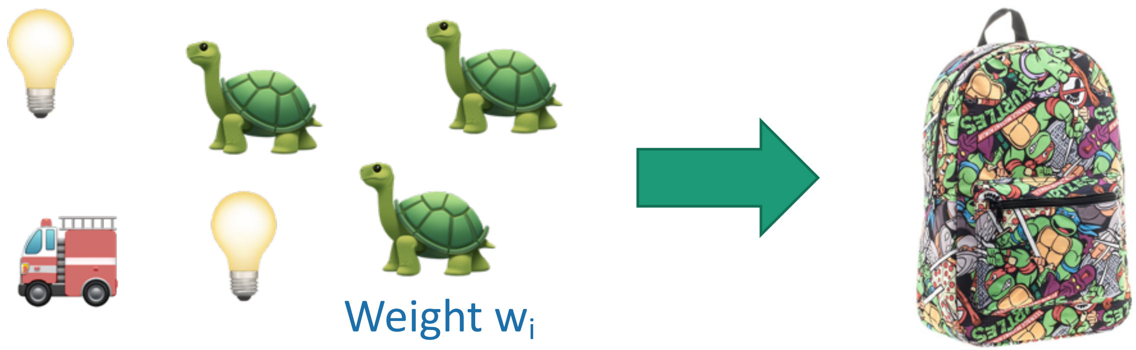
Then larger knapsacks

Optimal substructure


item i

- Suppose this is an optimal solution for capacity x:

Say that the optimal solution contains at least one copy of item i.

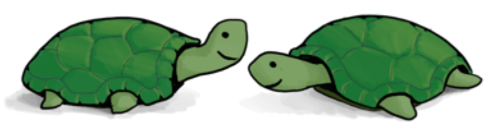


Weight w_i
Value v_i

Capacity x
Value V

- Then this is optimal for capacity x - w_i:

Why?
1 minute think
(wait) 1 minute share



Capacity x – w_i
Value V - v_i

Optimal substructure

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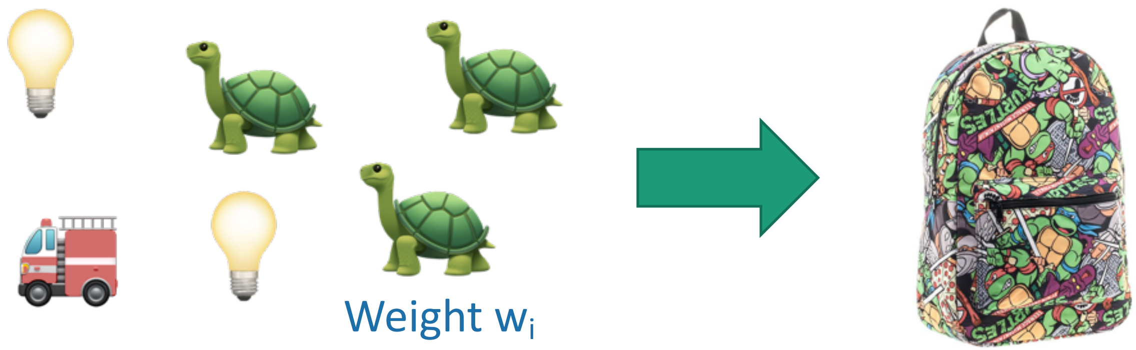

Weight w_i
Value v_i

Capacity x
Value V

- Then this is optimal for capacity x - w_i:

If I could do better than the second solution, then adding a turtle to that improvement would improve the first solution.

Capacity x – w_i
Value V - v_i

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Recursive relationship

- Let K[x] be the optimal value for capacity x.

$K[x] = \max_i \{$  $+$  $\}$

The maximum is over all i so that $w_i \leq x$.

Optimal way to fill the smaller knapsack

The value of item i.

$$K[x] = \max_i \{ K[x - w_i] + v_i \}$$

- (And K[x] = 0 if the maximum is empty).
  - That is, if there are no i so that $w_i \leq x$

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Let's write a bottom-up DP algorithm

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
  - **return** K[W]

Running time: O(nW)

$K[x] = \max_i \{$  $+$  $\}$

$= \max_i \{ K[x - w_i] + v_i \}$

Can we do better?

- Writing down W takes log(W) bits.
- Writing down all n weights takes at most nlog(W) bits.
- Input size: nlog(W).
  - Maybe we could have an algorithm that runs in time O(nlog(W)) instead of O(nW)?
  - Or even $O(\ n^{1000000} \log^{1000000}(W)\ )$?

- Open problem!
  - (But probably the answer is **no**…otherwise P = NP)

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    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
  - **return** K[W]

$K[x] = \max_i \{$  +  $\}$

$= \max_i \{ K[x - w_i] + v_i \}$

Let’s write a bottom-up DP algorithm

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = Ø
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{\, K[x], K[x - w_i] + v_i \,\}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] U { item i }
  - **return** ITEMS[W]

K[x] = $\max_i$ {  +  }

= $\max_i$ { K[x – w_i] + v_i }

Example

	0	1	2	3	4
K	0				
ITEMS					

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = Ø
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1			
ITEMS					

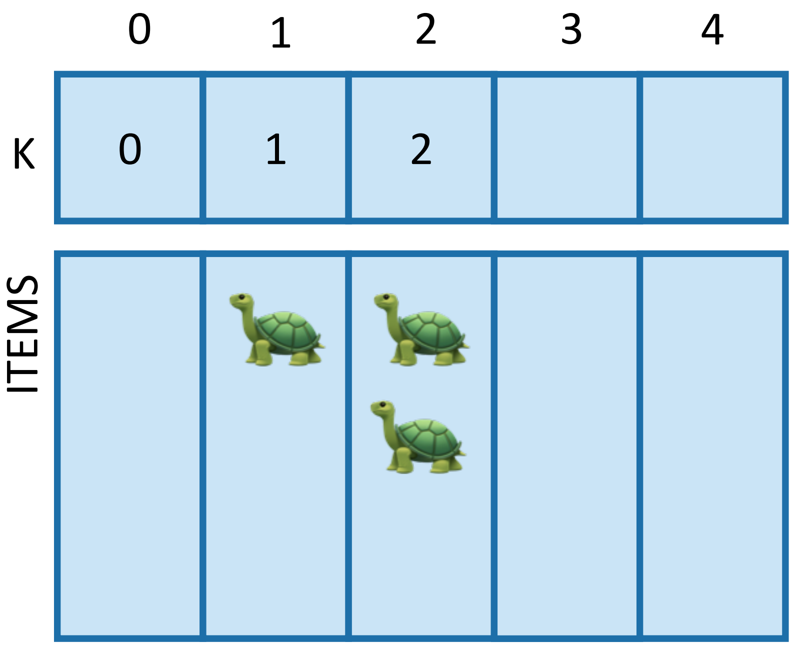
ITEMS[1] = ITEMS[0] + 

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example



ITEMS[2] = ITEMS[1] + 

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = Ø
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item: 

Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

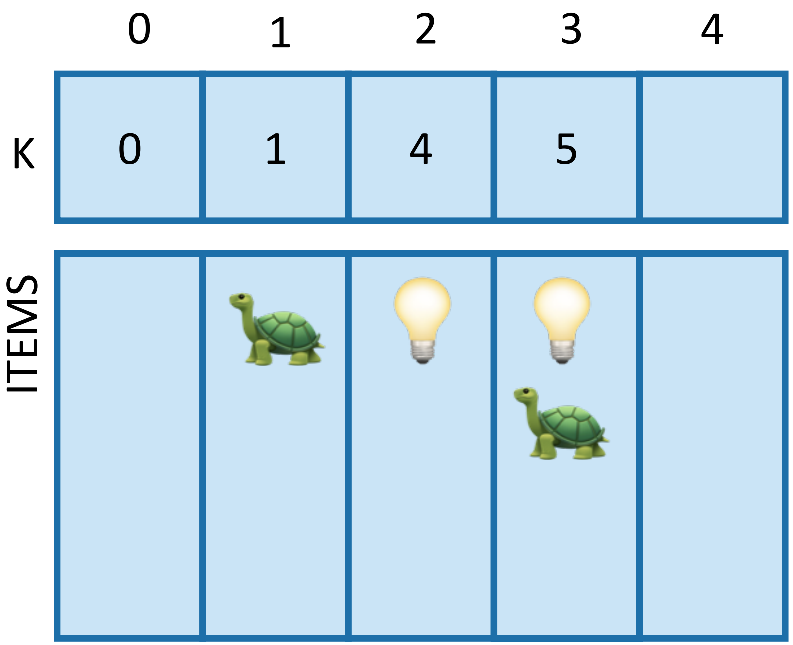
	0	1	2	3	4
K	0	1	4		
ITEMS					

ITEMS[2] = ITEMS[0] + 

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example



ITEMS[3] = ITEMS[2] + 

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = Ø
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = Ø
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{\, K[x], K[x - w_i] + v_i \,\}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

	0	1	2	3	4
K	0	1	4	6	
ITEMS					

ITEMS[3] = ITEMS[0] + 

Item:			
Weight:	1	2	3
Value:	1	4	6

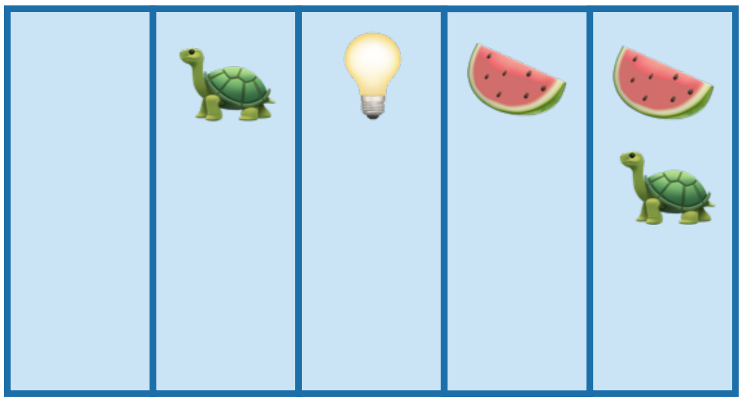
Capacity: 4

Example

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

	0	1	2	3	4
K	0	1	4	6	7

ITEMS



ITEMS[4] = ITEMS[3] + 

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

- UnboundedKnapsack(W, n, weights, values):
  - K[0] = 0
  - ITEMS[0] = ∅
  - **for** x = 1, …, W:
    - K[x] = 0
    - **for** i = 1, …, n:
      - **if** $w_i \leq x$:
        - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
        - If K[x] was updated:
          - ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
  - **return** ITEMS[W]

	0	1	2	3	4
K	0	1	4	6	8
ITEMS					

ITEMS[4] = ITEMS[2] + 

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

(Pass)

What have we learned?

- We can solve unbounded knapsack in time O(nW).
  - If there are n items and our knapsack has capacity W.
- We again went through the steps to create DP solution:
  - We kept a one-dimensional table, creating smaller problems by making the knapsack smaller.

Capacity: 10

Item:					
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- Unbounded Knapsack:
  - Suppose I have infinite copies of all of the items.
  - What's the most valuable way to fill the knapsack?

Total weight: 10
Total value: 42

- 0/1 Knapsack:
  - Suppose I have only one copy of each item.
  - What's the most valuable way to fill the knapsack?

Total weight: 9
Total value: 35

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

Optimal substructure: try 1

- Sub-problems:
  - Unbounded Knapsack with a smaller knapsack.

First solve the
problem for
small knapsacks

Then larger
knapsacks

Then larger
knapsacks

This won’t quite work…

- We are only allowed **one copy of each item**.
- The sub-problem needs to “know” what items we’ve used and what we haven’t.

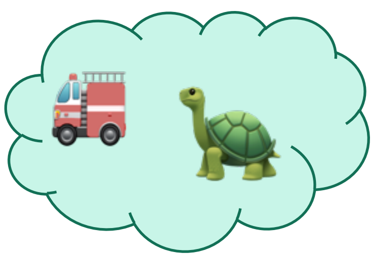
Optimal substructure: try 2

- Sub-problems:
  - 0/1 Knapsack with fewer items.

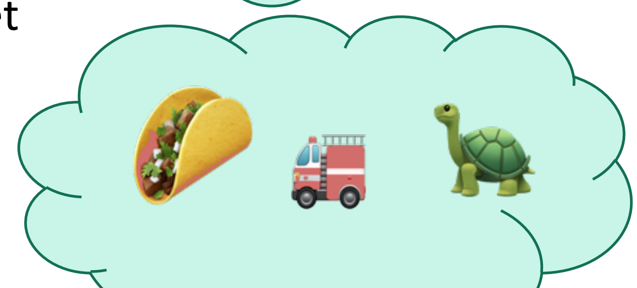
First solve the problem with few items

We'll still increase the size of the knapsacks.

Then more items



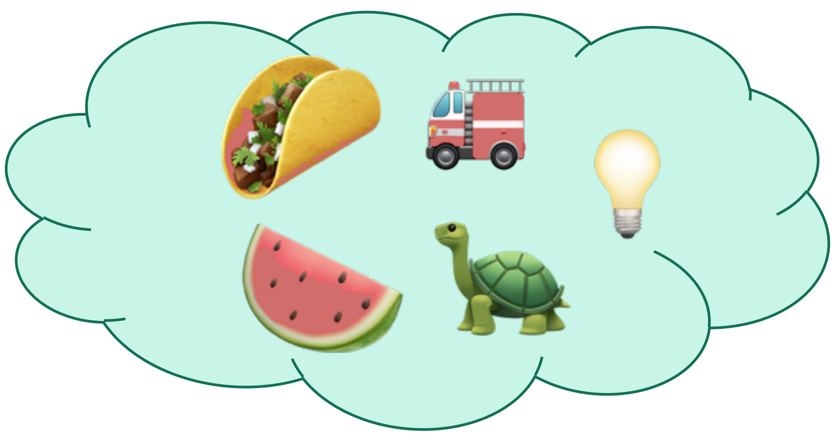
Then yet more items



(We'll keep a two-dimensional table).

Our sub-problems:

- Indexed by x and j



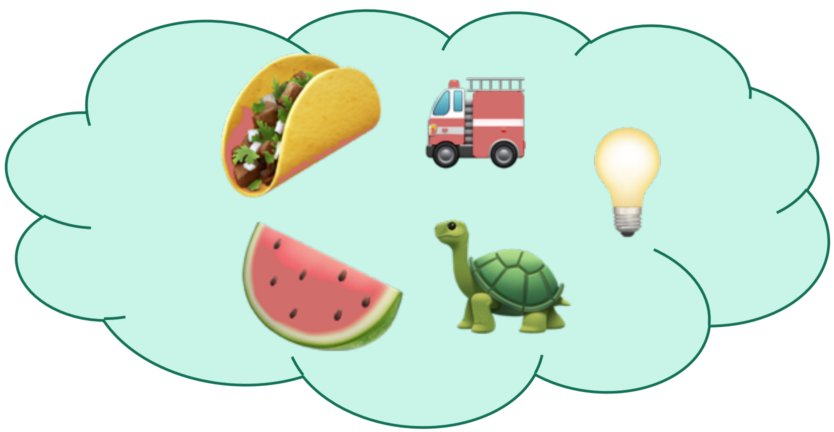
First j items

Capacity x

K[x,j] = optimal solution for a knapsack of size x using only the first j items.

Relationship between sub-problems

- Want to write K[x,j] in terms of smaller sub-problems.



First j items

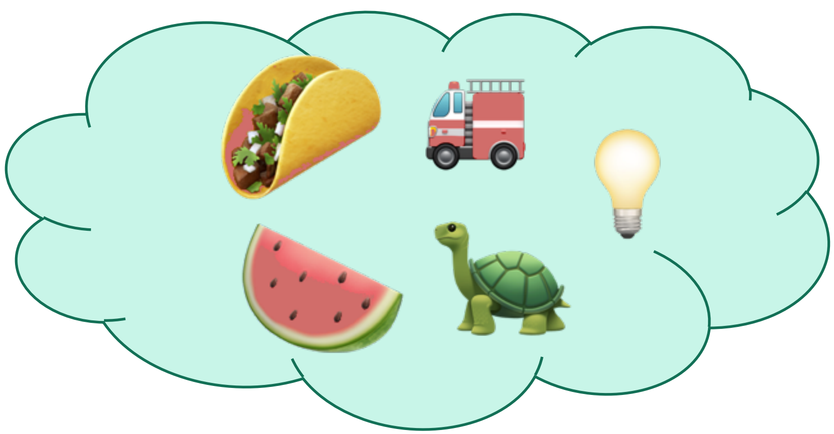
Capacity x

K[x,j] = optimal solution for a knapsack of size x using only the first j items.

Two cases


item j

- **Case 1**: Optimal solution for j items does not use item j.
- **Case 2**: Optimal solution for j items does use item j.


First j items

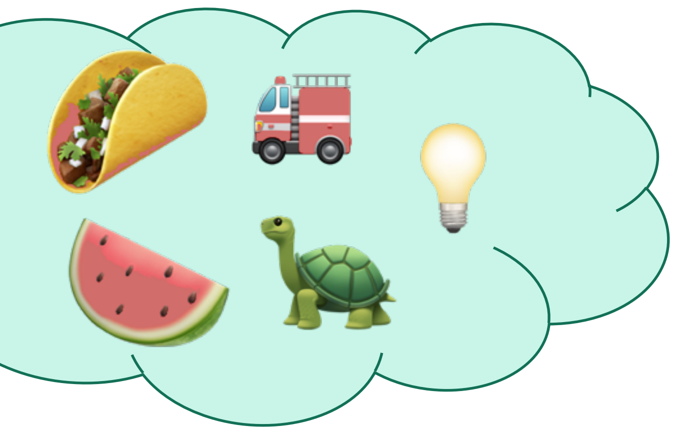

Capacity x

K[x,j] = optimal solution for a knapsack of size x using only the first j items.

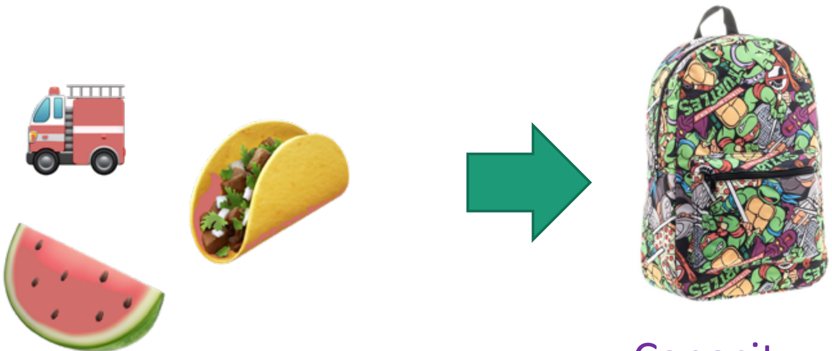
Two cases

item j

- **Case 1**: Optimal solution for j items does not use item j.

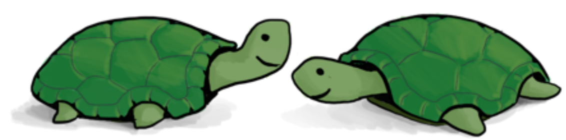


First j items



Capacity x
Value V
Use only the first j items

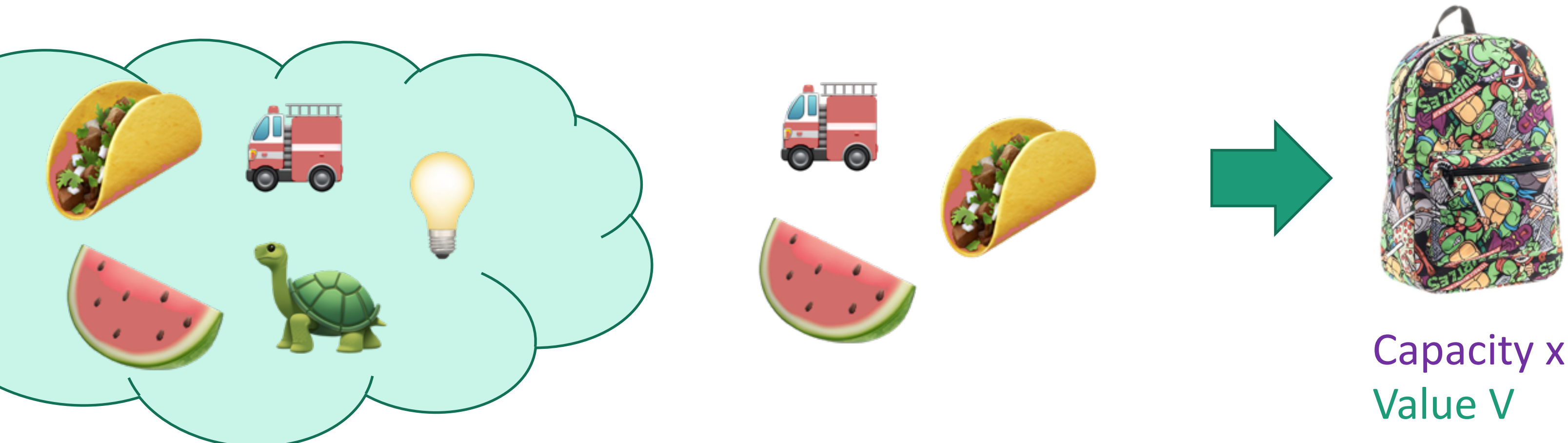
What lower-indexed problem should we solve to solve this problem?
1 min think; (wait) 1 min share



Two cases

item j

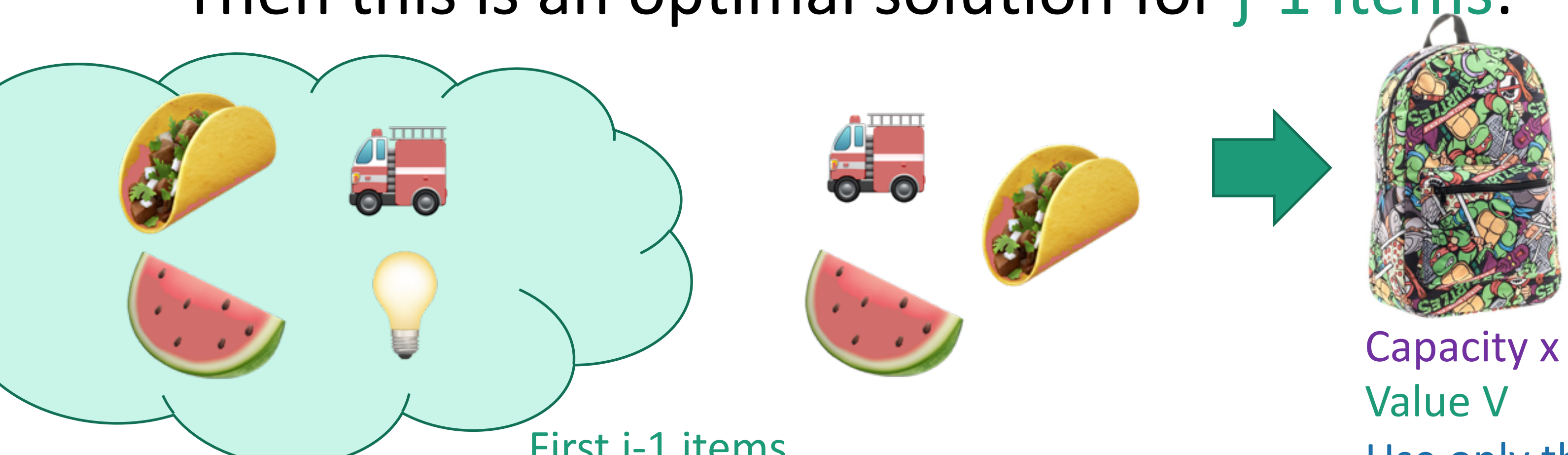
- **Case 1**: Optimal solution for j items does not use item j.



First j items

Capacity x
Value V
Use only the first j items

- Then this is an optimal solution for j-1 items:



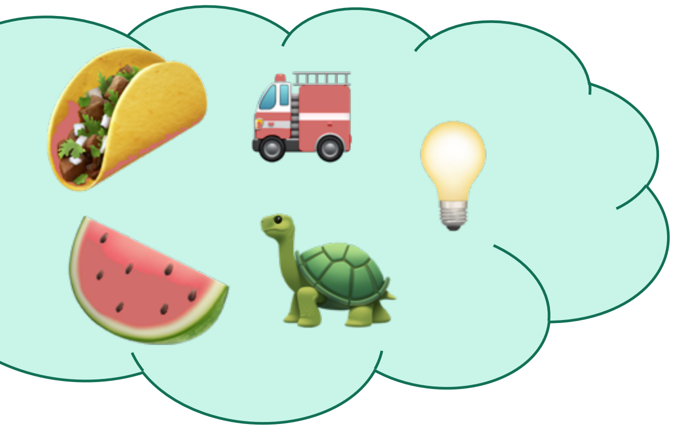
First j-1 items

Capacity x
Value V
Use only the first j-1 items.

Two cases

item j

- **Case 2**: Optimal solution for j items uses item j.

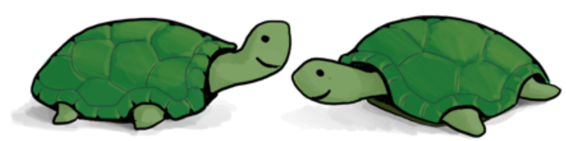


First j items

Weight w_j
Value v_j

Capacity x
Value V
Use only the first j items

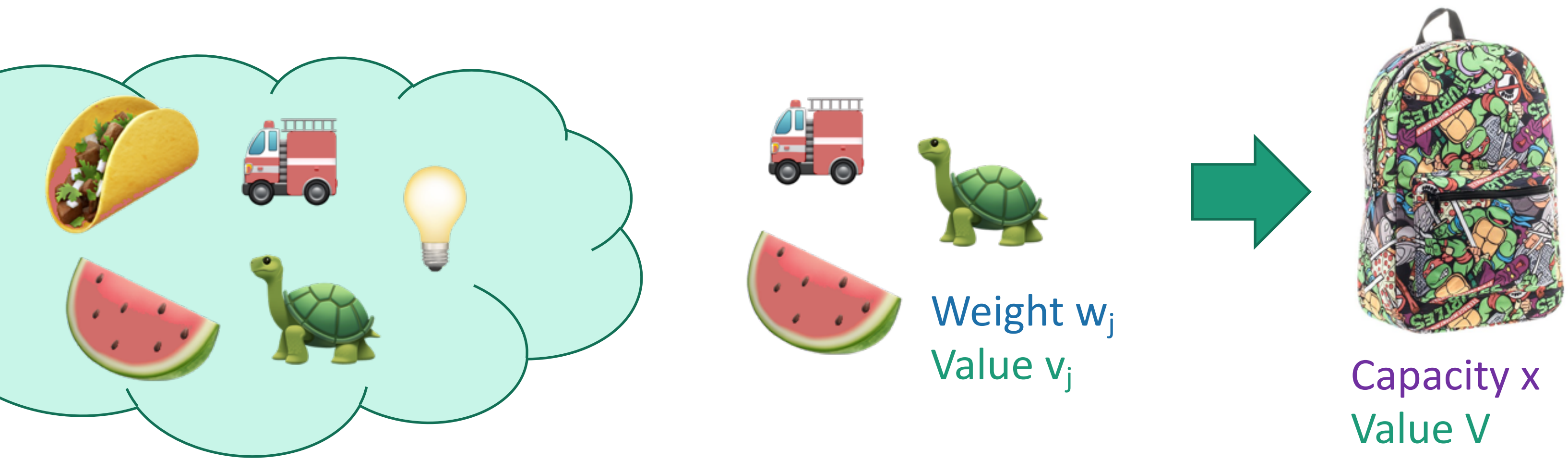
What lower-indexed problem should we solve to solve this problem?
1 min think; (wait) 1 min share



Two cases


item j

- **Case 2**: Optimal solution for j items uses item j.



First j items

Weight w_j
Value v_j

Capacity x
Value V
Use only the first j items

- Then this is an optimal solution for j-1 items and a smaller knapsack:

First j-1 items

Capacity $x - w_j$
Value $V - v_j$
Use only the first j-1 items.

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

Recursive relationship

- Let K[x,j] be the optimal value for:
  - capacity x,
  - with j items.

$$K[x,j] = \max\{ \underbrace{K[x, j-1]}_{\text{Case 1}} , \underbrace{K[x - w_{j,} j-1] + v_j}_{\text{Case 2}} \}$$

- (And K[x,0] = 0 and K[0,j] = 0).

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

Bottom-up DP algorithm

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1] Case 1
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j-1] + v_j$ } Case 2
  - **return** K[W,n]

Running time O(nW)

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

		x=0	x=1	x=2	x=3
	j=0	0	0	0	0
	j=1	0			
	j=2	0			
	j=3	0			

current entry

relevant previous entry

Item:				
Weight:	1	2	3	
Value:	1	4	6	Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[$x - w_j$, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	0		
j=2	0			
j=3	0			

current entry

relevant previous entry

Item:

Weight:	1	2	3	
Value:	1	4	6	Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j\text{-}1] + v_j$ }
  - **return** K[W,n]

		x=0	x=1	x=2	x=3
	j=0	0	0	0	0
🐢	j=1	0	1 🐢		
💡 🐢	j=2	0			
🍉 💡 🐢	j=3	0			

current entry

relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1		
j=2	0	1		
j=3	0			

current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1		
j=2	0	1		
j=3	0	1		

current entry

relevant previous entry

Item:

Weight: 1 2 3

Value: 1 4 6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j-1] + v_j$ }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	0	
j=2	0	1		
j=3	0	1		

current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	
j=2	0	1		
j=3	0	1		

current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	
j=2	0	1	1	
j=3	0	1		

current entry

relevant previous entry

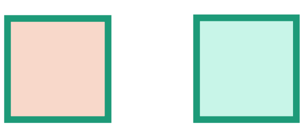
Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	
j=2	0	1	4	
j=3	0	1		



current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	
j=2	0	1	4	
j=3	0	1	4	

current entry

relevant previous entry

Item:				
Weight:	1	2	3	
Value:	1	4	6	Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
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  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	0
j=2	0	1	4	
j=3	0	1	4	

current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

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  - K[x,0] = 0 for all x = 0,…,W
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      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	1
j=2	0	1	4	
j=3	0	1	4	

current entry

relevant previous entry

Item:				
Weight:	1	2	3	
Value:	1	4	6	Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], K[x – w_j, j-1] + v_j }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	1
j=2	0	1	4	1
j=3	0	1	4	

current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
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  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j\text{-}1] + v_j$ }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	1
j=2	0	1	4	5
j=3	0	1	4	

current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j-1] + v_j$ }
  - **return** K[W,n]

Example

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
🐢 j=1	0	1 🐢	1 🐢	1 🐢
💡 🐢 j=2	0	1 🐢	4 💡	5 🐢 💡
🍉 💡 🐢 j=3	0	1 🐢	4 💡	5 🐢 💡

current entry

relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
    - **for** j = 1,…,n:
      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j-1] + v_j$ }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	1
j=2	0	1	4	5
j=3	0	1	4	6

current entry

relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):
  - K[x,0] = 0 for all x = 0,…,W
  - K[0,i] = 0 for all i = 0,…,n
  - **for** x = 1,…,W:
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      - K[x,j] = K[x, j-1]
      - **if** $w_j \leq x$:
        - K[x,j] = max{ K[x,j], $K[x - w_j, j-1] + v_j$ }
  - **return** K[W,n]

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	1
j=2	0	1	4	5
j=3	0	1	4	6

So the optimal solution is to put one watermelon in your knapsack!

current entry

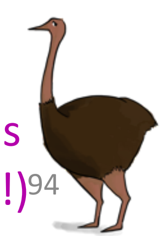
relevant previous entry

Item:				
Weight:	1	2	3	
Value:	1	4	6	Capacity: 3

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

You do this one!
(We did it on the slide in the previous example, just not in the pseudocode!)



What have we learned?

- We can solve 0/1 knapsack in time O(nW).
  - If there are n items and our knapsack has capacity W.
- We again went through the steps to create DP solution:
  - We kept a two-dimensional table, creating smaller problems by restricting the set of allowable items.

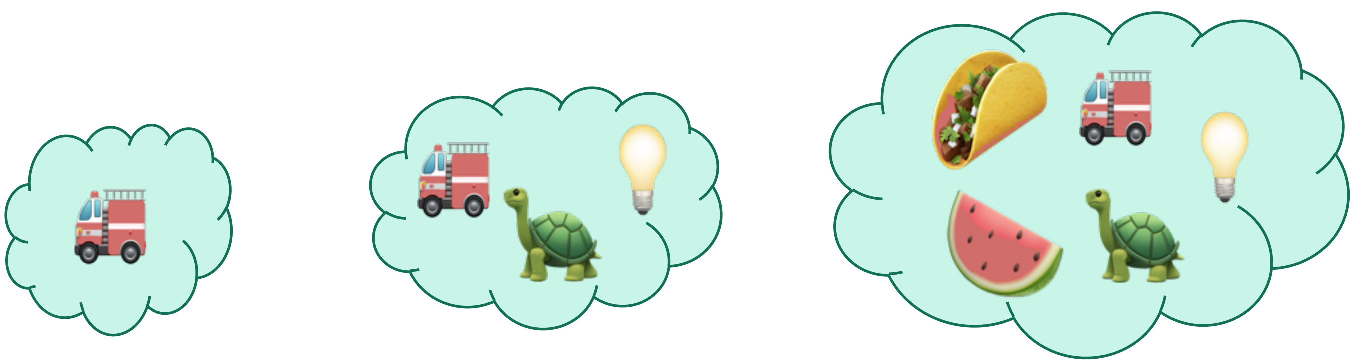
Question

- How did we know which substructure to use in which variant of knapsack?

Answer in retrospect:

This one made sense for unbounded knapsack because it doesn't have any memory of what items have been used.

vs.

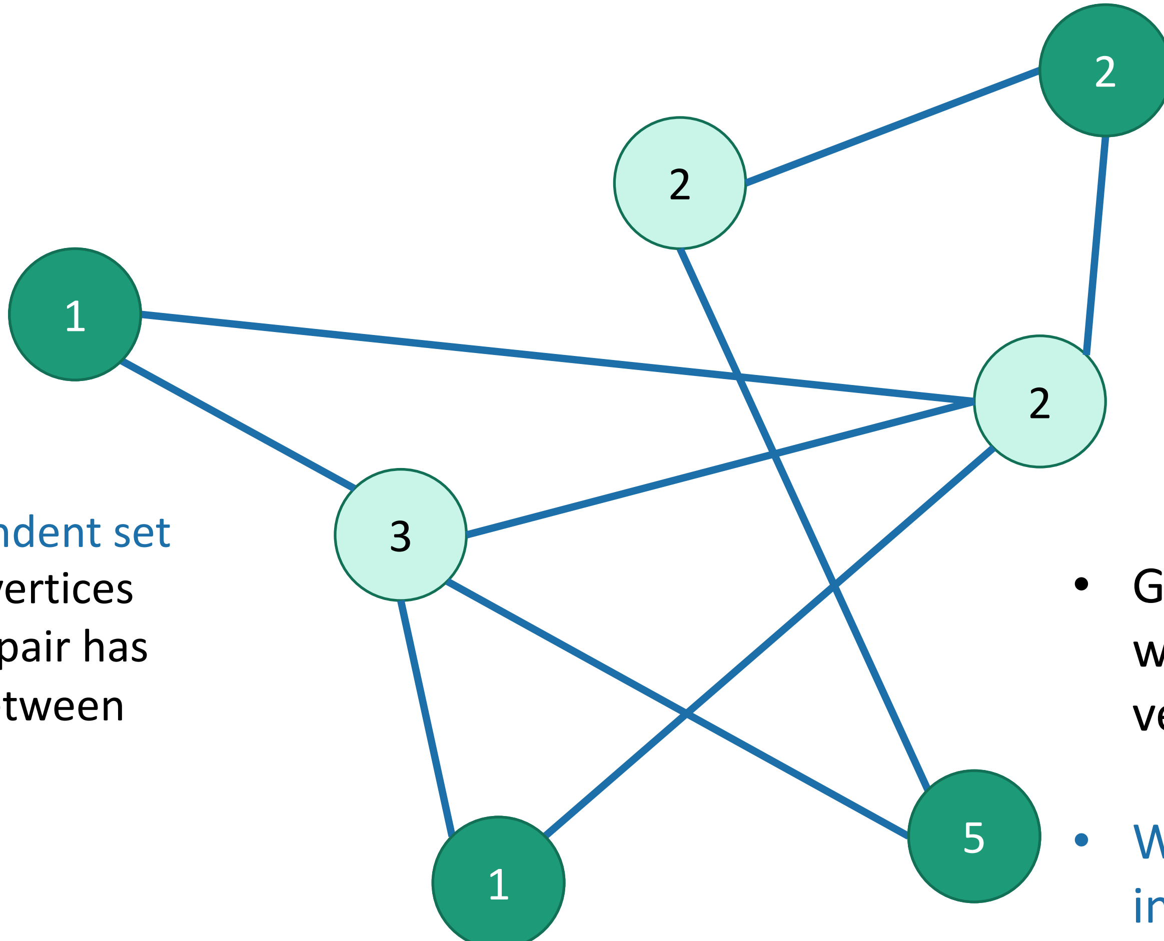


In 0/1 knapsack, we can only use each item once, so it makes sense to leave out one item at a time.

Operational Answer: try some stuff, see what works!

Example 3: Independent Set

if we still have time



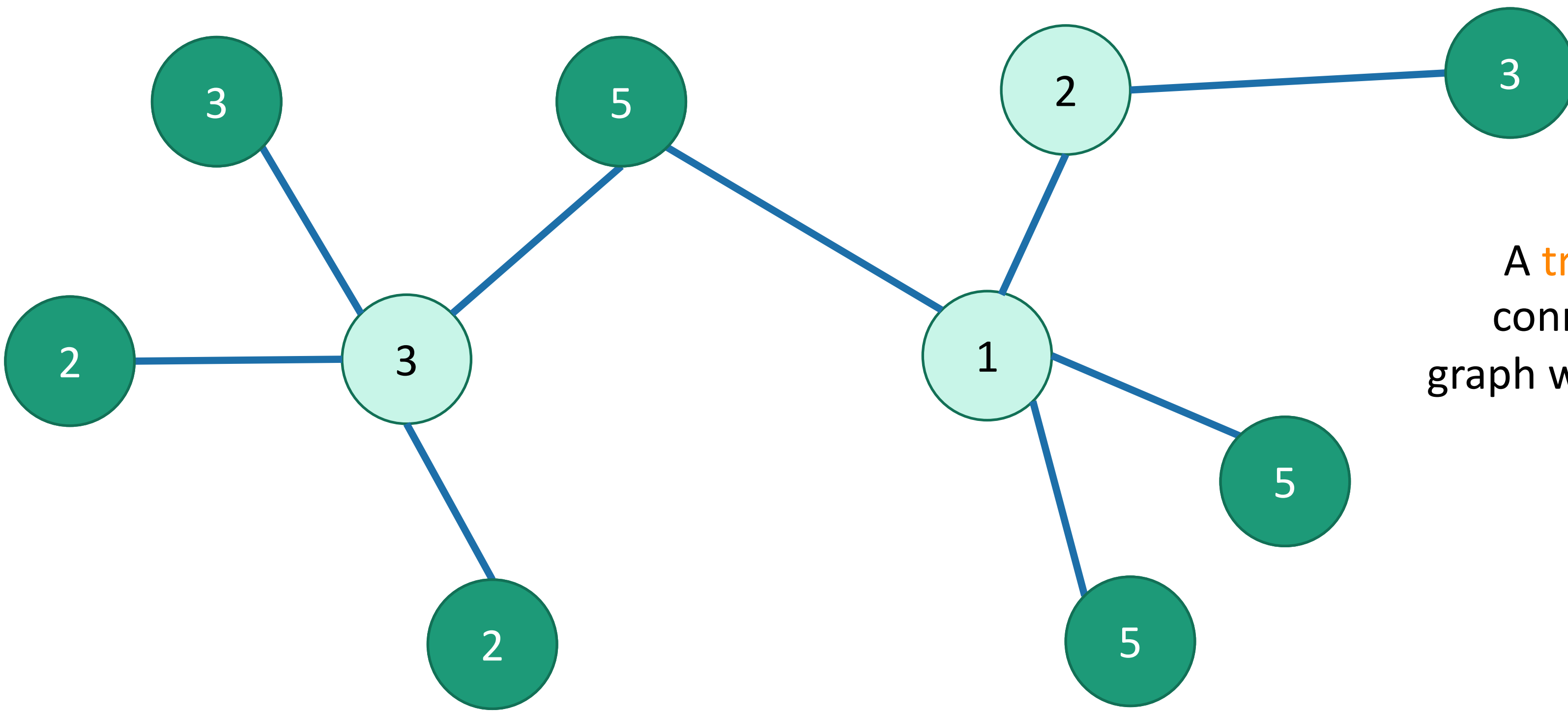
An independent set is a set of vertices so that no pair has an edge between them.

- Given a graph with weights on the vertices…
- What is the independent set with the largest weight?

Actually, this problem is NP-complete.

So, we are unlikely to find an efficient algorithm.

- But if we also assume that the graph is a tree…



A tree is a connected graph with no cycles.

Problem:

find a maximal independent set in a tree (with vertex weights).

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
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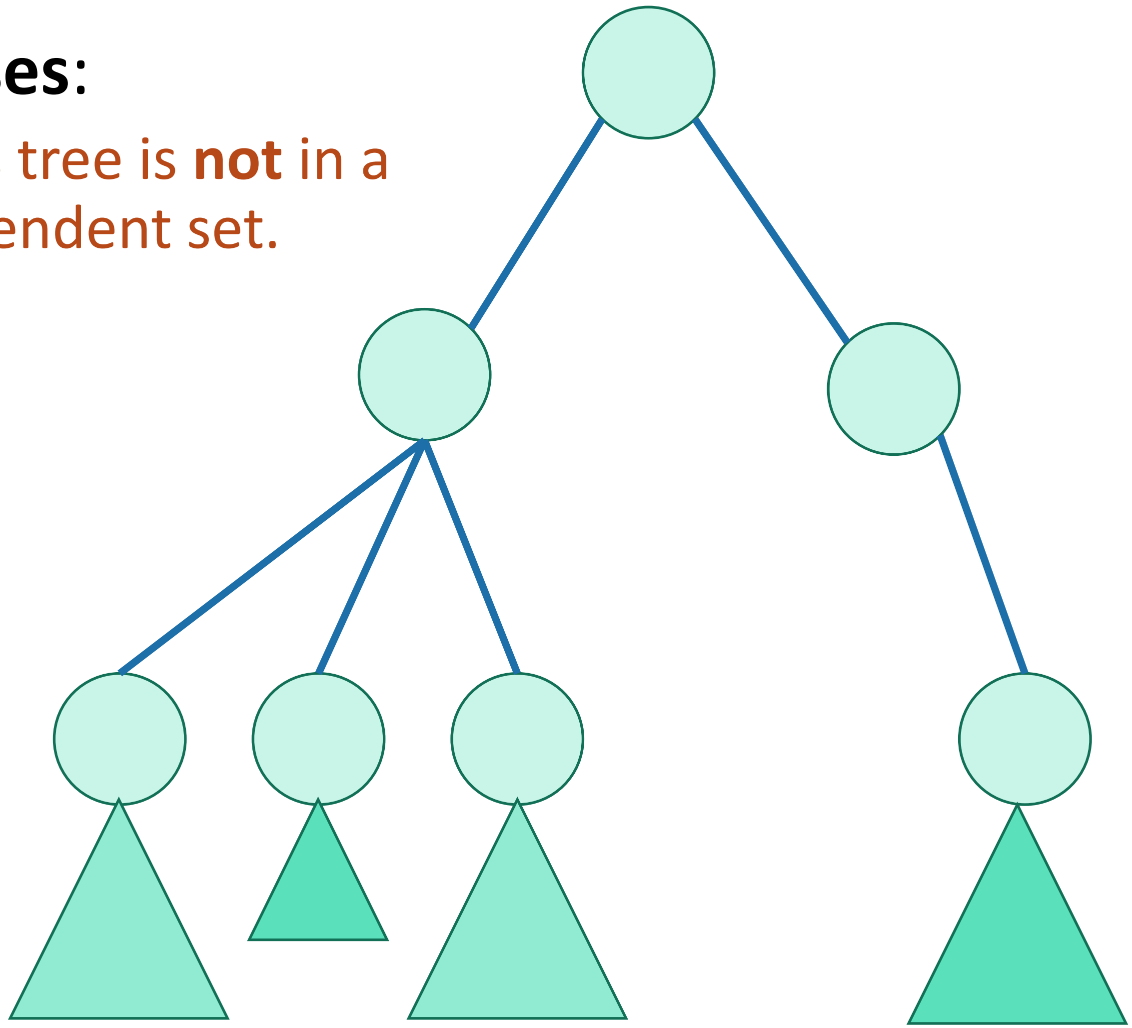
- **Step 5:** If needed, code this up like a reasonable person.

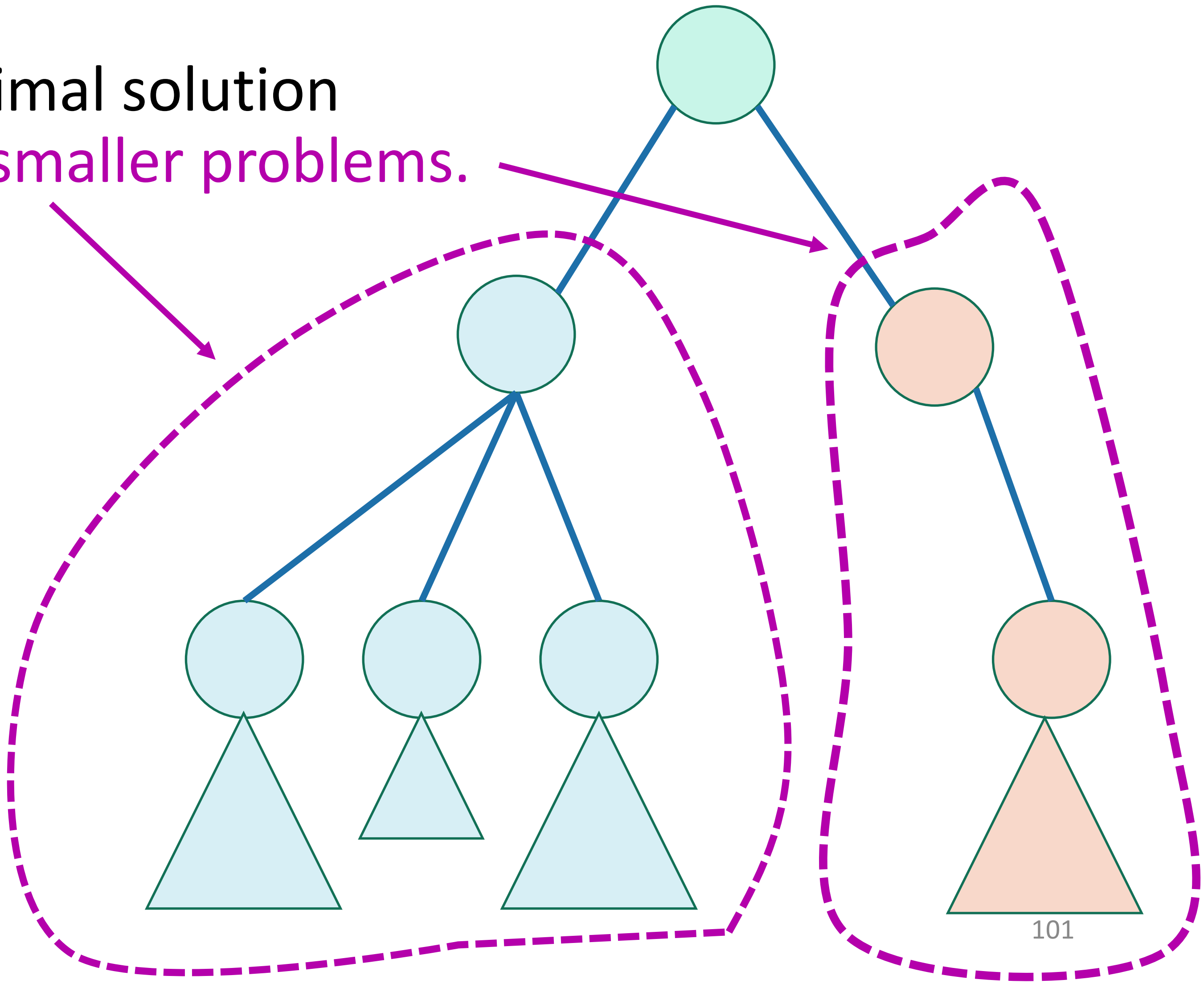
Optimal substructure

- **Subtrees** are a natural candidate.
- There are **two cases**:
  1. The root of this tree is **not** in a maximal independent set.
  2. Or it is.



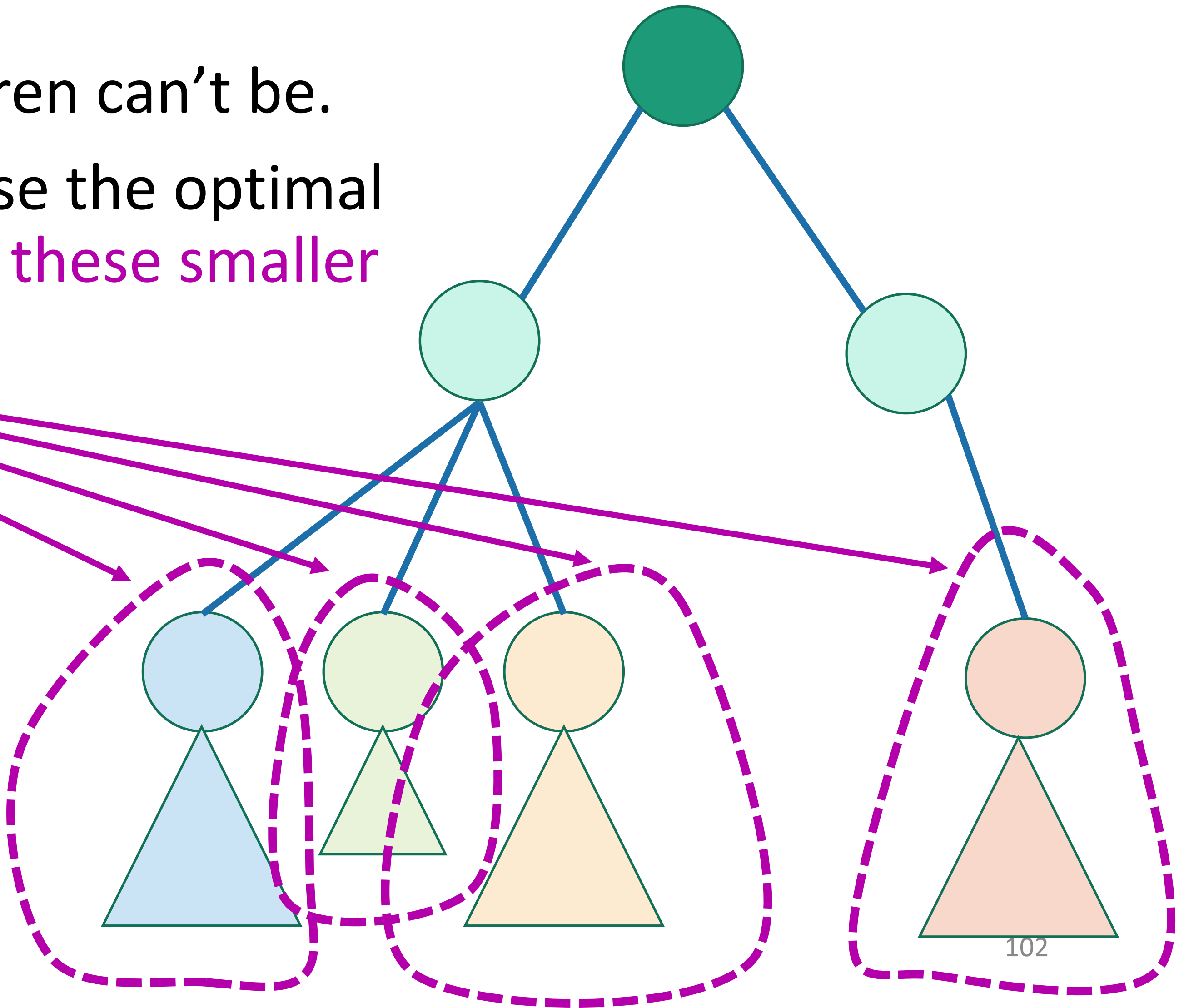
Case 1:
the root is **not** in a maximal independent set

- Use the optimal solution from these smaller problems.



Case 2:
the root is in an maximal independent set

- Then its children can't be.
- Below that, use the optimal solution from these smaller subproblems.



Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
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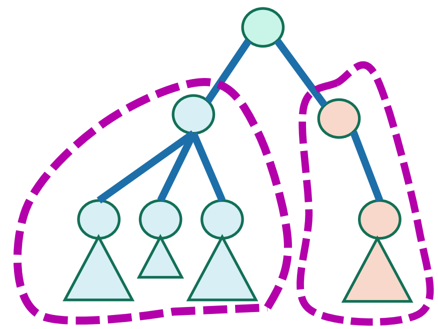
Recursive formulation: try 1

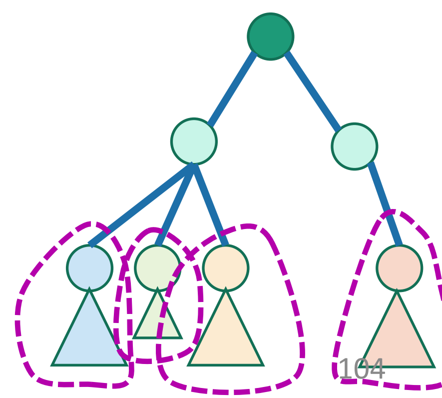
- Let A[u] be the weight of a maximal independent set in the tree rooted at u.



- $A[u] =$

$$\max \begin{cases} \sum_{v \in u.\mathrm{children}} A[v] \\ \mathrm{weight}(u) + \sum_{v \in u.\mathrm{grandchildren}} A[v] \end{cases}$$

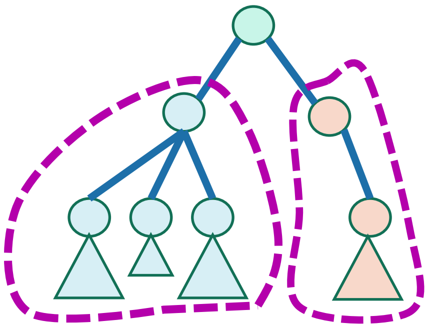
When we implement this, how do we keep track of this term?



Recursive formulation: try 2

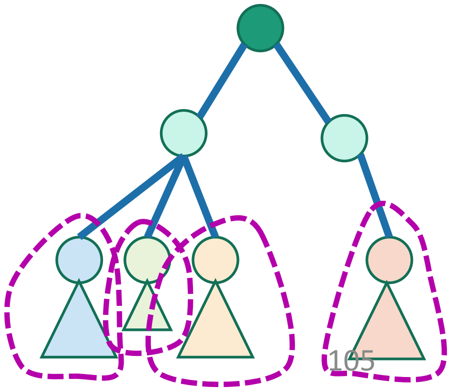
Keep two arrays!

- Let A[u] be the weight of a maximal independent set in the tree rooted at u.



- Let B[u] = $\sum_{v \in u.\text{children}} A[v]$

- $A[u] = \max \begin{cases} \sum_{v \in u.\text{children}} A[v] \\ \text{weight}(u) + \sum_{v \in u.\text{children}} B[v] \end{cases}$



Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
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- **Step 5:** If needed, code this up like a reasonable person.

A top-down DP algorithm

- MIS_subtree(u):
  - **if** u is a leaf:
    - A[u] = weight(u)
    - B[u] = 0
  - **else**:
    - **for** v in u.children:
      - MIS_subtree(v)
    - $A[u] = \max\{ \sum_{v \in u.\text{children}} A[v] , \text{weight}(u) + \sum_{v \in u.\text{children}} B[v] \}$
    - $\text{B}[u] = \sum_{v \in u.\text{children}} A[v]$

Initialize global arrays A, B that we will use in all of the recursive calls.

- MIS(T):
  - MIS_subtree(T.root)
  - **return** A[T.root]

Running time?

- We visit each vertex once, and for every vertex we do O(1) work:
  - Make a recursive call
  - Participate in summations of parent node
- Running time is O(|V|)

Why is this different from divide-and-conquer?

That's always worked for us with tree problems before…

- MIS_subtree(u):
  - **if** u is a leaf:
    - **return** weight(u)
  - **else**:
    - **return** max{ $\sum_{v \in u.\text{children}} \text{MIS\_subtree}(v)$,

      $\text{weight}(u) + \sum_{v \in u.\text{grandchildren}} \text{MIS\_subtree}(v)$ }

This is exactly the same pseudocode, except we've ditched the table and are just calling MIS_subtree(v) instead of looking up A[v] or B[v].

- MIS(T):
  - **return** MIS_subtree(T.root)

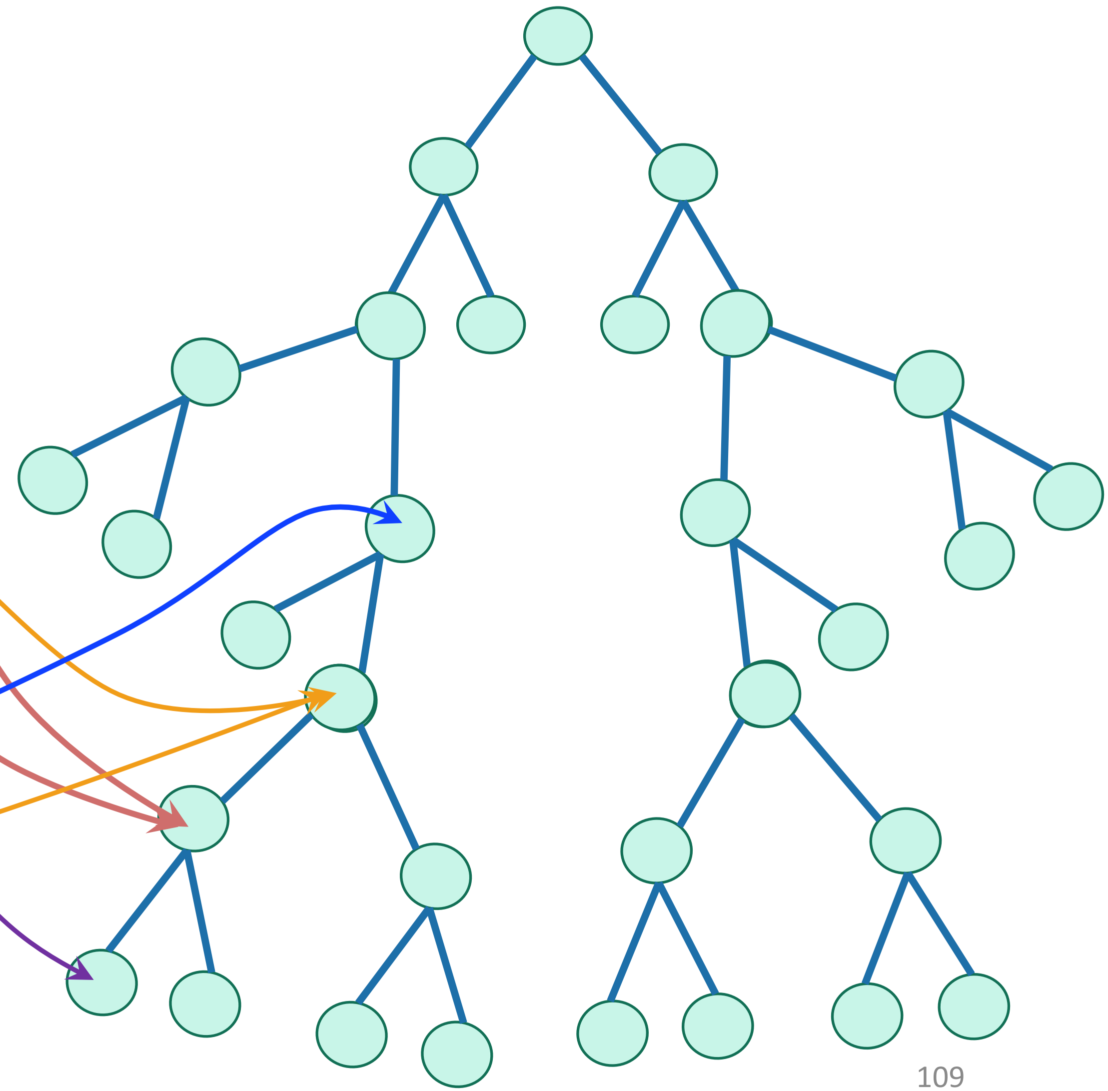
Why is this different from divide-and-conquer?

That’s always worked for us with tree problems before…

How often would we ask about the subtree rooted **here**?

Once for **this node** and once for **this one**.

But we then ask about **this node** twice, **here** and **here**.



This will blow up exponentially without using dynamic programming to take advantage of **overlapping subproblems**.

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
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- **Step 5:** If needed, code this up like a reasonable person.

You do this one!

What have we learned?

- We can find maximal independent sets in trees in time O(|V|) using dynamic programming!
- For this example, it was natural to implement our DP algorithm in a top-down way.

Recap

- Today we saw examples of how to come up with dynamic programming algorithms.
  - Longest Common Subsequence
  - Knapsack two ways
  - (If time) maximal independent set in trees.
- There is a **recipe** for dynamic programming algorithms.

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
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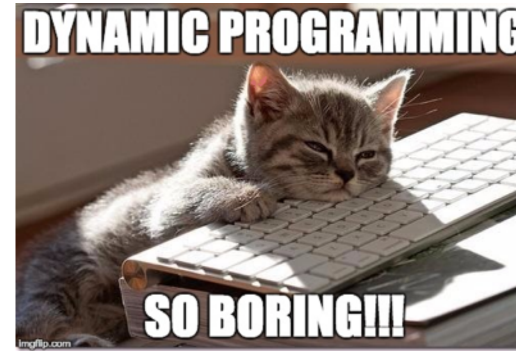
Recap



- Today we saw examples of how to come up with dynamic programming algorithms.
  - Longest Common Subsequence
  - Knapsack two ways
  - (If time) maximal independent set in trees.
- There is a **recipe** for dynamic programming algorithms.
- Sometimes coming up with the right substructure takes some creativity
  - Practice on homework! ☺
  - For even more practice check out additional examples/practice problems in CLRS or section!

Next time

- Greedy algorithms!

Before next time

- Pre-lecture exercise: Greed is good!
- Good luck on exam 3.