CS 161 Winter 2021 Section 9

March 11, 2021

Exercise 1

Let G be a connected weighted undirected graph. In class, we defined a minimum spanning tree of G as a spanning tree T of G which minimizes the quantity

$$X = \sum_{e \in T} w_e,$$

where the sum is over all the edges in T, and w_e is the weight of edge e. Define a "minimum-maximum spanning tree" to be a spanning tree that minimizes the quantity

$$Y = \max_{e \in T} w_e.$$

That is, a minimum-maximum spanning tree has the smallest maximum edge weight out of all possible spanning trees.

1. Give an example of a graph G which has a minimum-maximum spanning tree T so that T is *not* a minimum spanning tree.
2. Prove that a minimum spanning tree in a connected weighted undirected graph G is always a minimum-maximum spanning tree for G.

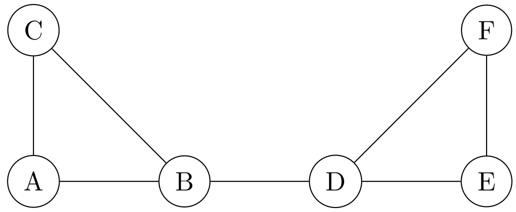
   Hint: Suppose toward a contradiction that T is an MST but not a minimum-maximum spanning tree, and say that T' is a minimum-maximum spanning tree. How can you use T' to modify T, to come up with a cheaper MST than T (and hence a contradiction)? (Sub-hint: consider the heaviest edge in T).

Exercise 2

Given a set of n cities, we would like to build a transportation system such that there is some path from any city i to any other city j. There are two ways to travel: by driving or by flying. Initially all of the cities are disconnected. It costs r_{ij} build a road between city i and city j. It costs a_i to build an airport in city i. For any two cities i and j, we can fly directly from i to j if there is an airport in both cities. Give an efficient algorithm for determining which roads and airports to build to minimize the cost of connecting the cities.

Exercise 3

Consider the following graph:



1. What is the global mimimum cut of this graph?
2. What is the probability that Karger's algorithm chooses an edge of the minimum cut with its first choice?
3. What is the probability that one run of Karger's algorithm returns a minimum cut on this graph? How does it compare to the bound of $1/\binom{n}{2}$ that we saw in class?