# CS 161 (Stanford, Winter 2022) Homework 1

**Style guide and expectations:** Please see the "Homework" part of the "Resources" section on the webpage for guidance on what we are looking for in homework solutions. We will grade according to these standards. You should cite all sources you used outside of the course material.
**What we expect:** Make sure to look at the "**We are expecting**" blocks below each problem to see what we will be grading for in each problem!

**Exercises.** The following questions are exercises. We suggest you do these on your own. As with any homework question, though, you may ask the course staff for help.

## 1 Exercise: Course policies

**(1 pt.)** Have you thoroughly read the course policies on the webpage?
**[We are expecting:** The answer "yes."**]**

## 2 Exercise: Big-Oh Basics

**(2 pt.)** Which of the following functions are $O(n^2)$?
No explanation is required, but you might want to prove your answer to yourself to convince yourself that you are correct.

1. $f_1(n) = 2n + 4$
2. $f_2(n) = 2n^2 + 4$
3. $f_3(n) = 2n^3 + 4$
4. $f_4(n) = n \log n$
5. $f_5(n) = \sin(n) + 5$
6. $f_6(n) = 2^n$
7. $f_7(n) = (100^3)^4$

**[We are expecting:** A list of which functions are $O(n^2)$. No explanation is required and no partial credit will be given.**]**

# 3 Exercise: Big-Oh Definitions

Terry the Terrific Tiger wants to prove, from the definition of big-Oh, that $f(n) = O(g(n))$, where $f(n) = 6n$ and $g(n) = n^2 - 3n$.

## 3.1 "c = 1"

**(2 pt.)** In the definition of big-Oh, Terry really wants to take $c = 1$. Give a proof that $f(n) = O(g(n))$ in which "$c$" is chosen to be 1.

## 3.2 "$n_0 = 4$"

**(2 pt.)** Terry has changed their mind. Now, they don't care what $c$ is, but would like to take $n_0 = 4$. Give a different proof that $f(n) = O(g(n))$ in which "$n_0$" is chosen to be 4.
**[We are expecting:** For both parts, a short but formal proof.**]**

# 4 Exercise: Wrapping Up

**(10 pt.)** For each blank, indicate whether $A_i$ is in $O$, $\Omega$, or $\Theta$ of $B_i$. More than one space per row can be valid.
**[We are expecting:** All valid spaces in the table to be marked (checkmark, "x", etc.). No explanation is required.**]**

| **A** | **B** | $O$ | $\Omega$ | $\Theta$ |
|---|---|---|---|---|
| $\log^7 n$ | $n^{0.7}$ | | | |
| $2n^{1.5}$ | $2^n$ | | | |
| $3^{3n}$ | $3^{4n}$ | | | |
| $\ln n$ | $\log n$ | | | |
| $\log(n!)$ | $\log(n^n)$ | | | |
| $(13/12)^n$ | $(12/13)^n$ | | | |
| $n^2$ | $4^{\log_2 n}$ | | | |
| $n^{0.1}$ | $(0.1)^n$ | | | |
| $\log\log n$ | $\sqrt{\log n}$ | | | |
| $n^{1/\log n}$ | 1 | | | |

**Problems.** The following questions are problems. You may talk with your fellow CS 161-ers about the problems. However:

- Try the problems on your own *before* collaborating.
- Write up your answers yourself, in your own words. You should never share your typed-up solutions with your collaborators.
- If you collaborated, list the names of the students you collaborated with at the beginning of each problem.

# 5 Big-Oh and Big-Omega

Consider the function $f(n)$ defined by

$$f(n) = \frac{1}{n+1}\binom{2n}{n}.$$

Using the definitions given in lecture, show:

(a) **(5 pt.)** That $f(n) = \Omega(4^n/n^2)$, and

(b) **(5 pt.)** that $f(n) = O(4^n)$.
**[We are expecting:** Formal proofs for both parts.**]**

# 6 Minerals in a cave

**(10 pt.)** You are given a picture of a cave that is $n$ pixels tall and $m$ pixels wide. Each pixel is either a 0 or a 1. Each column of the picture belongs to exactly one of the following categories:

- A column is a *mineral-column* if every pixel in it is a 1.
- A column is a *stalactite* if it has both 0s and 1s, and all the ones appear above all the 0s.
- A column is a *stalagmite* if it has both 0s and 1s, and all the ones appear below all the 0s.
- A column is *nothing* if every pixel in it is a 0s.

The *length* of a stalactite or stalagmite is the number of 1s in that column. Given such a picture, devise a $O(m \log n)$ algorithm that outputs the following value:

$$\max(\text{length of the longest stalactite}, \text{length of the longest stalagmite}).$$

**[We are expecting:** Pseudocode for your algorithm and a clear English description of what your algorithm is doing and why it is correct. You do not need to prove that your algorithm is correct.**]**

# 7 Math on Mars?!

**(10 pt.)** Elon Tusk has discovered a colony of aliens living on Mars, and has been learning more about their arithmetic. Interestingly, he has discovered that they use the same base 10 numerical system, addition, and subtraction as we do, but their system of multiplication is completely different. The symbol they use is $\star$, and he has discovered a 1-digit multiplication table they use in elementary school. Here are some examples:

$$1 \star 1 = 3 \qquad 2 \star 3 = 19 \qquad 3 \star 5 = 49 \qquad 9 \star 9 = 243$$

and so on. After some time, he has discovered that their multiplication corresponds to the following in human arithmetic:

$$x \star y = x^2 + xy + y^2.$$

Wanting to gain more followers on Martian Tweeeter, Elon wants to send out a tweeet detailing an efficient way to compute the Martian product of two $n$-digit numbers, without referencing or explaining *any* human multiplication (even implicitly as a sum). For example, he can't tell them to compute $x^2, xy$, and $y^2$ separately since they don't understand what these terms mean. Help Elon find an algorithm to compute the Martian product of two $n$-digit numbers that runs in $O(n^{\log_2 3})$ time. For example, your algorithm might take as input $x = 1234$ and $y = 4321$, and it should return $x \star y = 25525911$. You may assume that $n$ is a power of 2, and that the alien computers can shift digits, add, and subtract just like human computers can. You may also assume that the alien computers have easy access to the $\star$-multiplication table you received.
**[We are expecting:** Pseudocode for your algorithm and a clear English description of what your algorithm is doing and why it is correct. You do not need to prove that your algorithm is correct.**]**