Lecture 11

Weighted Graphs: Dijkstra and Bellman-Ford

NOTE: We may not get to Bellman-Ford!
We will spend more time on it next time.

Announcements

- The midterm is over!
  - for most of you
- Don’t talk about it just yet – we will tell you when it is ok to discuss the midterm!
- HW5 is out today!

Ed Heroes

- < Your name here >

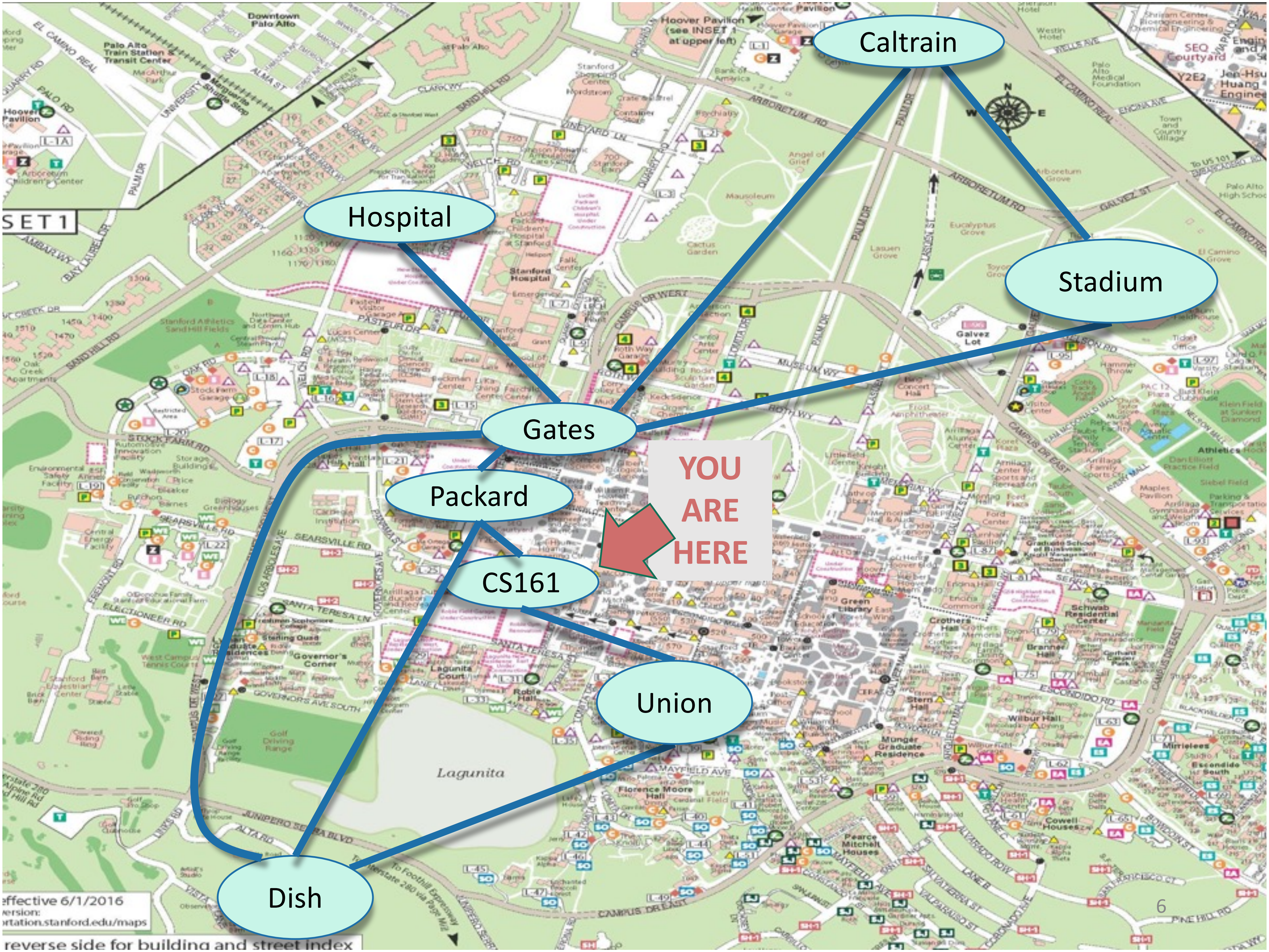
- Bonus points for the most endorsed students on Ed

Previous two lectures

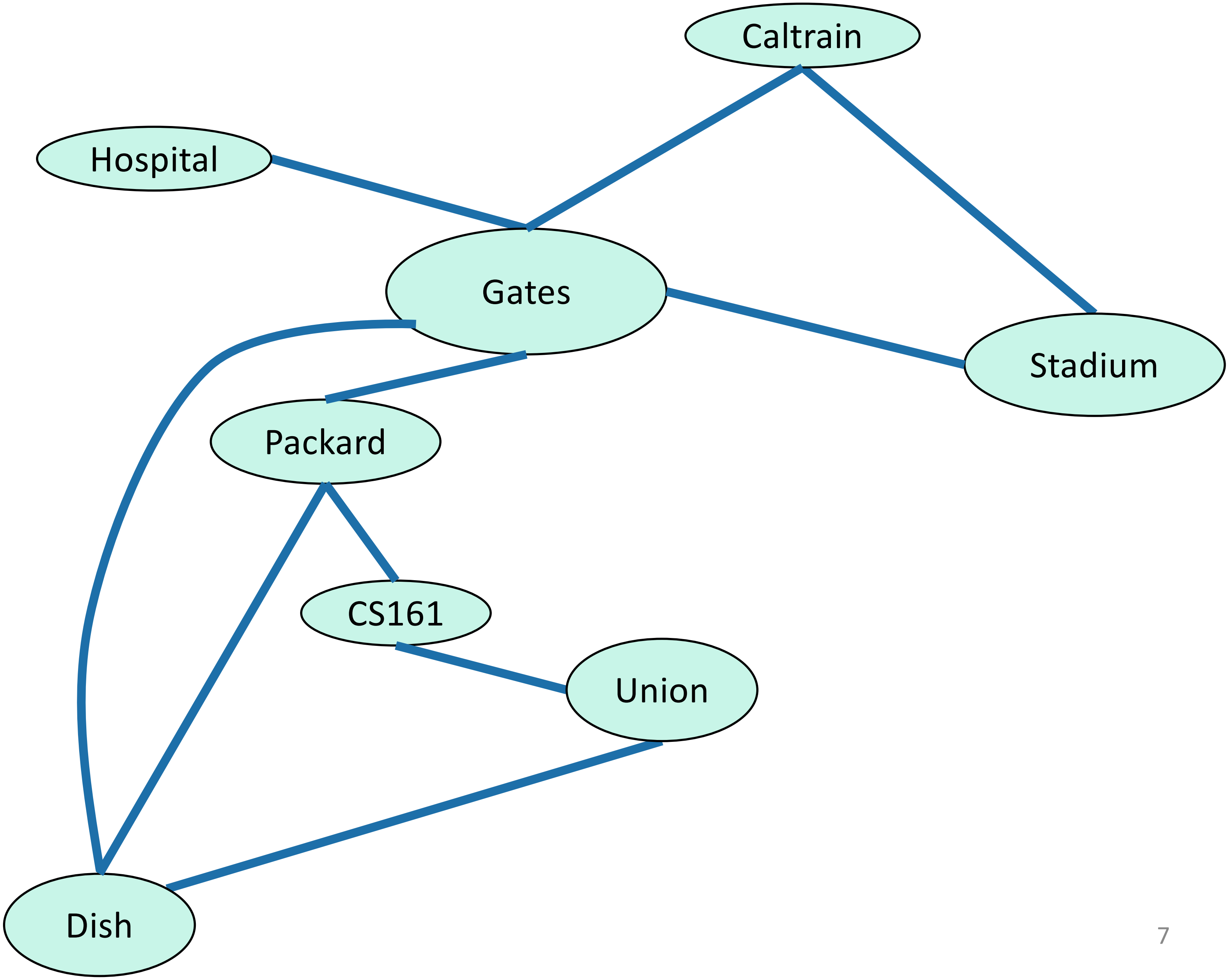
- Graphs!
- DFS
  - Topological Sorting
  - Strongly Connected Components
- BFS
  - Shortest Paths in unweighted graphs

Today

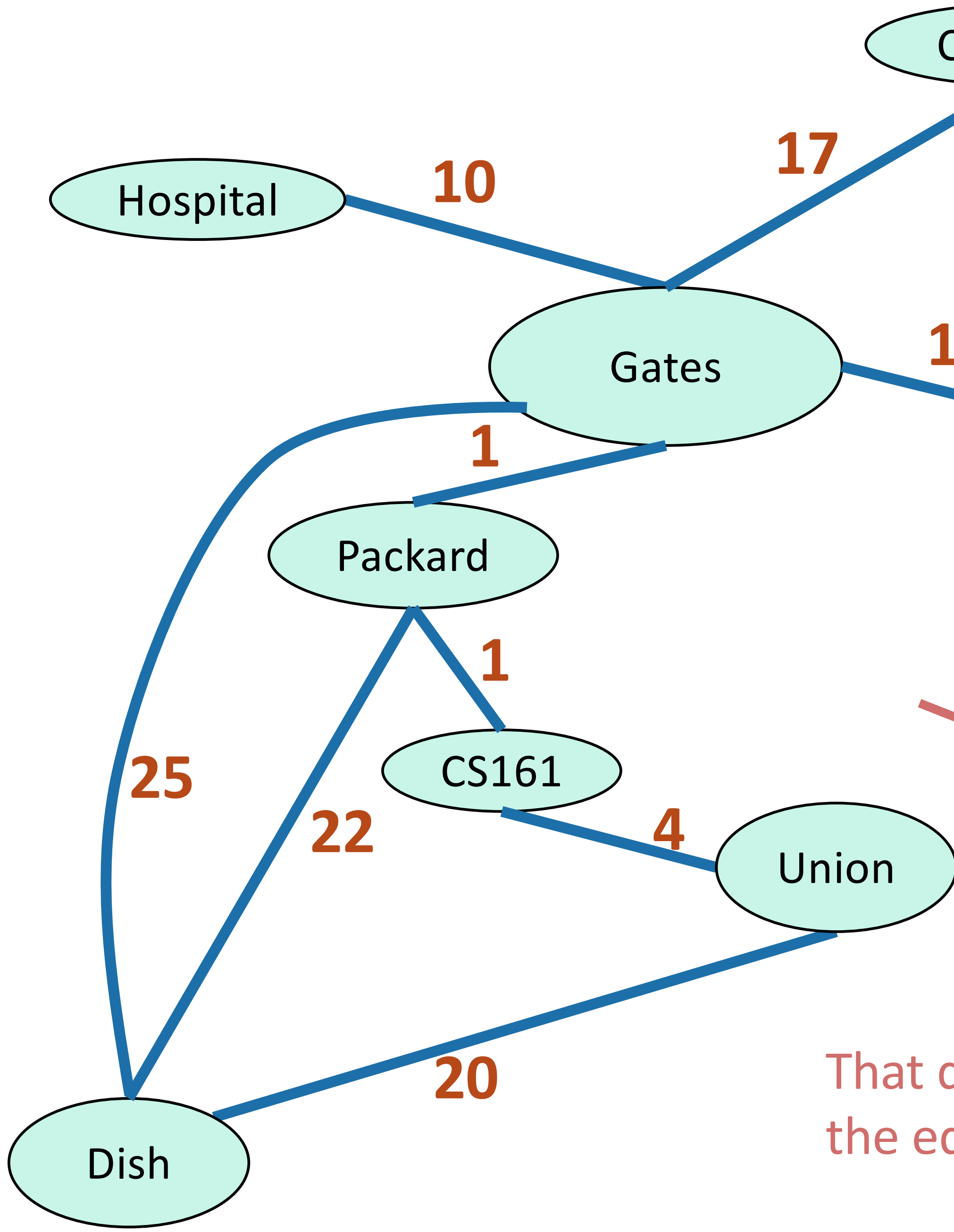
- What if the graphs are weighted?
- Part 1: Dijkstra!
  - This will take most of today's class
- Part 2: Bellman-Ford!
  - Real quick at the end if we have time!
  - We'll come back to Bellman-Ford in more detail, so today is just a taste.


Caltrain
Hospital
Stadium
Gates
Packard
CS161
Union
Dish
YOU ARE HERE

Just the graph



Shortest path from Gates to the Union?



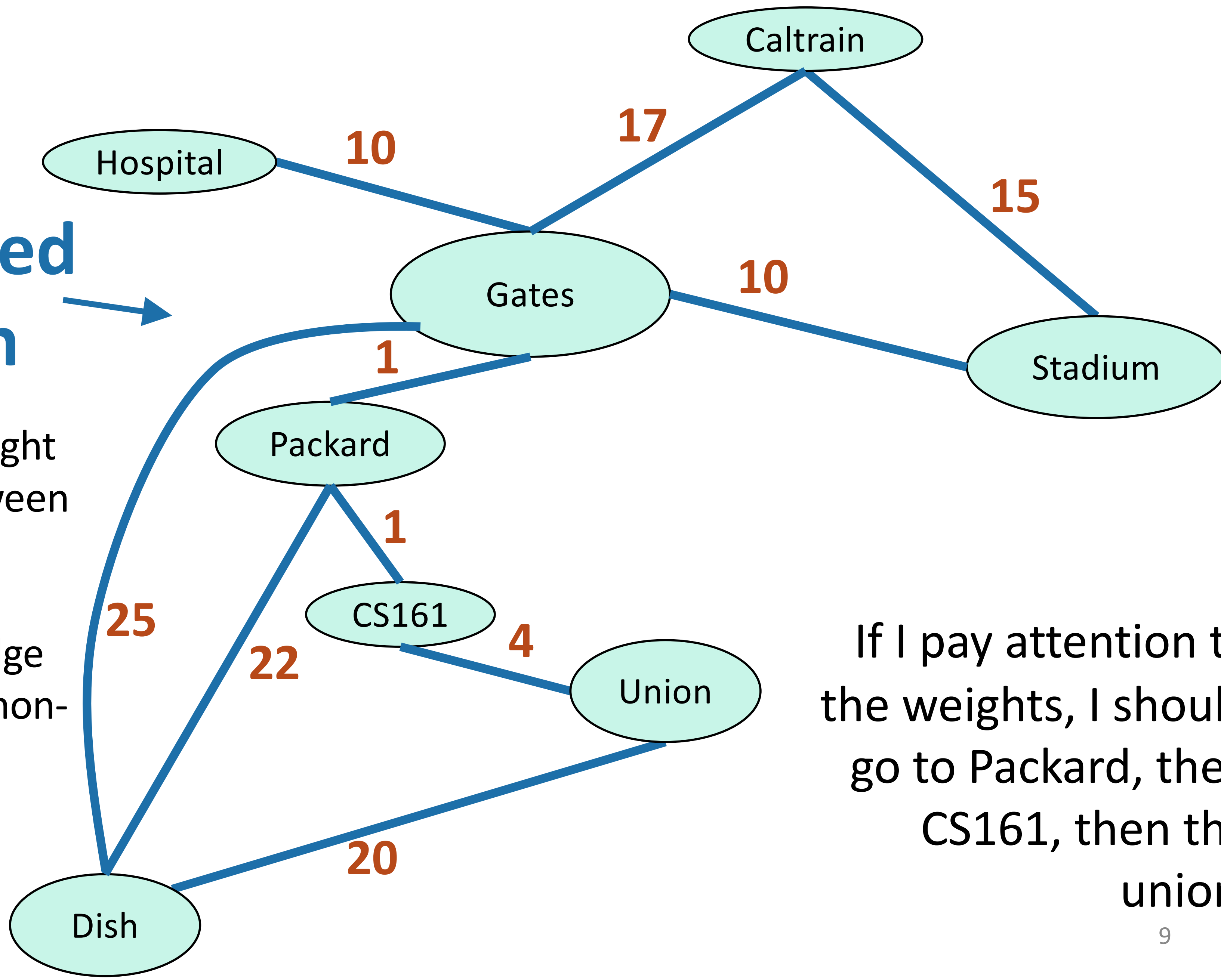
~~Run BFS …~~

~~I should go to the dish and then back to the union!~~

That doesn't make sense if I label the edges by walking time.

Shortest path from Gates to the Union?

weighted graph



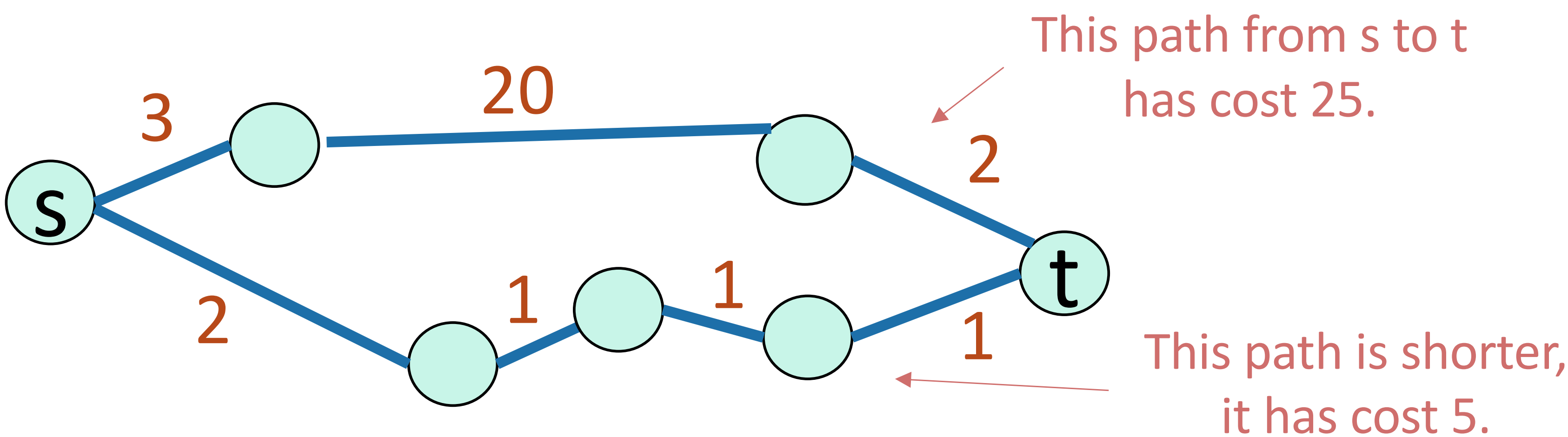
w(u,v) = weight of edge between u and v.

For now, edge weights are non-negative.

If I pay attention to the weights, I should go to Packard, then CS161, then the union.

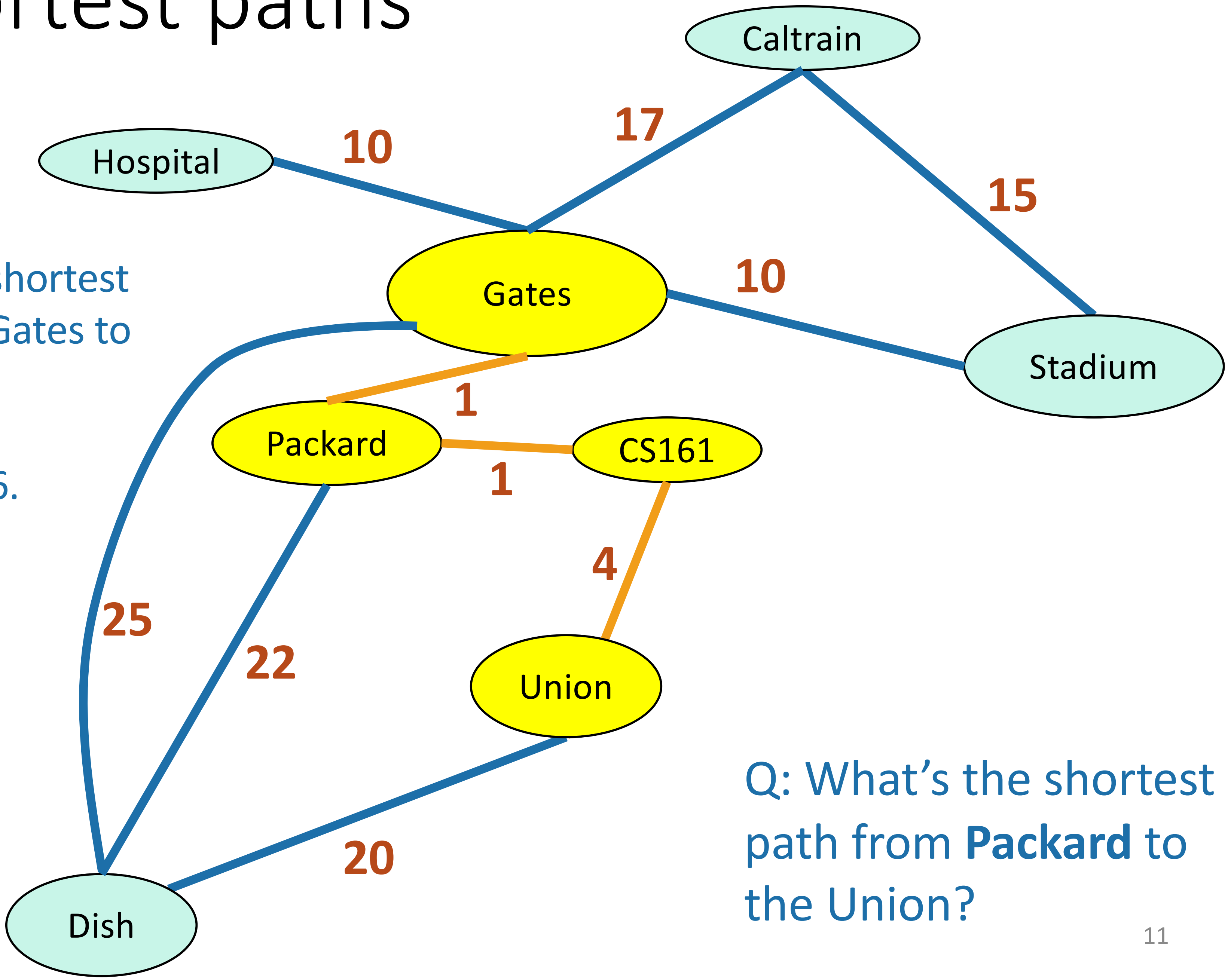
Shortest path problem

- What is the shortest path between u and v in a weighted graph?
  - the **cost** of a path is the sum of the weights along that path
  - The **shortest path** is the one with the minimum cost.



- The **distance** d(u,v) between two vertices u and v is the cost of the the shortest path between u and v.
- For this lecture **all graphs are directed**, but to save on notation I'm just going to draw undirected edges.

Shortest paths



This is the shortest path from Gates to the Union.

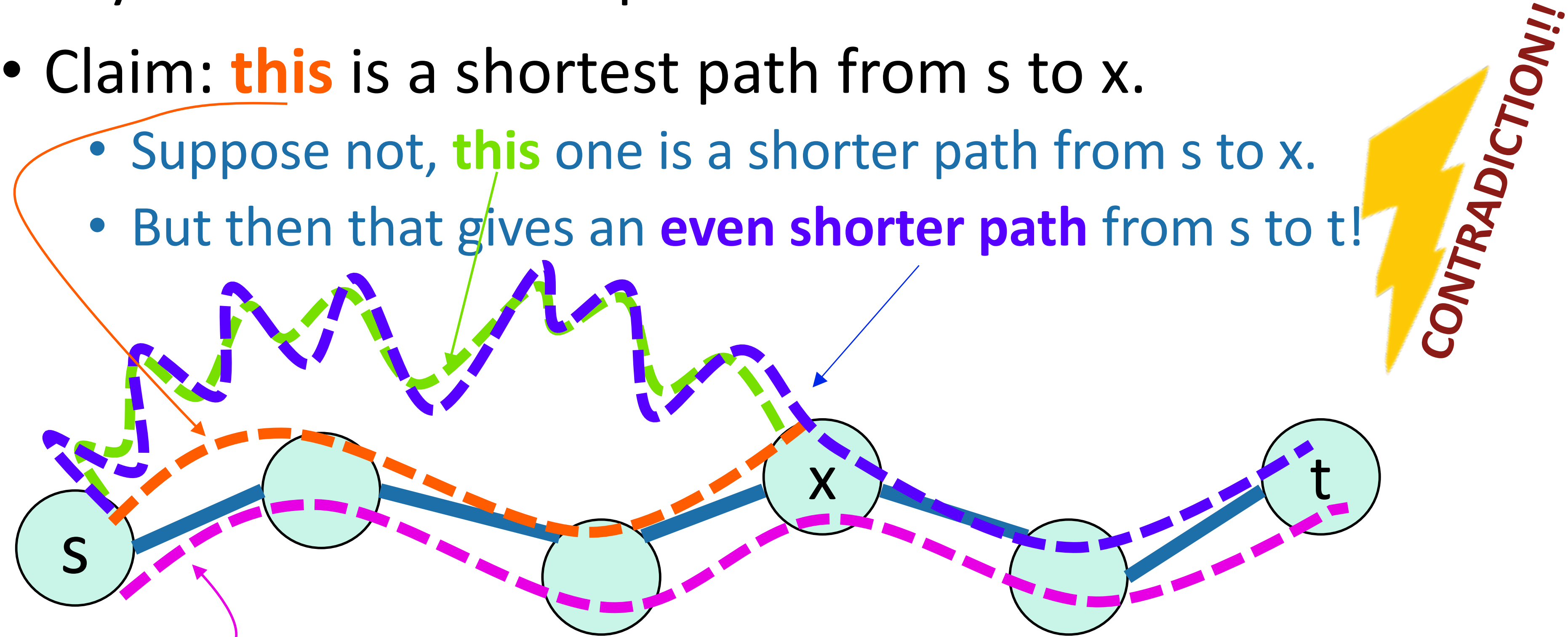
It has cost 6.

Q: What's the shortest path from **Packard** to the Union?

Warm-up

- A sub-path of a shortest path is also a shortest path.

- Say **this** is a shortest path from s to t.
- Claim: **this** is a shortest path from s to x.
  - Suppose not, **this** one is a shorter path from s to x.
  - But then that gives an **even shorter path** from s to t!



Single-source shortest-path problem

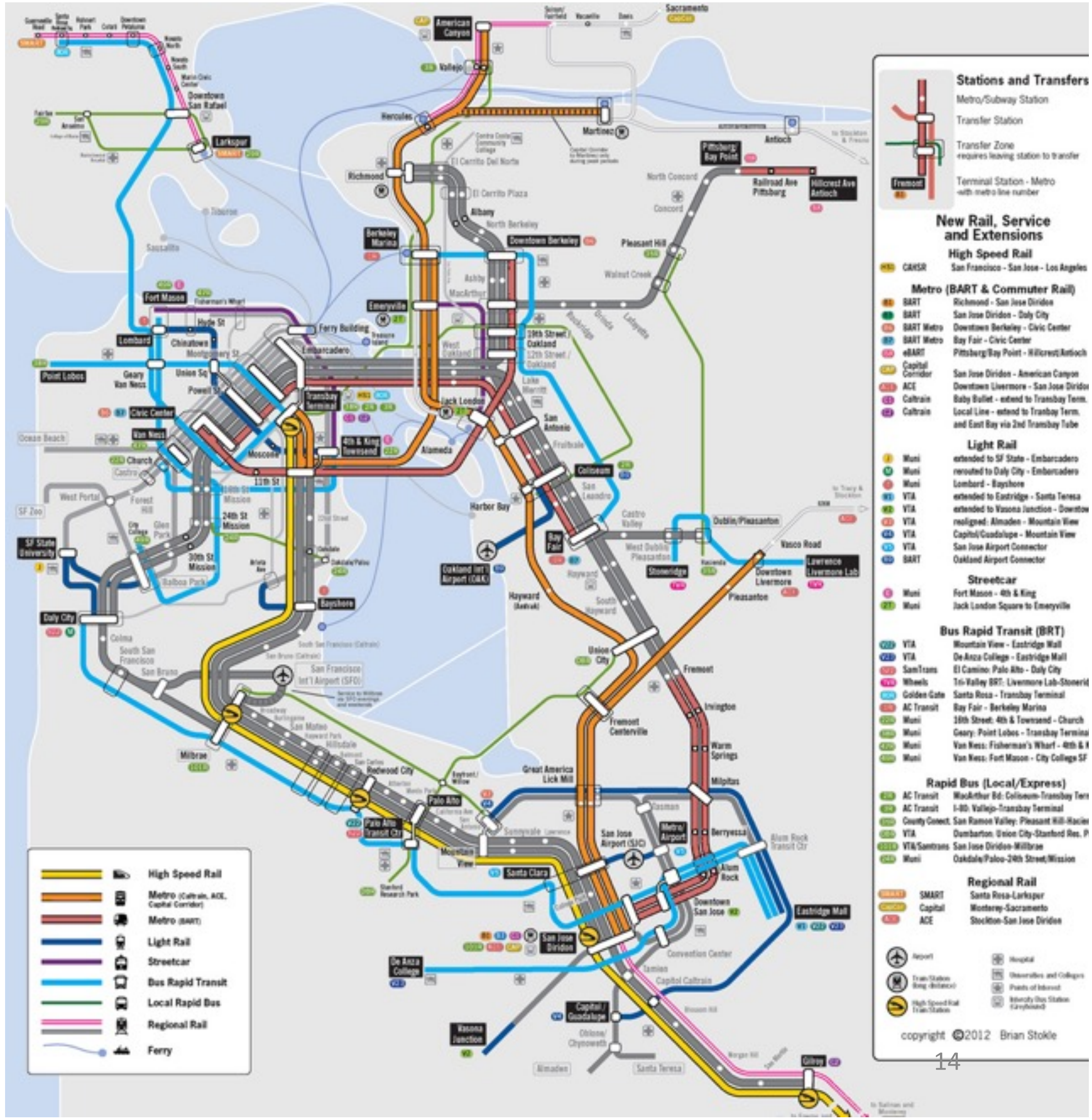
- I want to know the shortest path from one vertex (Gates) to all other vertices.

Destination	Cost	To get there
Packard	1	Packard
CS161	2	Packard-CS161
Hospital	10	Hospital
Caltrain	17	Caltrain
Union	6	Packard-CS161-Union
Stadium	10	Stadium
Dish	23	Packard-Dish

(Not necessarily stored as a table – how this information is represented will depend on the application)

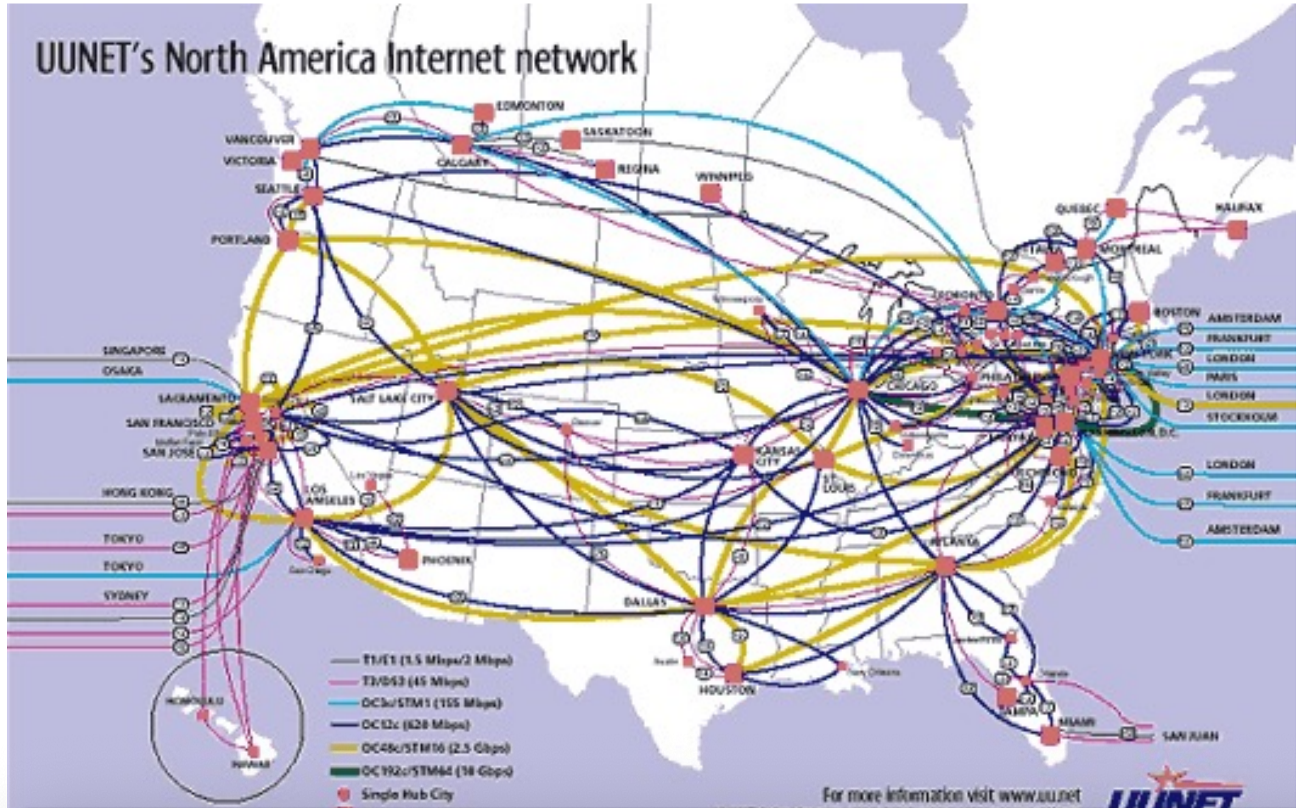
Example

- **"what is the shortest path from Palo Alto to [anywhere else]"** using BART, Caltrain, lightrail, MUNI, bus, Amtrak, bike, walking, uber/lyft.
- Edge weights have something to do with time, money, hassle.



Example

- **Network routing**
- I send information over the internet, from my computer to to all over the world.
- Each path has a cost which depends on link length, traffic, other costs, etc..
- How should we send packets?



moses — traceroute -a www.ethz.ch — 103×19

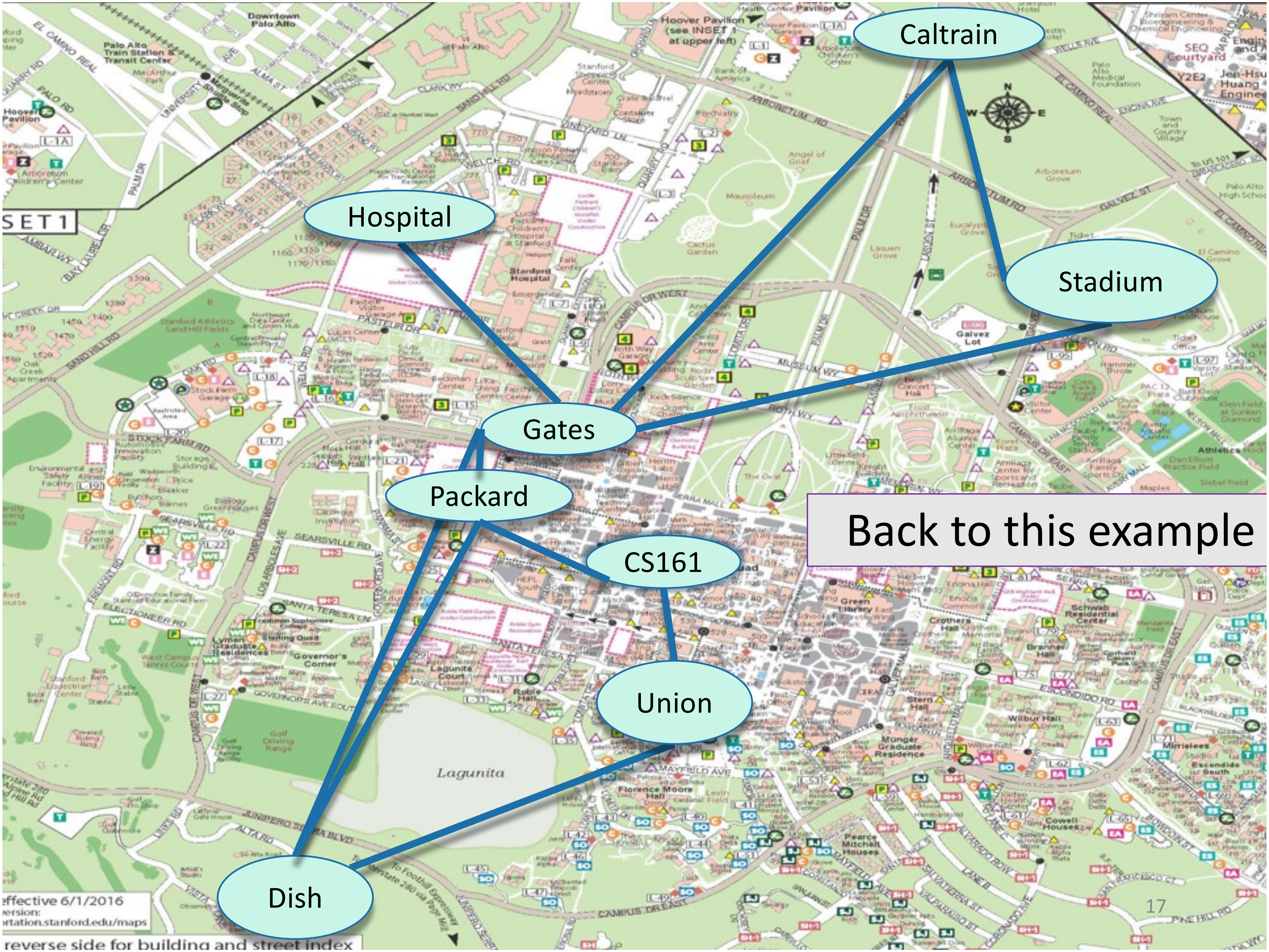
```
Last login: Mon Feb  7 09:27:47 on ttys003
moses@Mosess-MacBook-Pro ~ % traceroute -a www.ethz.ch
traceroute to www.ethz.ch (129.132.19.216), 64 hops max, 52 byte packets
 1  [AS0] 192.168.7.1 (192.168.7.1)  3.898 ms  2.066 ms  2.881 ms
 2  [AS0] 192.168.0.1 (192.168.0.1)  2.897 ms  4.720 ms  3.108 ms
 3  [AS0] 10.127.252.2 (10.127.252.2)  57.256 ms  5.571 ms  4.268 ms
 4  [AS32] he-rtr.stanford.edu (128.12.0.209)  4.039 ms  11.471 ms  4.628 ms
 5  [AS6939] 100gigabitethernet5-1.core1.pao1.he.net (184.105.177.237)  4.648 ms  3.
 6  [AS6939] 100ge9-2.core1.sjc2.he.net (72.52.92.157)  5.949 ms  5.291 ms  4.980 ms
 7  [AS6939] 100ge10-2.core1.nyc4.he.net (184.105.81.217)  69.007 ms  66.575 ms  67.
 8  [AS6939] 100ge7-1.core1.lon2.he.net (72.52.92.165)  268.329 ms  191.401 ms  203.
 9  [AS6939] port-channel2.core3.lon2.he.net (184.105.64.2)  205.515 ms  350.183 ms
10  [AS6939] port-channel12.core2.ams1.he.net (72.52.92.214)  144.263 ms  143.638 ms
11  [AS1200] swice1-100ge-0-3-0-1.switch.ch (80.249.208.33)  161.119 ms  208.169 ms
12  [AS559] swice4-b4.switch.ch (130.59.36.70)  219.228 ms  203.833 ms  204.402 ms
13  [AS559] swibf1-b2.switch.ch (130.59.36.113)  184.671 ms  204.955 ms  204.671 ms
14  [AS559] swiez3-b5.switch.ch (130.59.37.6)  205.079 ms  164.116 ms  245.086 ms
15  [AS559] rou-gw-lee-tengig-to-switch.ethz.ch (192.33.92.1)  204.296 ms  164.770 m
16  [AS559] rou-fw-rz-rz-gw.ethz.ch (192.33.92.169)  165.148 ms  322.839 ms  204.627
```

Aside: These are difficult problems

- Costs may change
  - If it's raining the cost of biking is higher
  - If a link is congested, the cost of routing a packet along it is higher
- The network might not be known
  - My computer doesn't store a map of the internet
- We want to do these tasks really quickly
  - I have time to bike to Berkeley, but not to think about whether I should bike to Berkeley…
  - More seriously, **the internet.**

This is a joke.

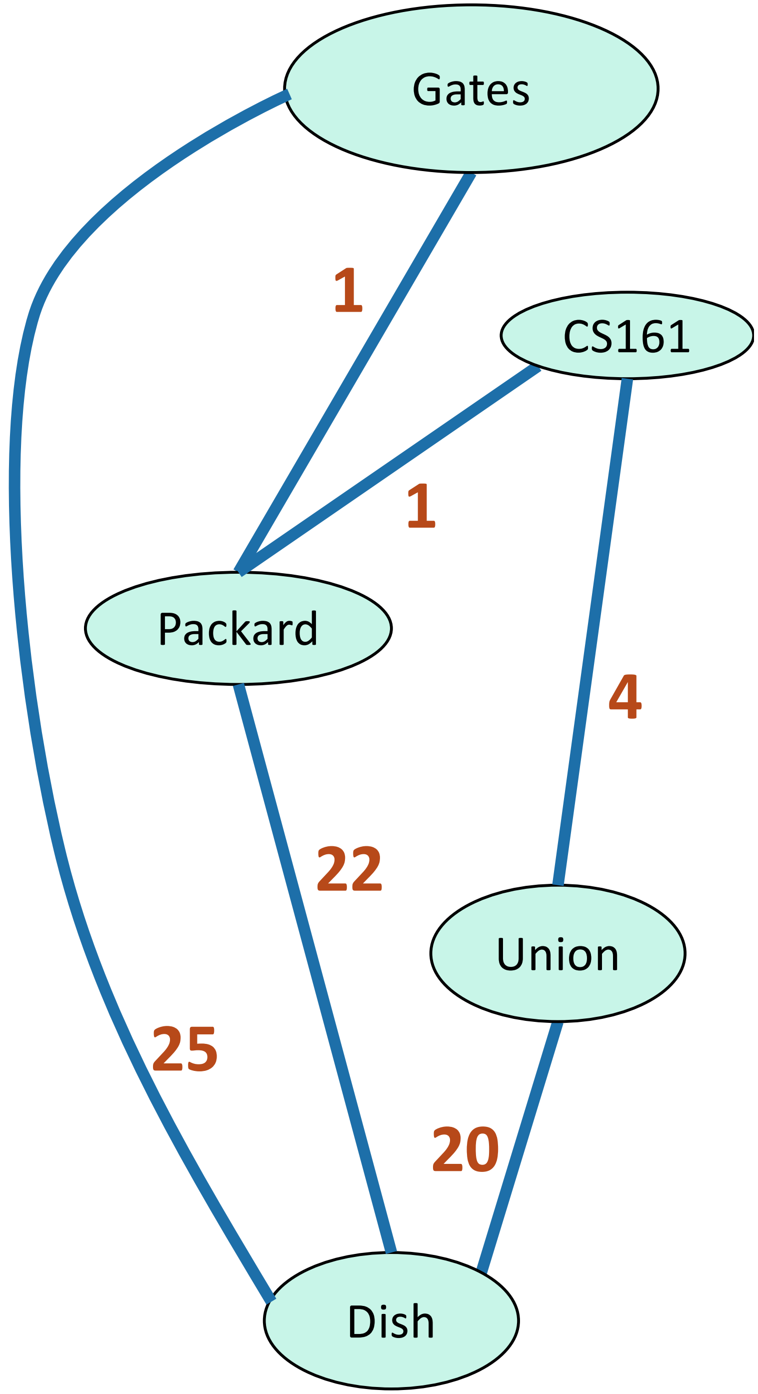
But let's ignore them for now.



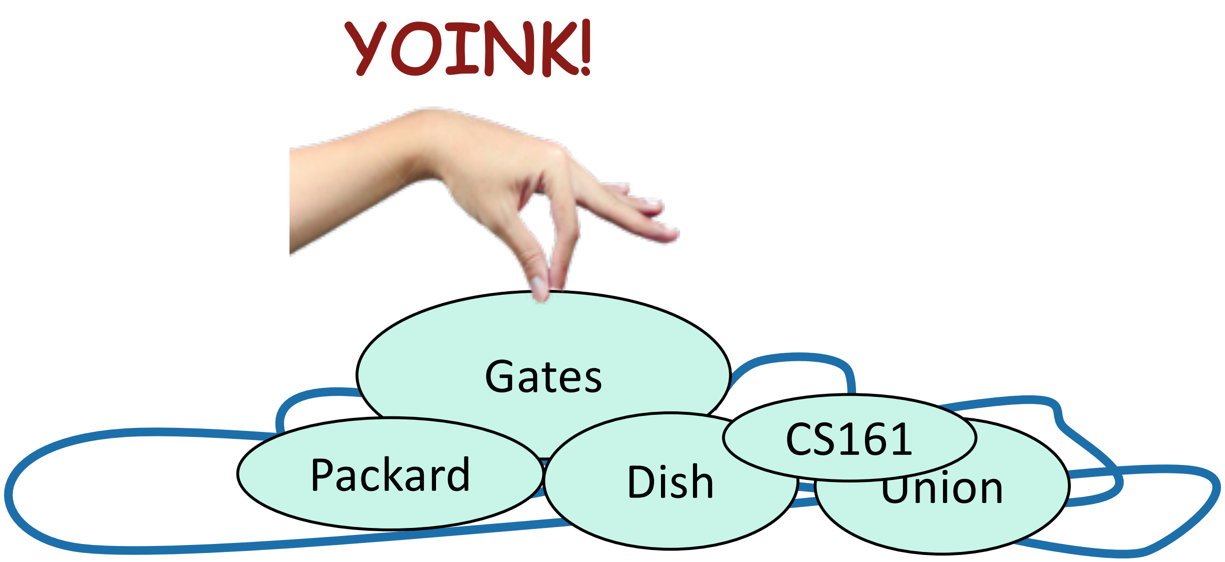
Back to this example

Dijkstra's algorithm

- Finds shortest paths from Gates to everywhere else.

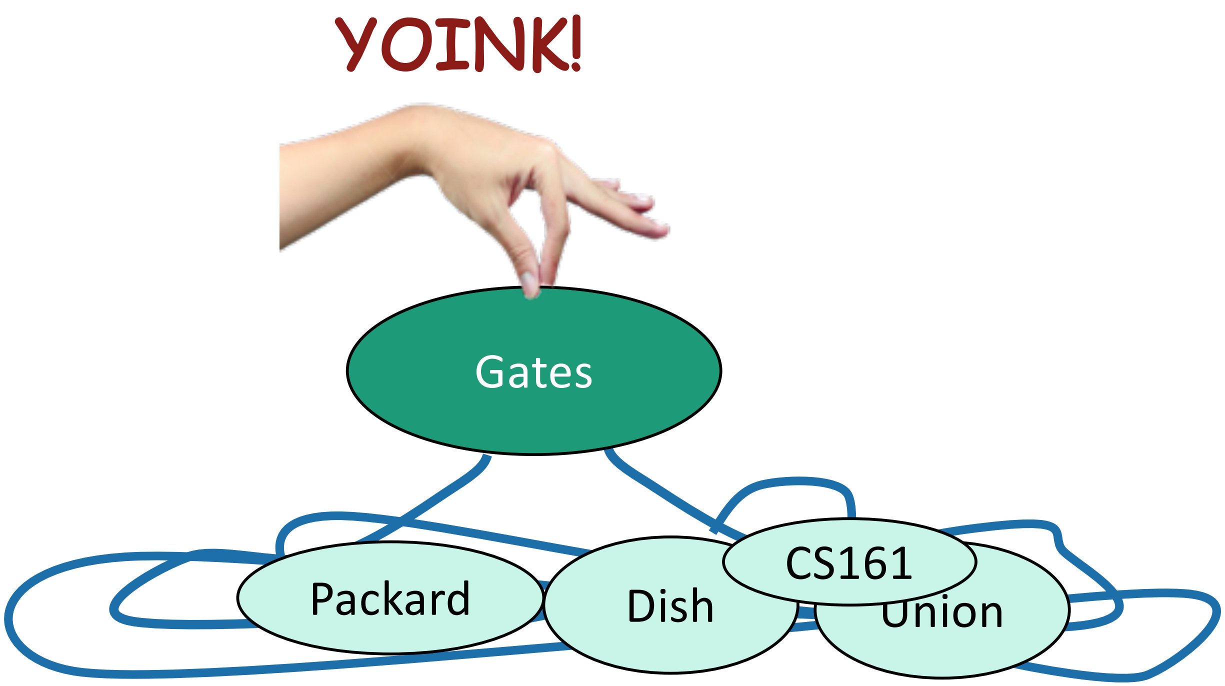


Dijkstra
intuition



Dijkstra
intuition

A vertex is done when it's not on the ground anymore.



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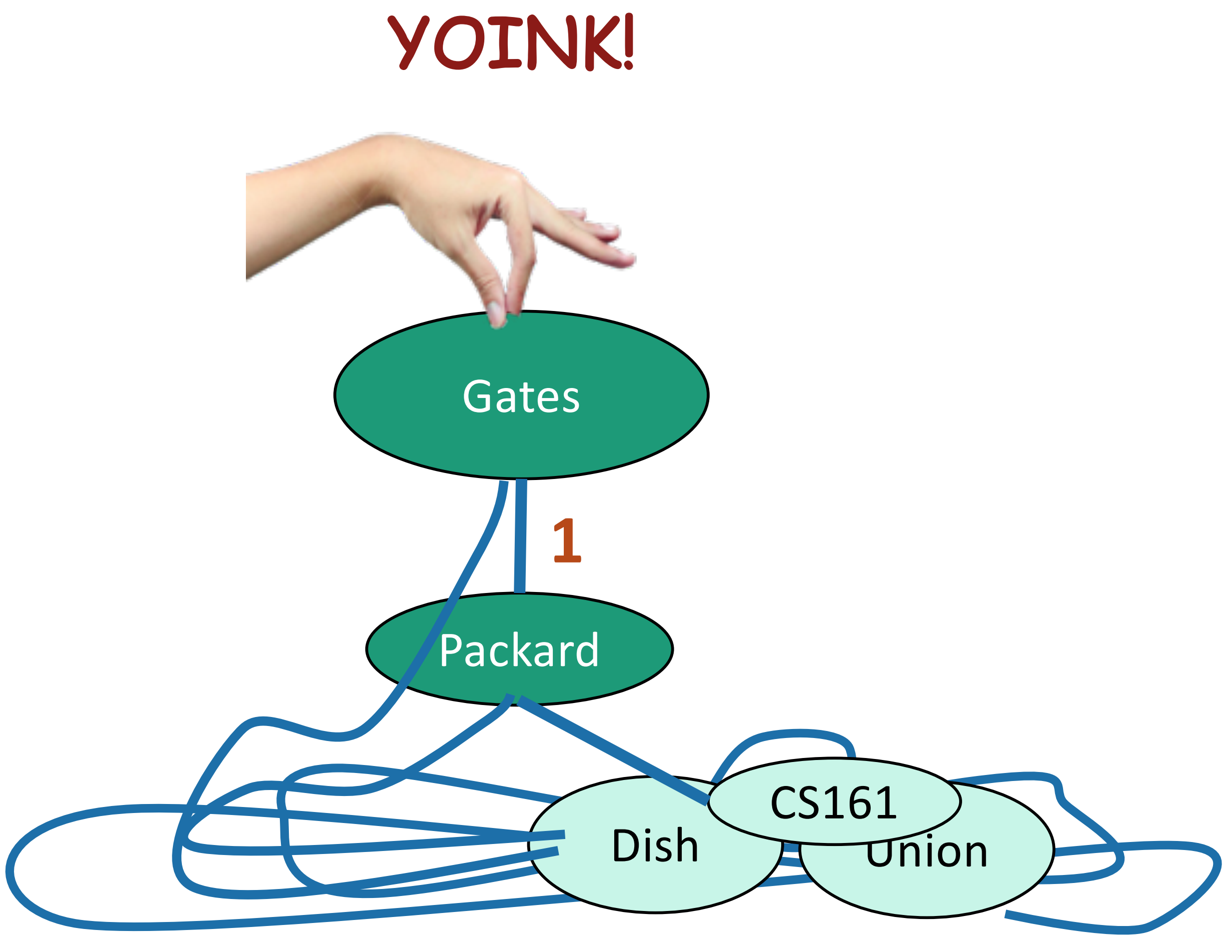

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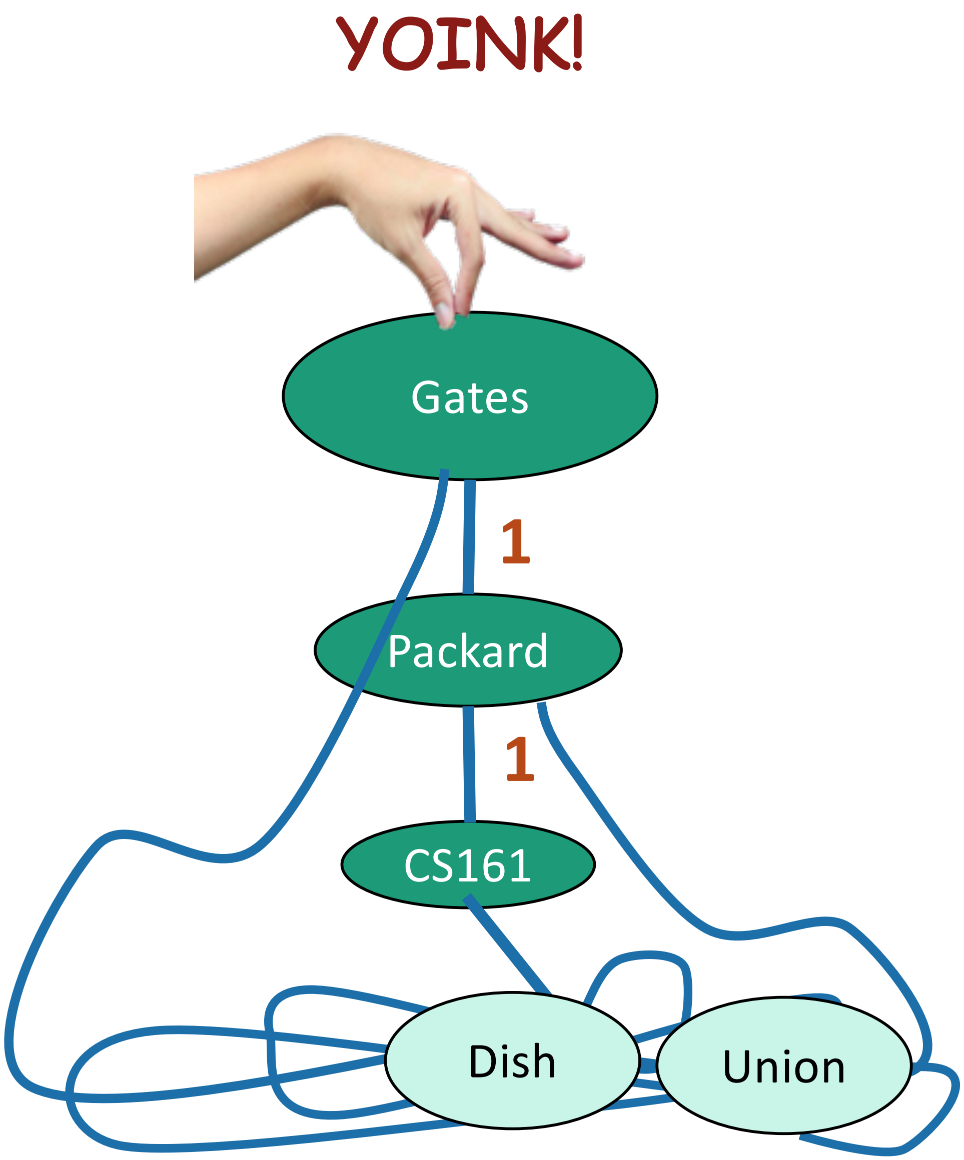

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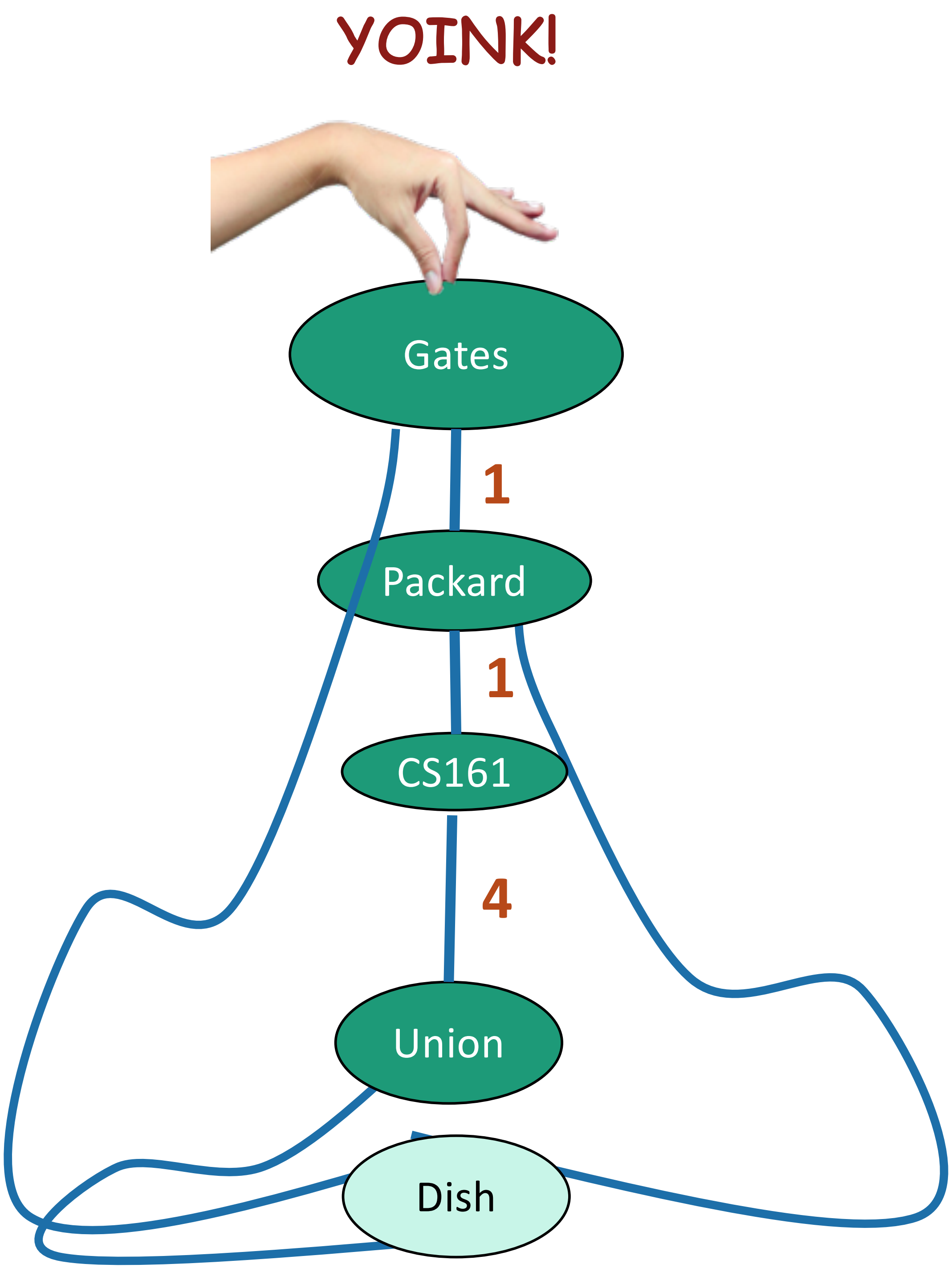

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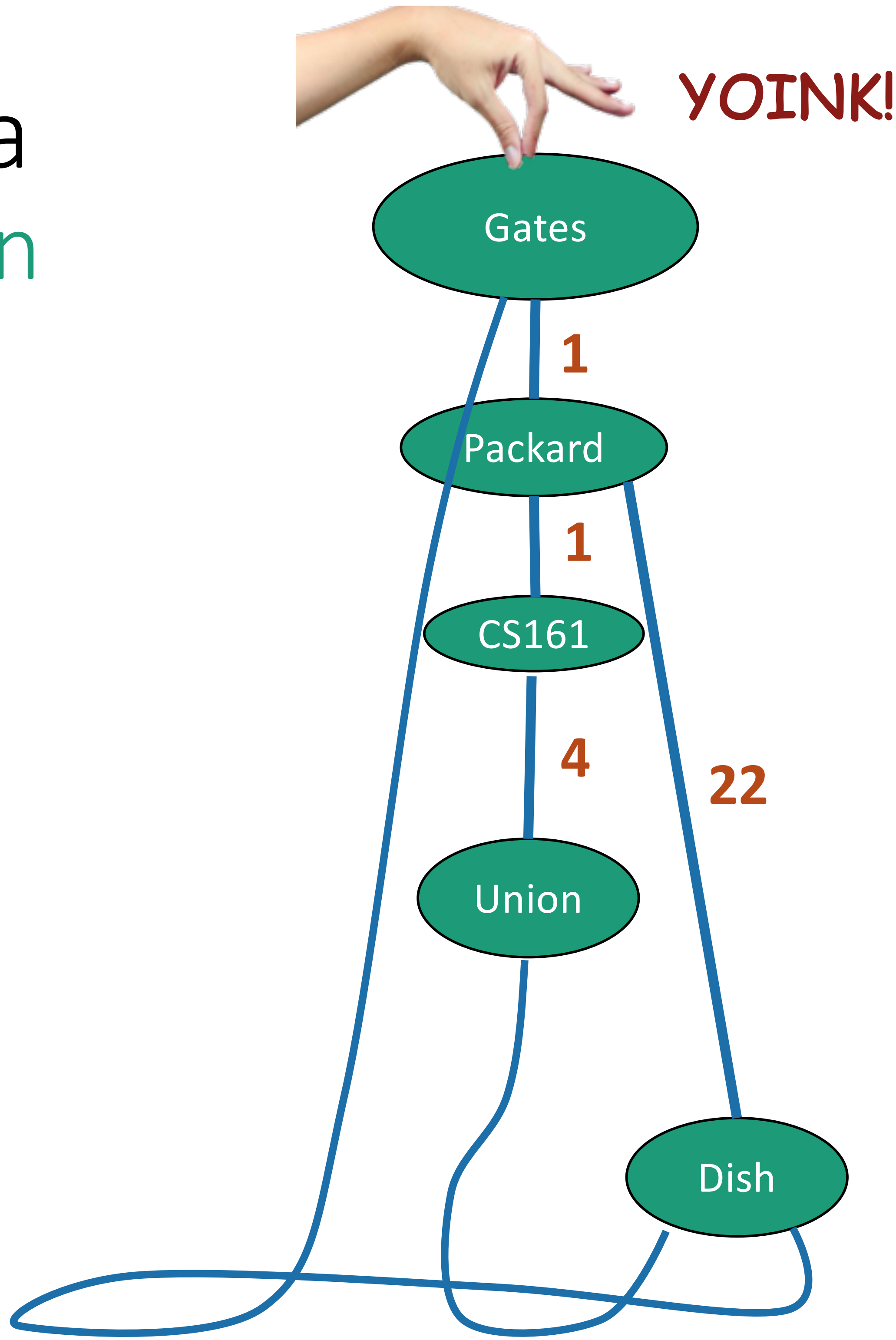

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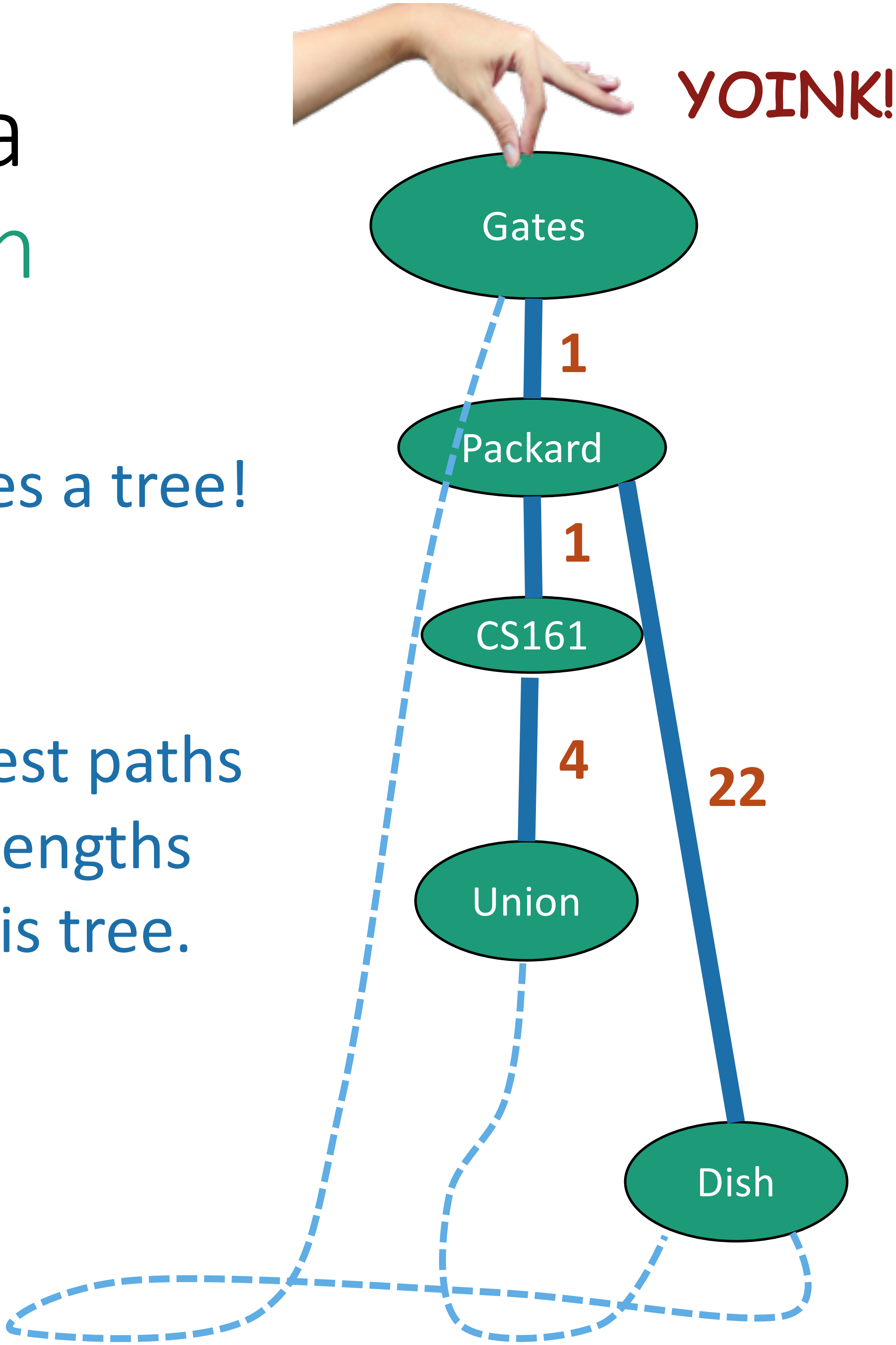
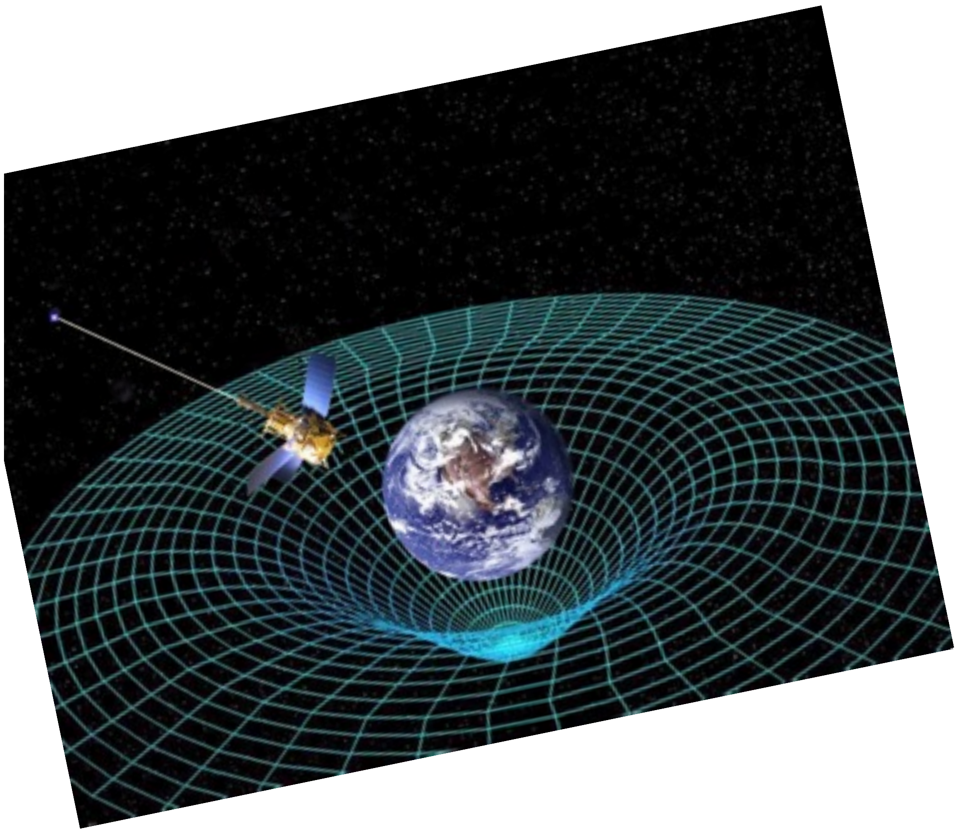


This creates a tree!

The shortest paths are the lengths along this tree.

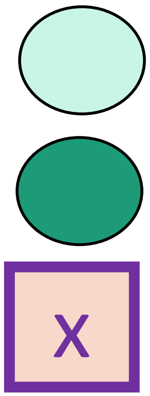
How do we actually implement this?

- **Without** string and gravity?



Dijkstra by example

How far is a node from Gates?



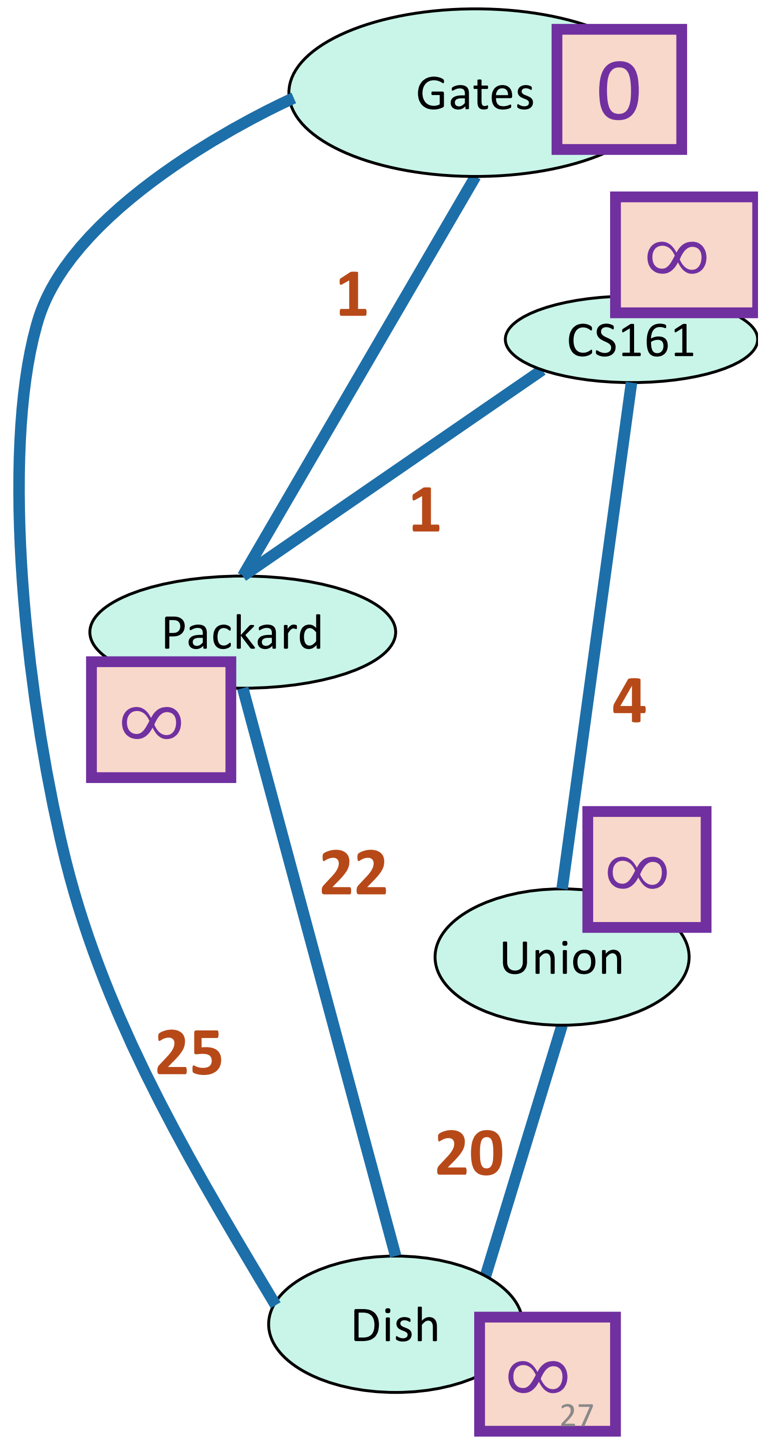
I'm not sure yet

I'm sure

x = d[v] is my best **over-estimate** for dist(Gates,v).

Initialize d[v] = ∞
for all non-starting vertices v,
and d[Gates] = 0

- Pick the **not-sure** node u with the smallest estimate **d[u].**



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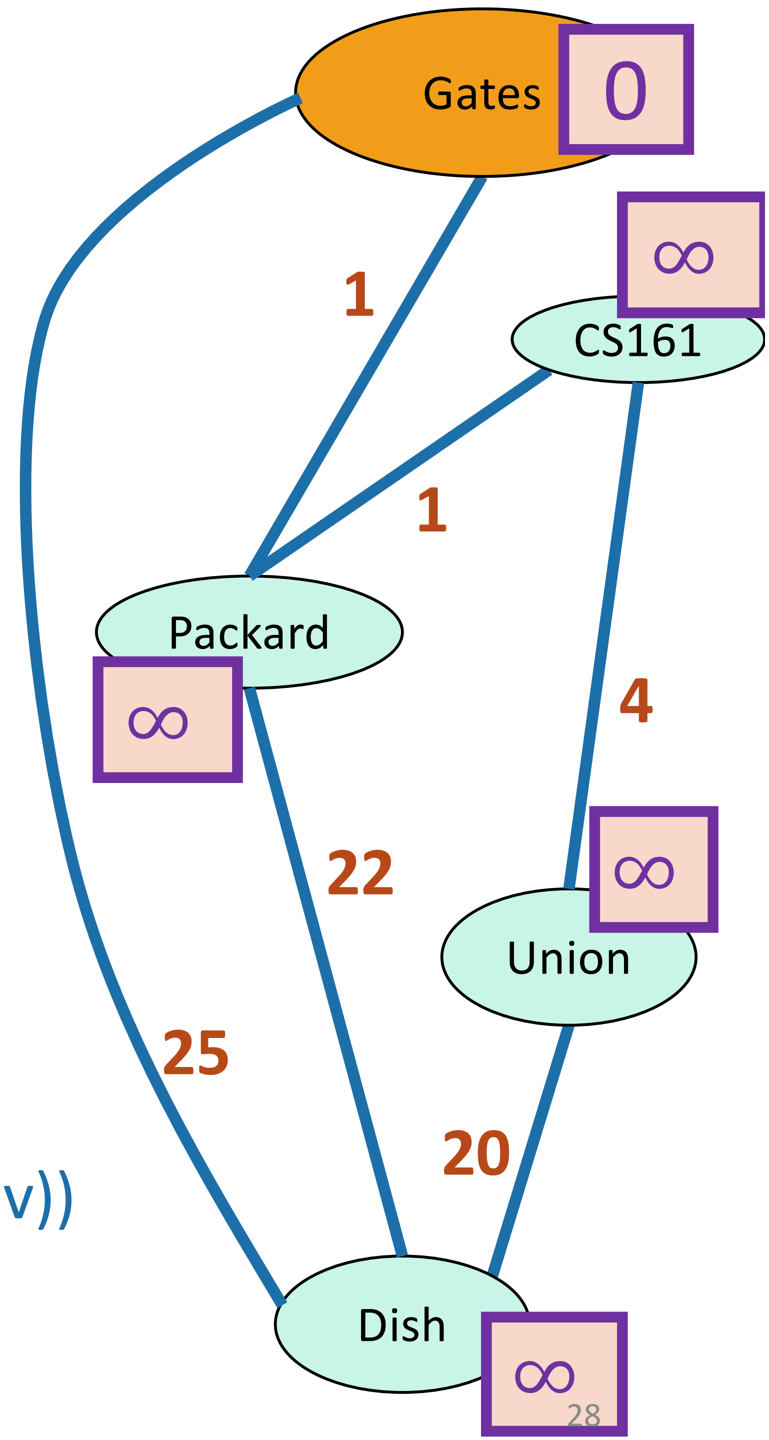

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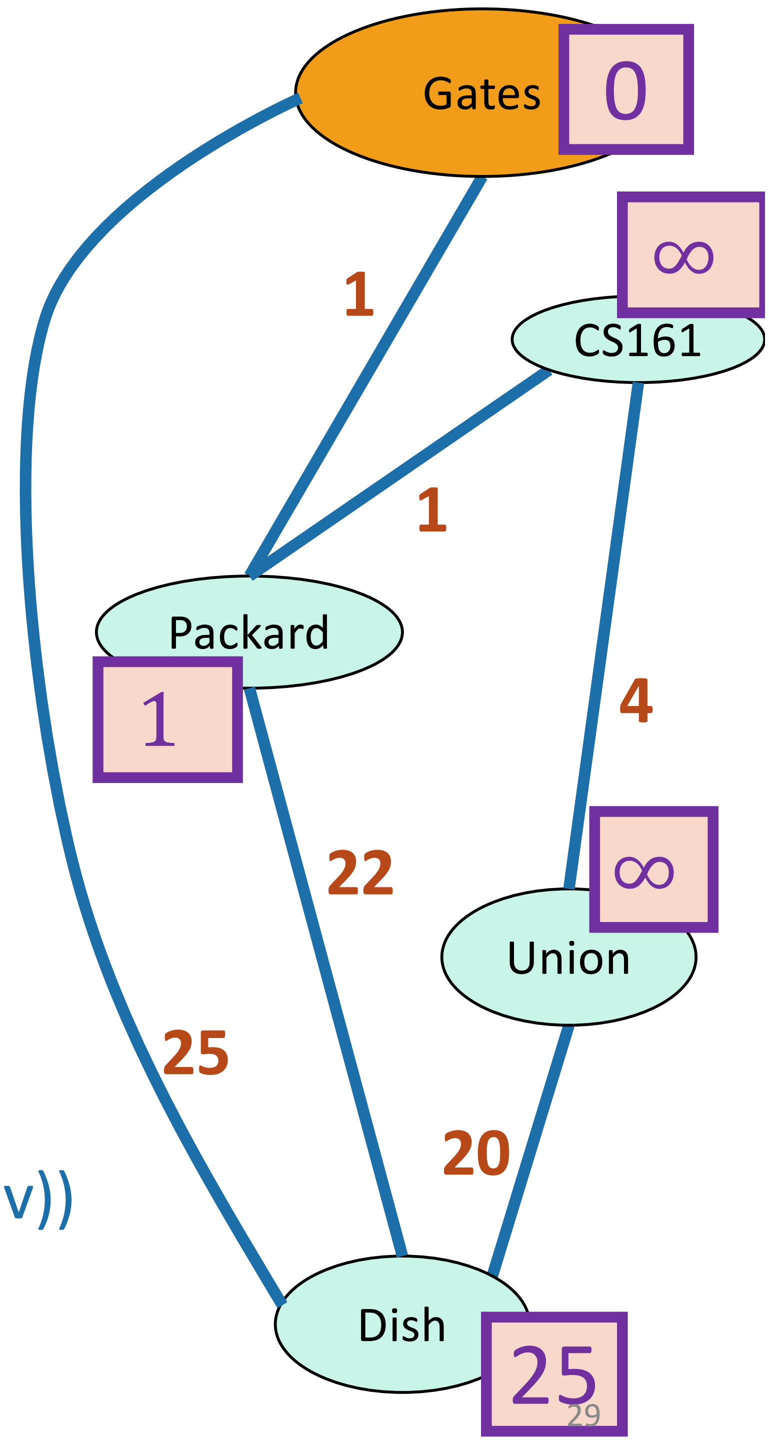

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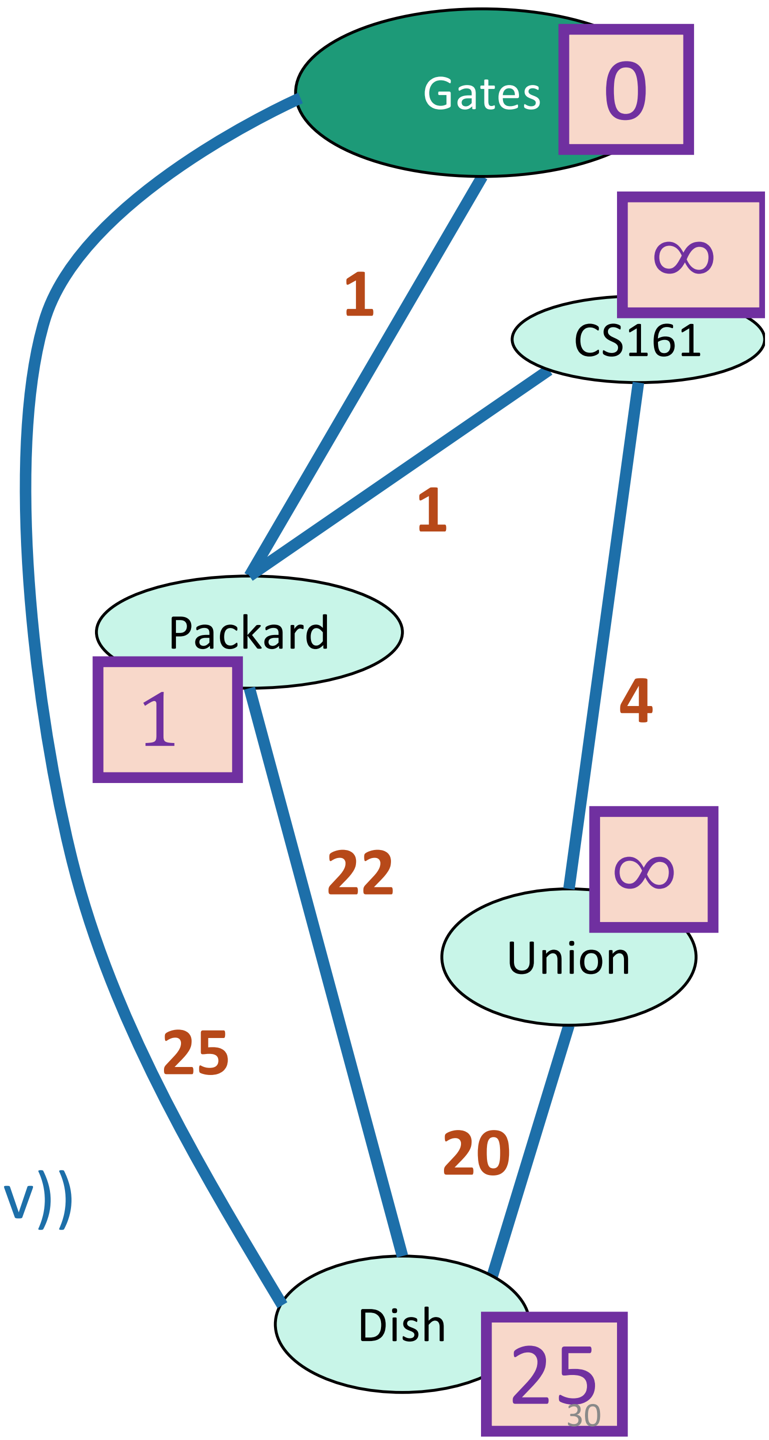

Dijkstra by example

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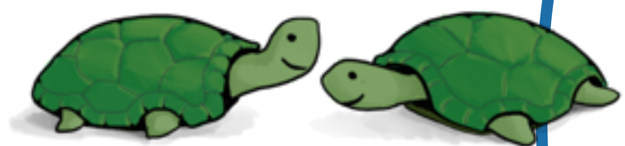
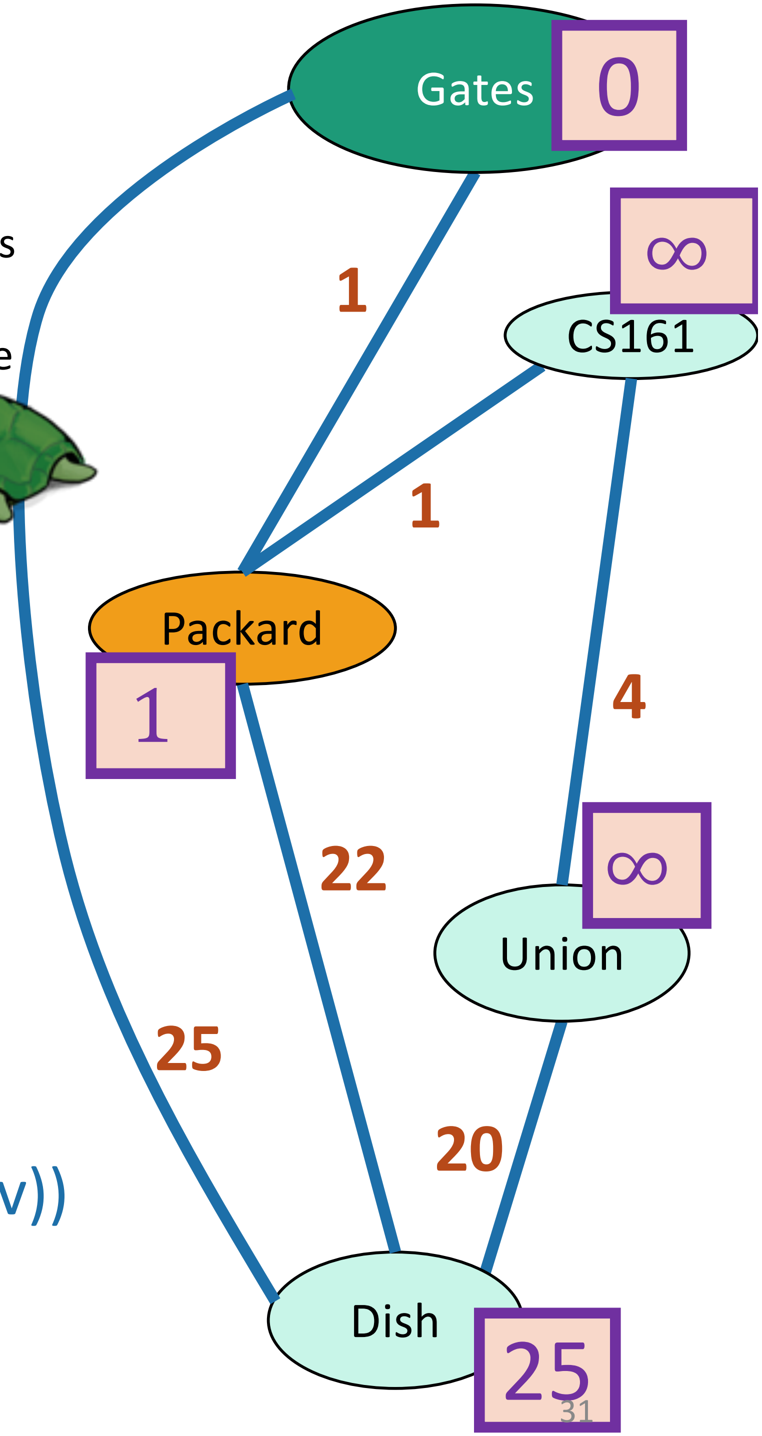
Current node u

Packard has three neighbors. What happens when we update them?
1 min. think; 1 min. share





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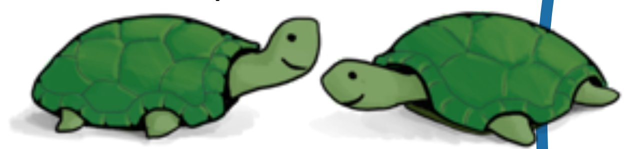

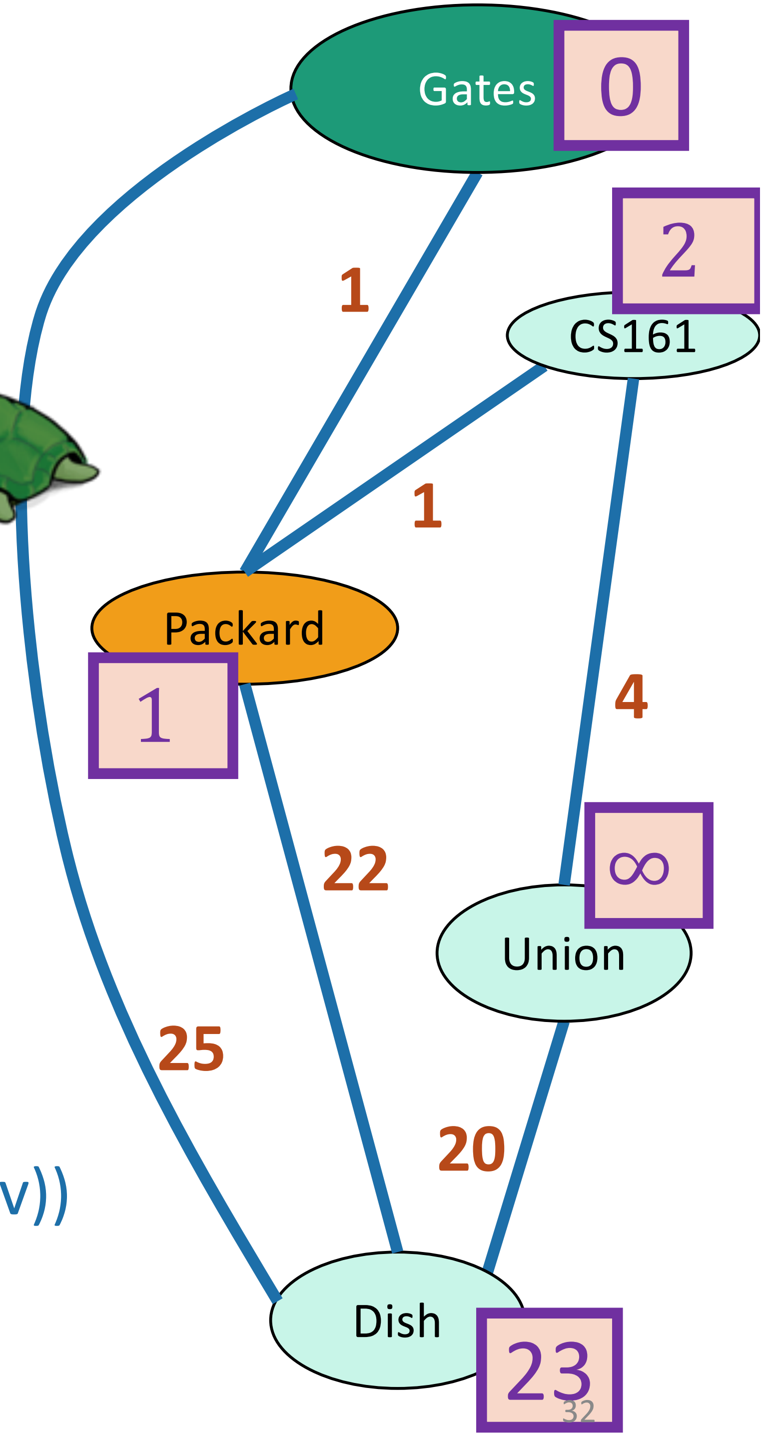

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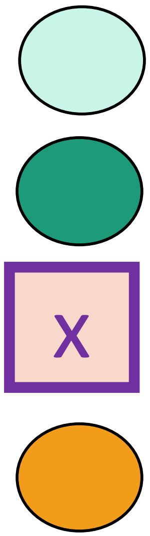

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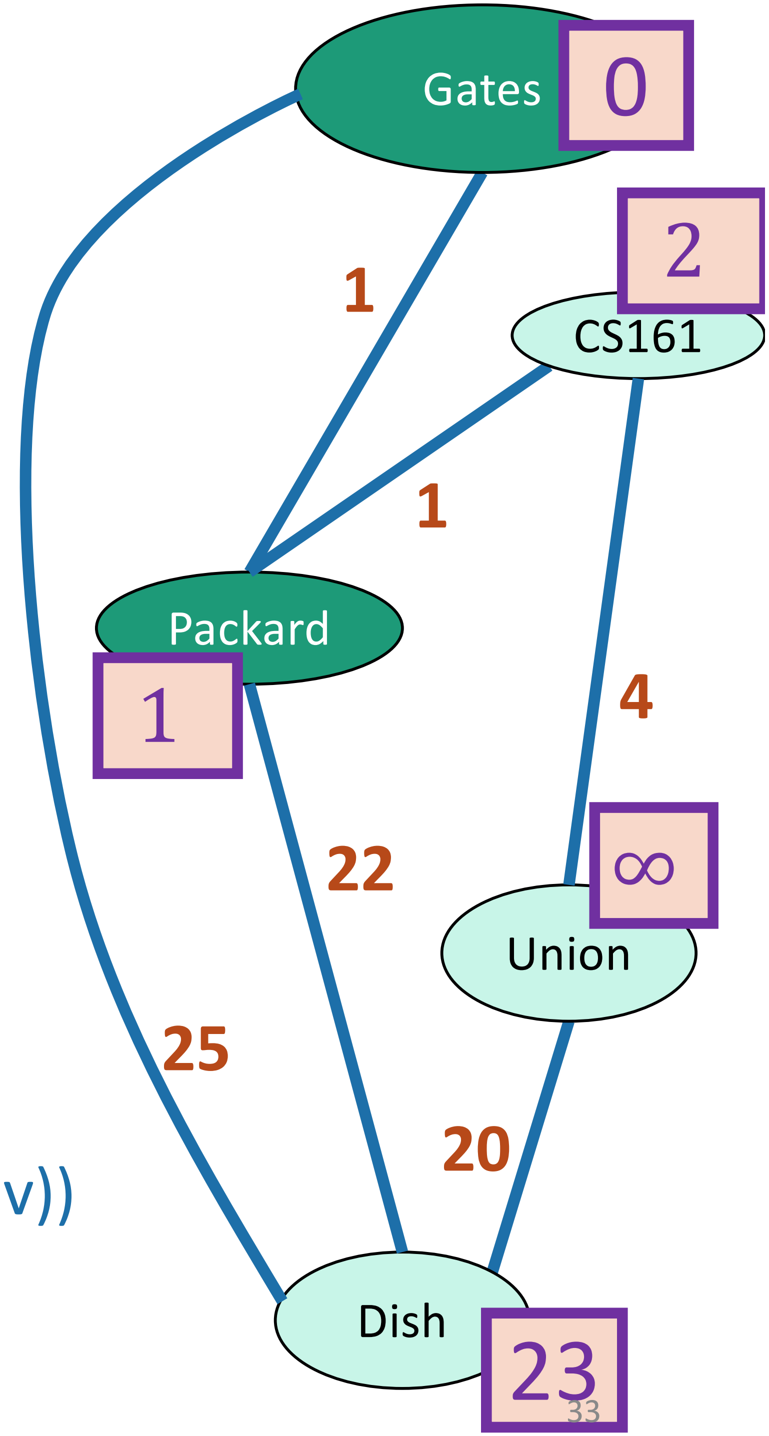

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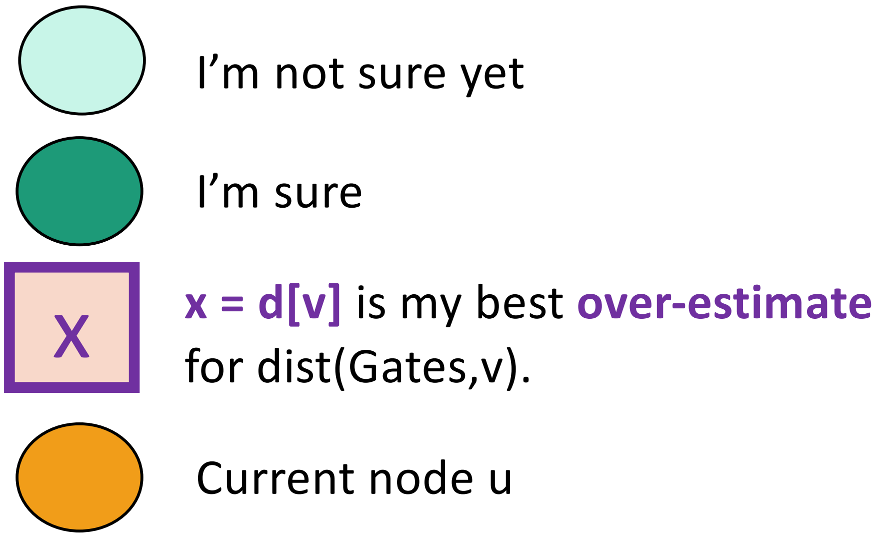

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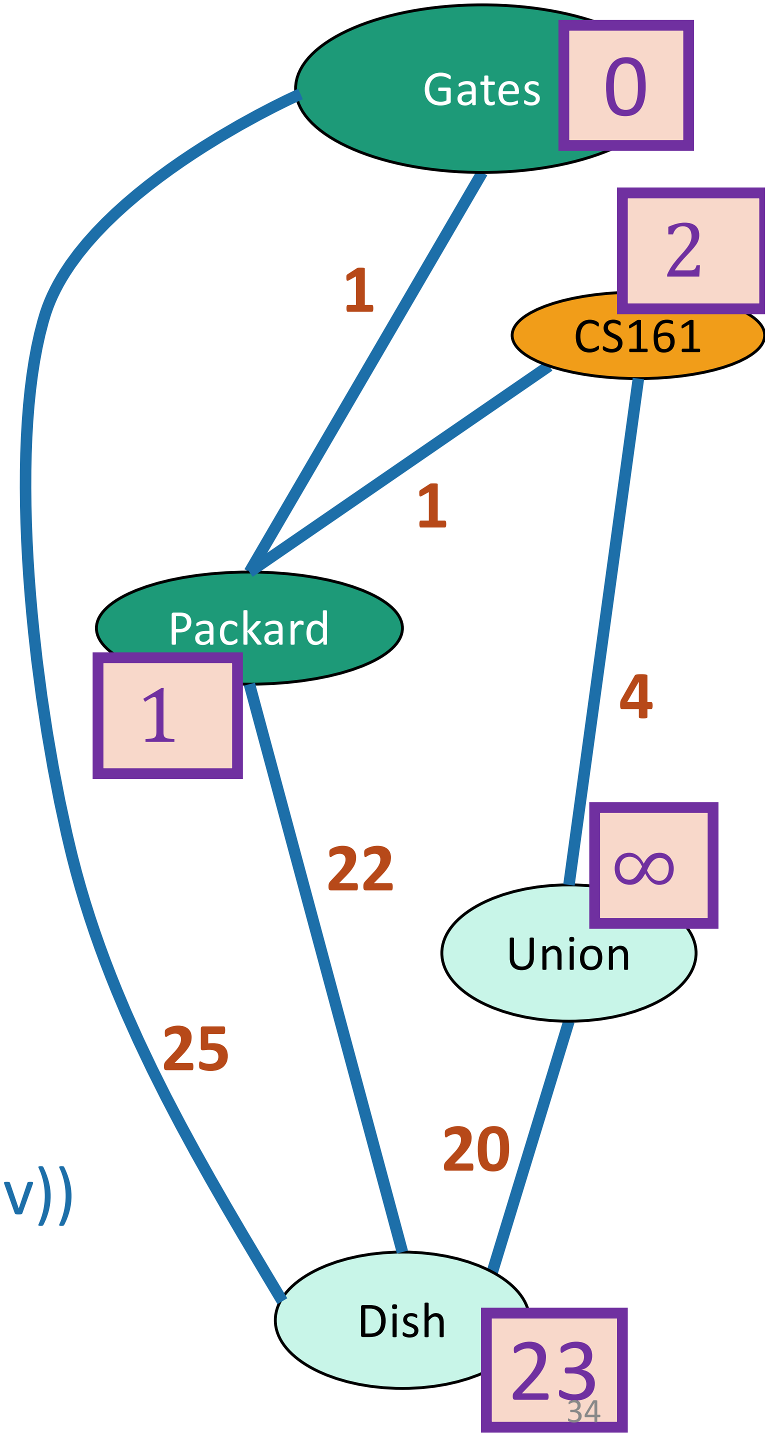

Dijkstra by example

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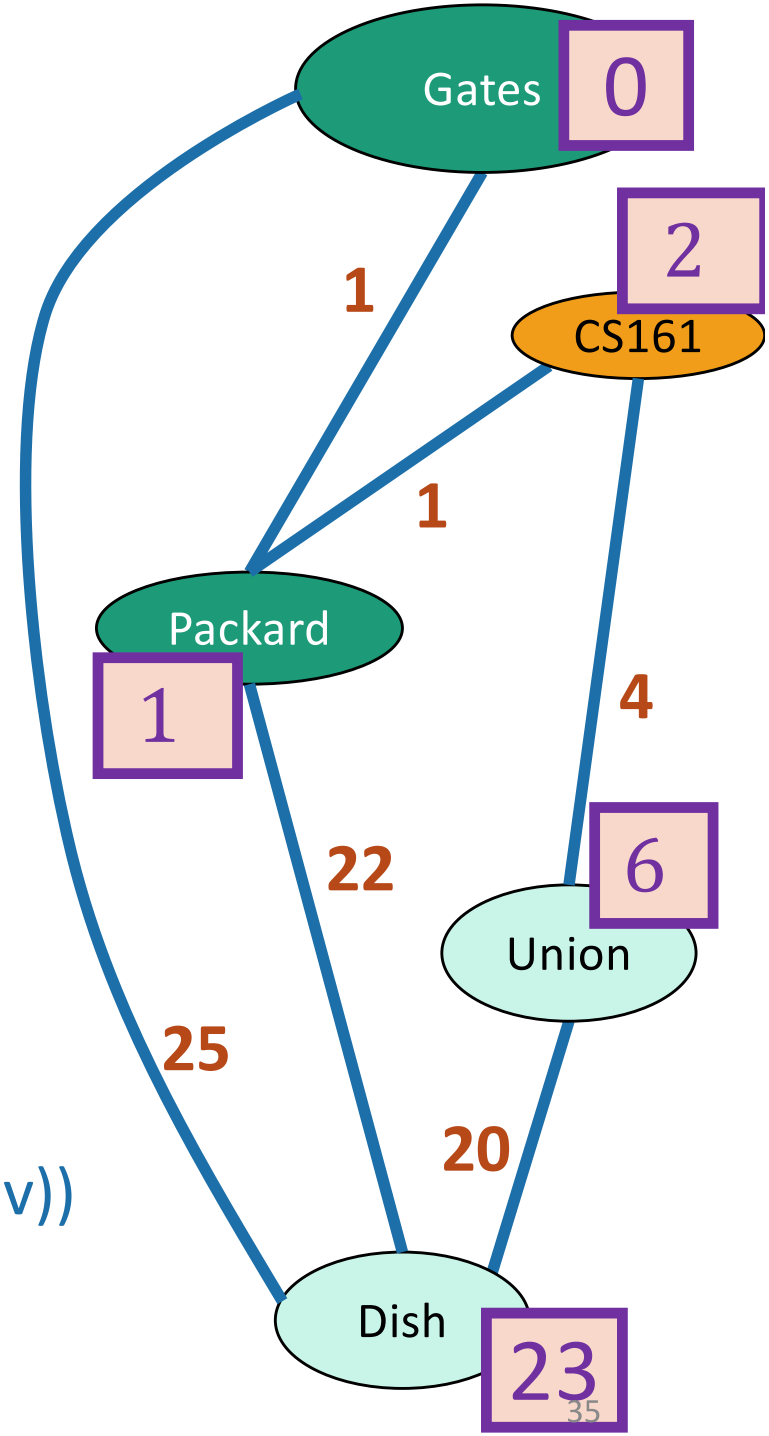

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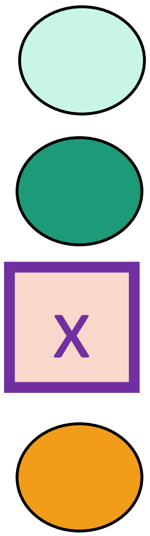

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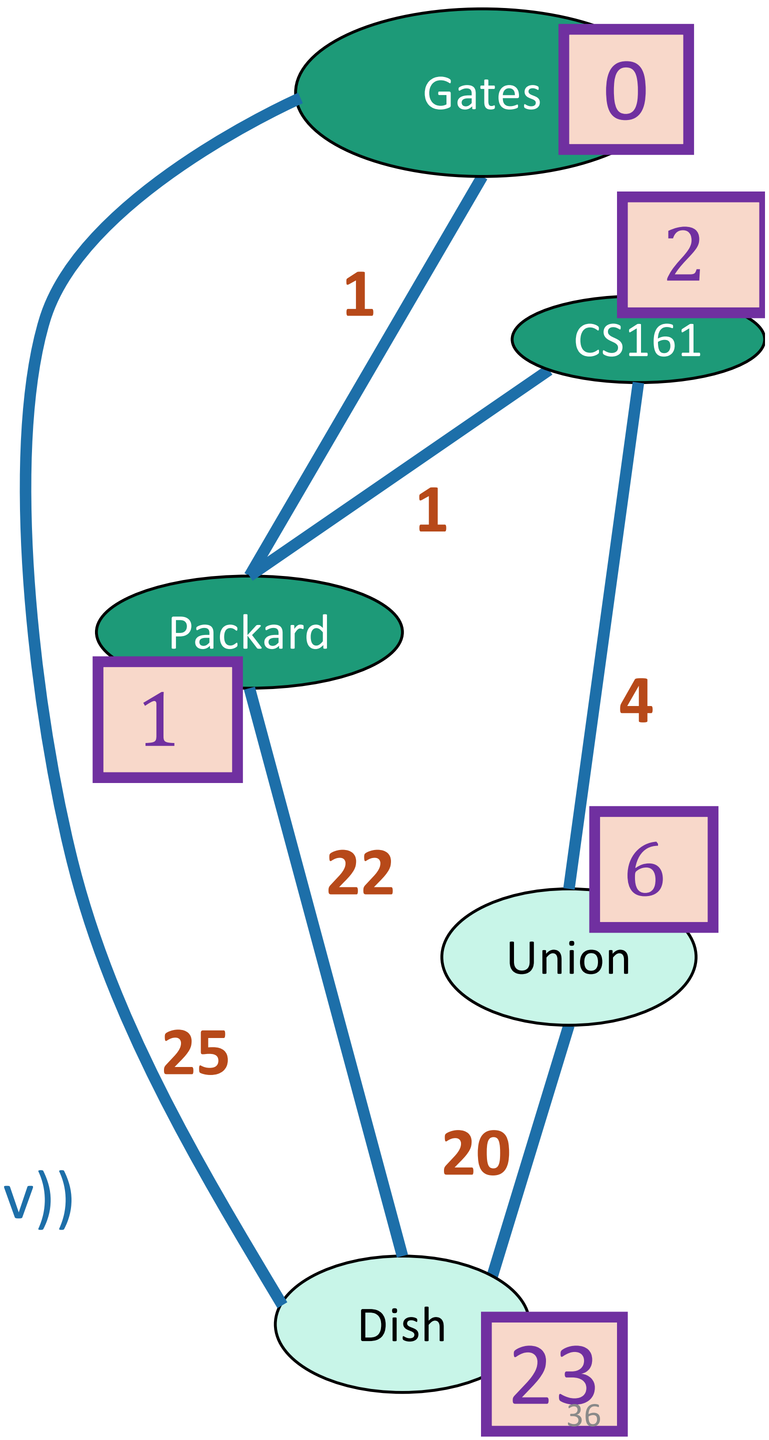

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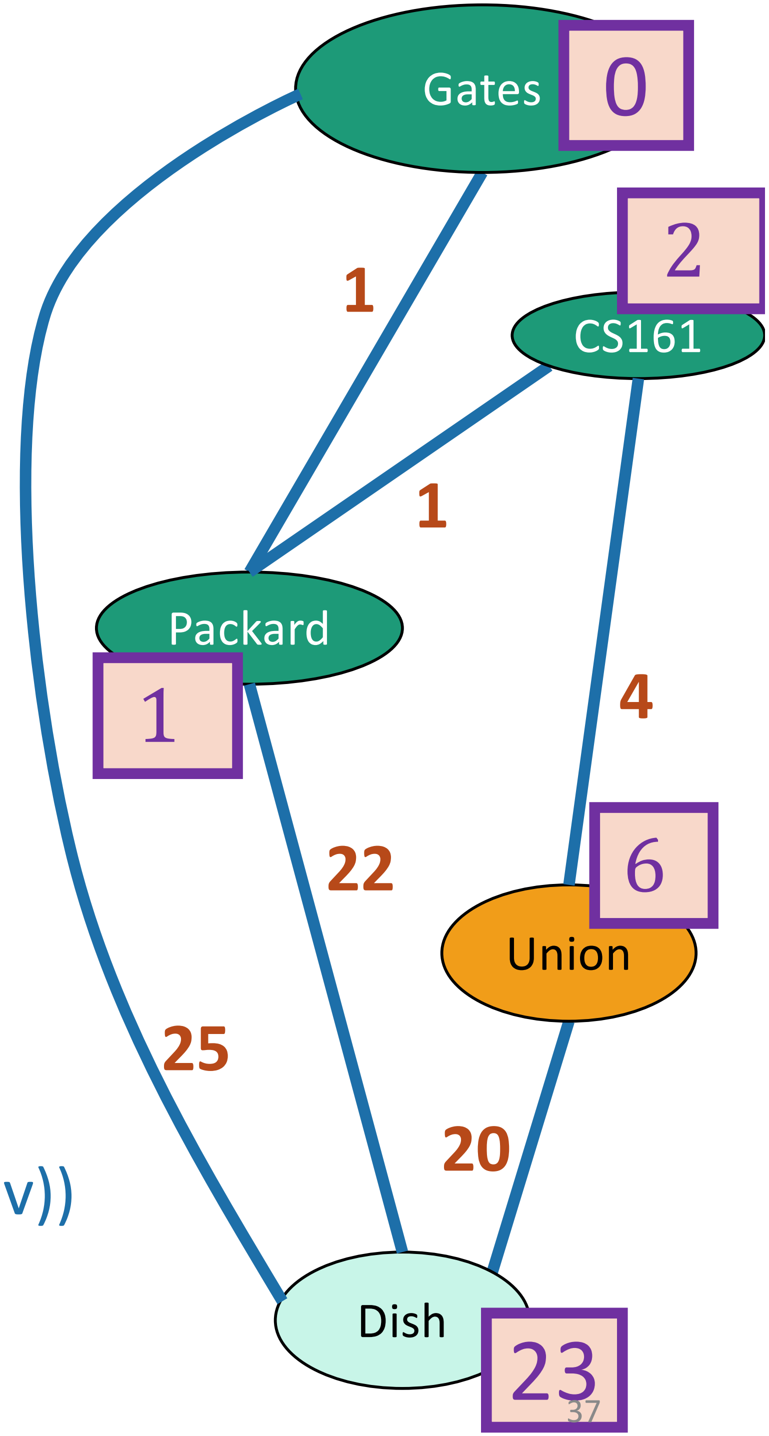

Dijkstra by example

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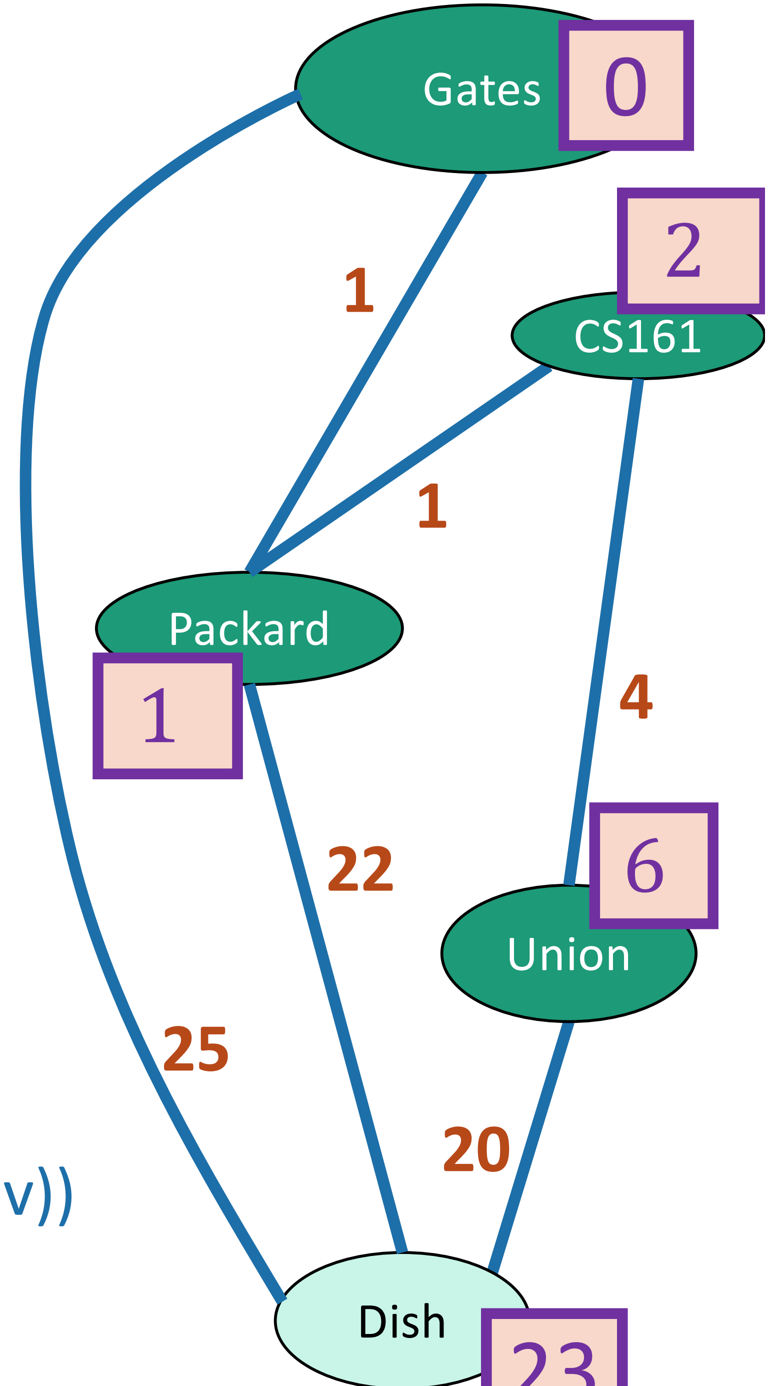

Dijkstra by example

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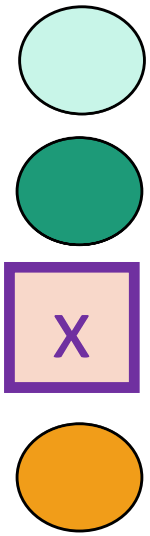

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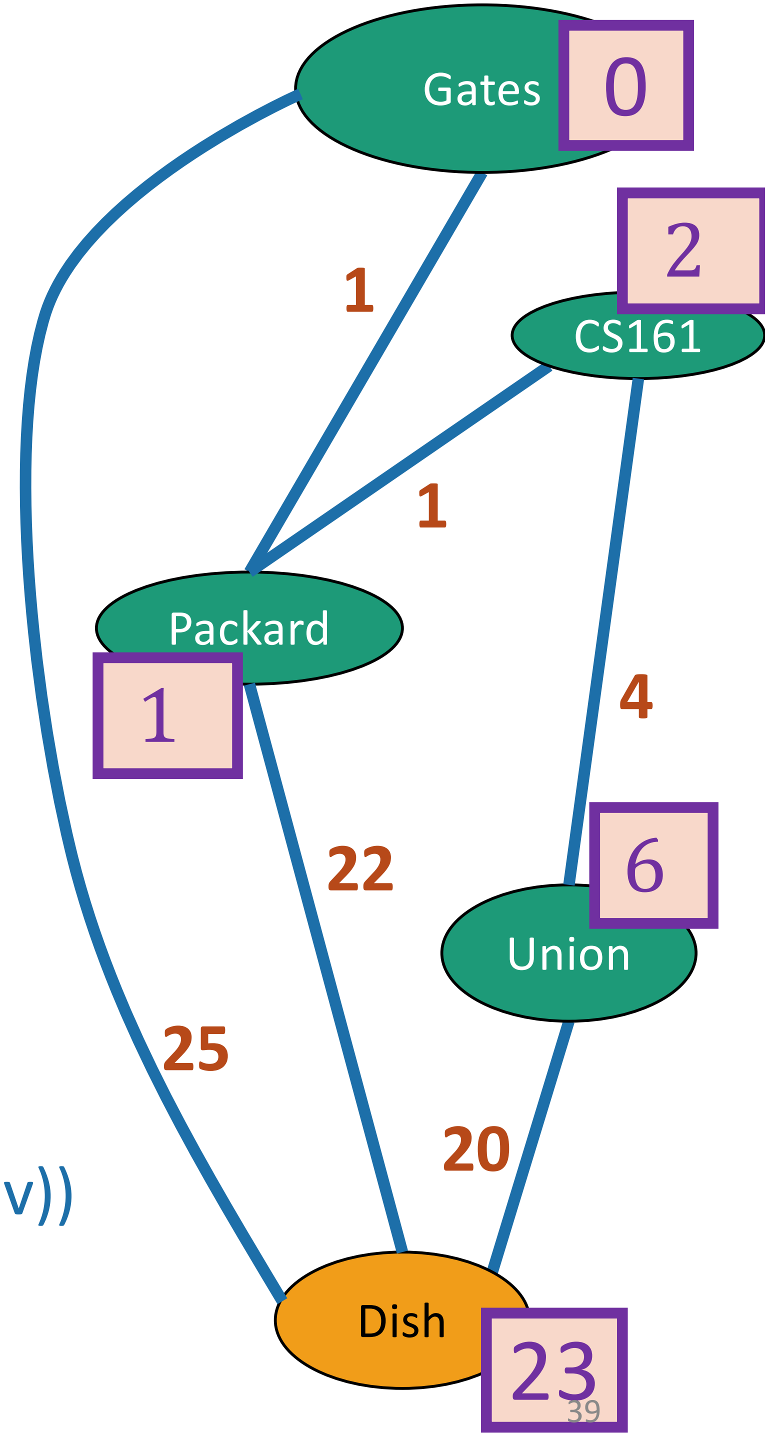

Dijkstra by example

How far is a node from Gates?

- I’m not sure yet
- I’m sure
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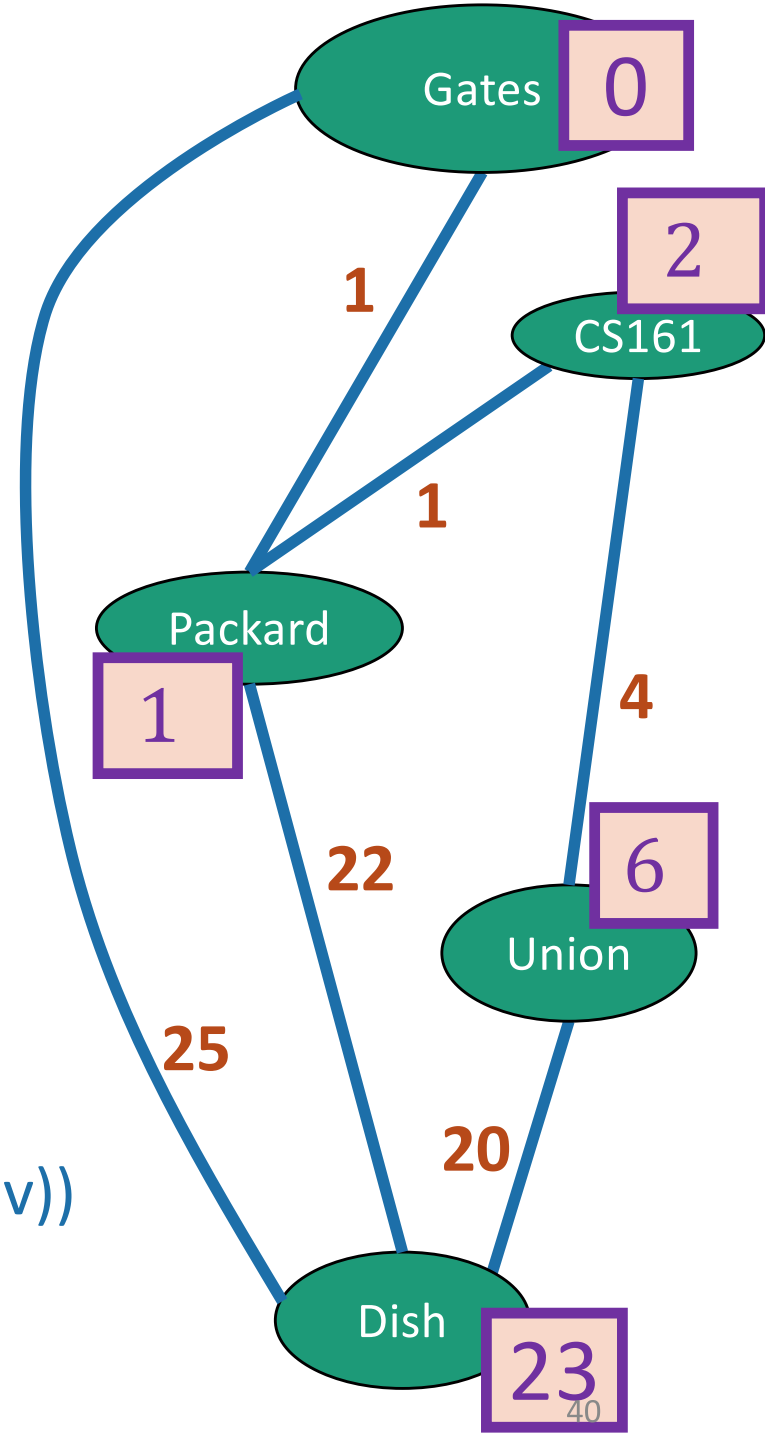
- Current node u



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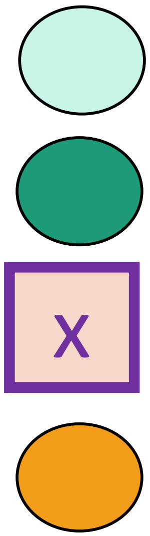


I'm not sure yet

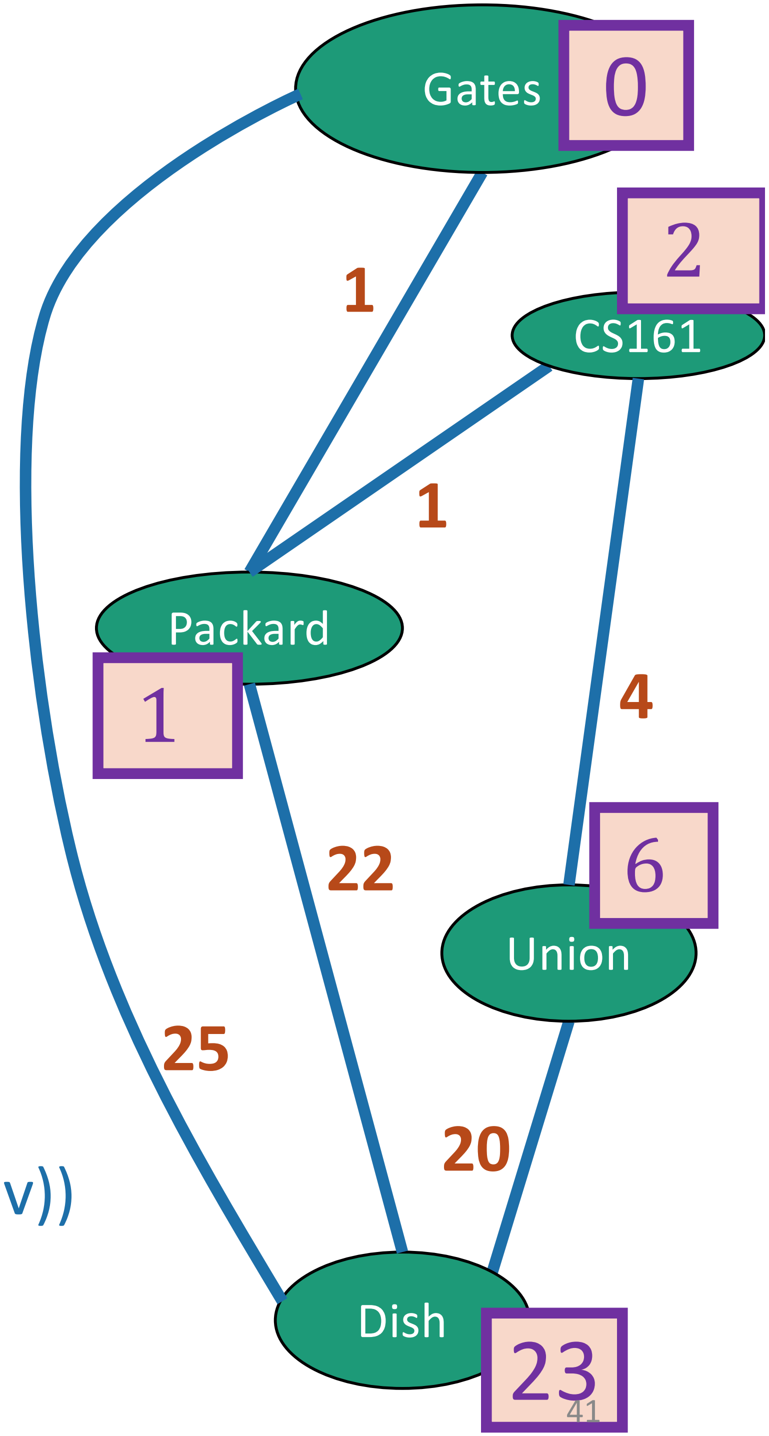
I'm sure

x = d[v] is my best **over-estimate** for dist(Gates,v).

Current node u

- Pick the **not-sure** node u with the smallest estimate **d[u].**
- Update all u's neighbors v:
  - d[v] = min(d[v] , d[u] + edgeWeight(u,v))
- Mark u as **sure**.
- Repeat

- After all nodes are **sure**, say that d(Gates, v) = d[v] for all v



Dijkstra's algorithm

Dijkstra(G,s):

- Set all vertices to **not-sure**
- d[v] = ∞ for all v in V
- d[s] = 0
- **While** there are **not-sure** nodes:
  - Pick the **not-sure** node u with the smallest estimate **d[u].**
  - **For** v in u.neighbors:
    - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
  - Mark u as **sure**.
- Now d(s, v) = d[v]

Lots of implementation details left un-explained.
We'll get to that!

See IPython Notebook for code!

As usual

- Does it work?
  - Yes.

- Is it fast?
  - Depends on how you implement it.

Why does this work?

- **Theorem**:
  - Suppose we run Dijkstra on G =(V,E), starting from s.
  - At the end of the algorithm, the estimate **d[v]** is the actual distance d(s,v).

Let's rename "Gates" to "s", our starting vertex.

- Proof outline:
  - **Claim 1**: For all v, **d[v] $\geq$ d(s,v).**
  - **Claim 2**: When a vertex v is marked **sure**, **d[v] = d(s,v)**.

- **Claims 1 and 2** imply the **theorem.**

Claim 2

Claim 1 + def of algorithm

  - When v is marked **sure**, **d[v] = d(s,v).**
  - **d[v] $\geq$ d(s,v)** and never increases, so after v is **sure**, **d[v]** stops changing.
  - This implies that at any time *after* v is marked **sure**, **d[v] = d(s,v).**
  - All vertices are **sure** at the end, so all vertices end up with **d[v] = d(s,v).**

Next let's prove the claims!

Claim 1

d[v] ≥ d(s,v) for all v.

Informally:

- Every time we update d[v], we have a path in mind:

$$d[v] \leftarrow \min(\ d[v]\ ,\ d[u] + \text{edgeWeight}(u,v)\ )$$

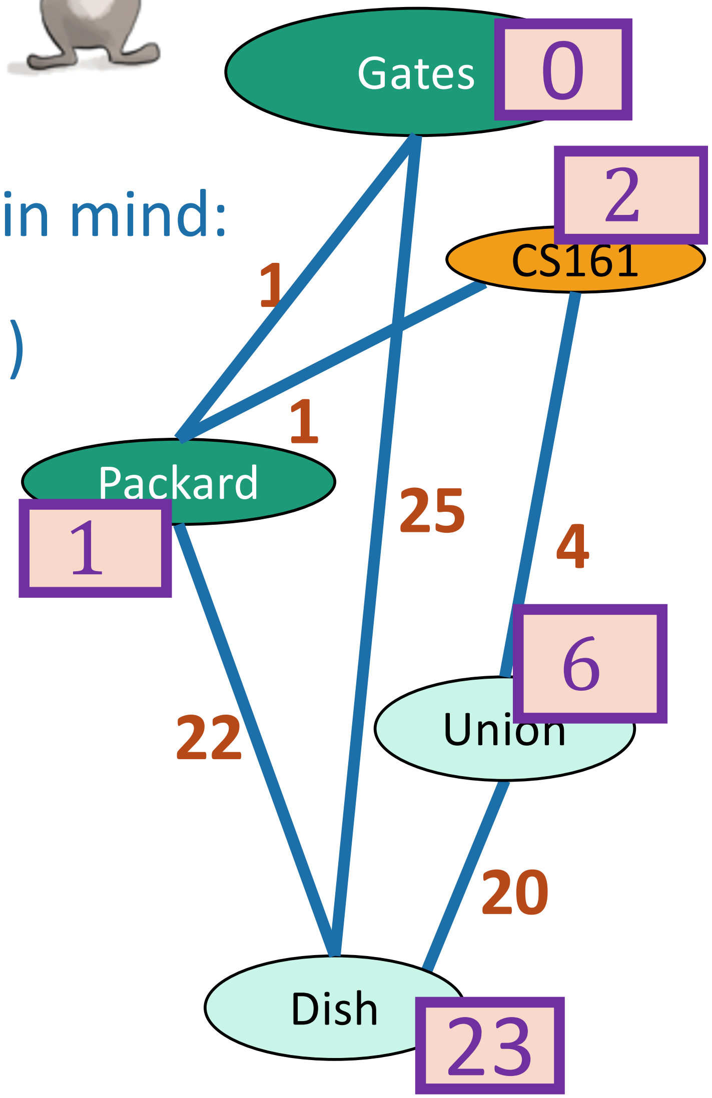
Whatever path we had in mind before

The shortest path to u, and then the edge from u to v.

- d[v] = length of the path we have in mind

  ≥ length of shortest path

  = d(s,v)



Formally:

- We should prove this by induction.
  - (See skipped slide or do it yourself)

Claim 1

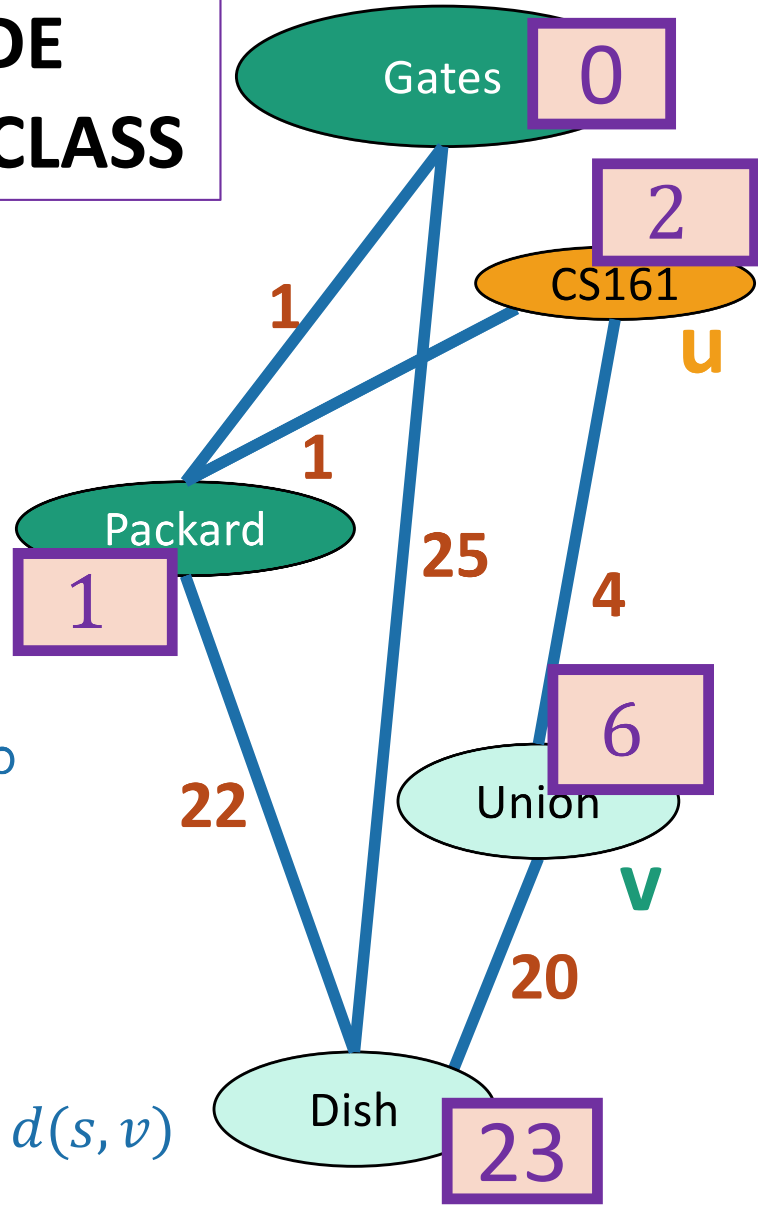
$d[v] \geq d(s,v)$ for all v.

THIS SLIDE SKIPPED IN CLASS

- Inductive hypothesis.
  - After t iterations of Dijkstra, $d[v] \geq d(s,v)$ for all v.
- Base case:
  - At step 0, $d(s,s) = 0$, and $d(s,v) \leq \infty$
- Inductive step: say hypothesis holds for t.
  - At step t+1:
    - Pick **u**; for each neighbor **v**:
    - $d[v] \leftarrow \min(d[v] , d[u] + w(u,v)) \geq d(s,v)$

By induction, $d(s,v) \leq d[v]$

$d(s,v) \leq d(s,u) + d(u,v)$
$\leq d[u] + w(u,v)$
using induction again for d[u]

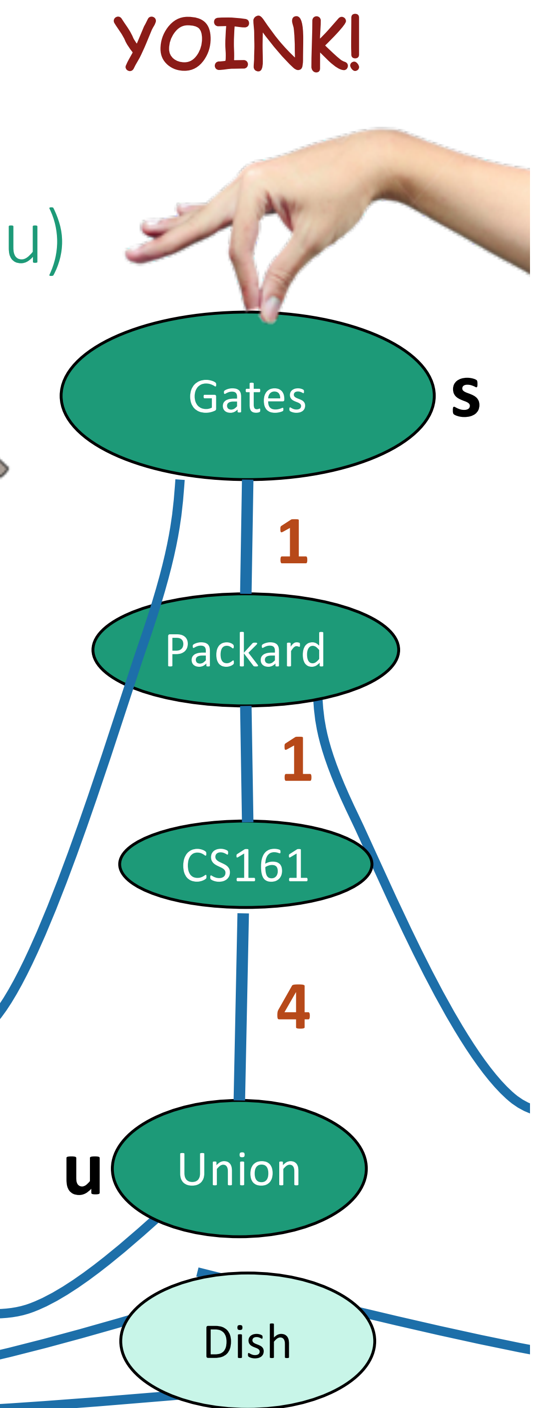


So the inductive hypothesis holds for t+1, and Claim 1 follows.

Intuition for Claim 2

When a vertex u is marked sure, d[u] = d(s,u)

- The first path that lifts **u** off the ground is the shortest one.



- Let's prove it!
  - Or at least see a proof outline.

Informal outline!

Claim 2

When a vertex u is marked sure, d[u] = d(s,u)

- Inductive Hypothesis:
  - When we mark the t'th vertex v as sure, d[v] = dist(s,v).
- Base case (t=1):
  - The first vertex marked **sure** is s, and d[s] = d(s,s) = 0.

(Assuming edge weights are non-negative!)

- Inductive step:
  - Assume by induction that every v already marked **sure** has d[v] = d(s,v).
  - Suppose that we are about to add u to the **sure** list.
  - That is, we picked u in the first line here:

> - Pick the **not-sure** node u with the smallest estimate **d[u].**
> - Update all u's neighbors v:
>   - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
> - Mark u as **sure**.
> - Repeat

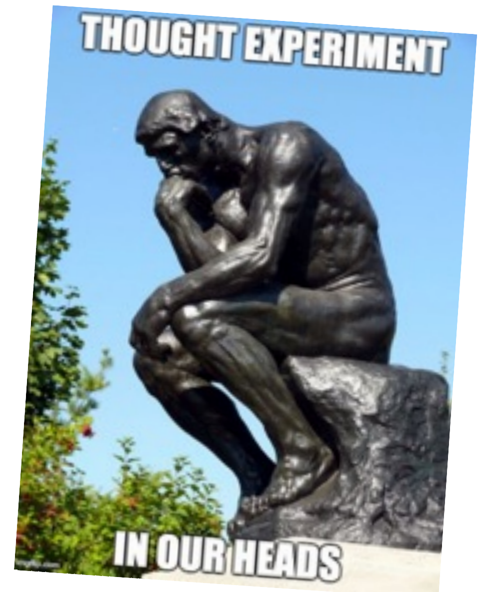
- Want to show that d[u] = d(s,u).

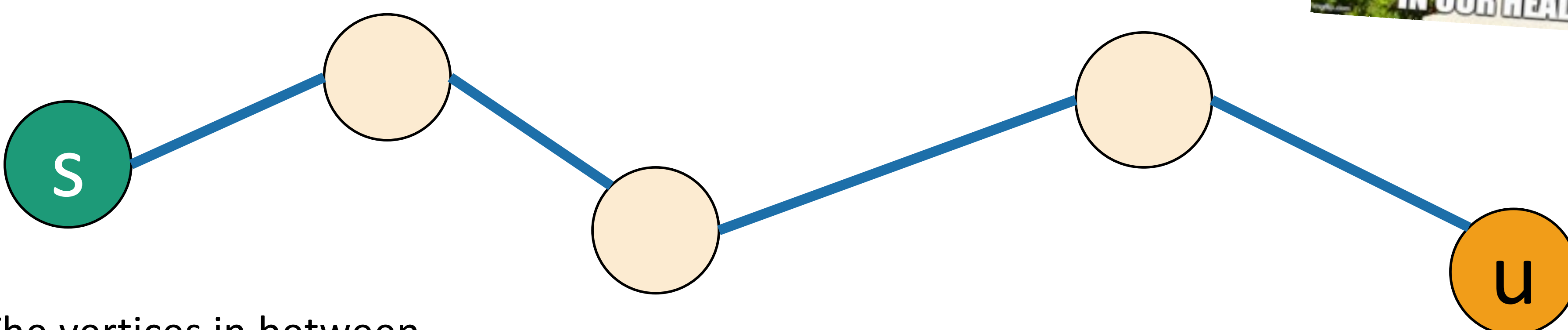
Temporary definition:
v is "good" means that d[v] = d(s,v)

Claim 2

Inductive step

- Want to show that u is good.
- Consider a **true** shortest path from s to u:





The vertices in between are beige because they may or may not be **sure.**

True shortest path.

Claim 2

Inductive step

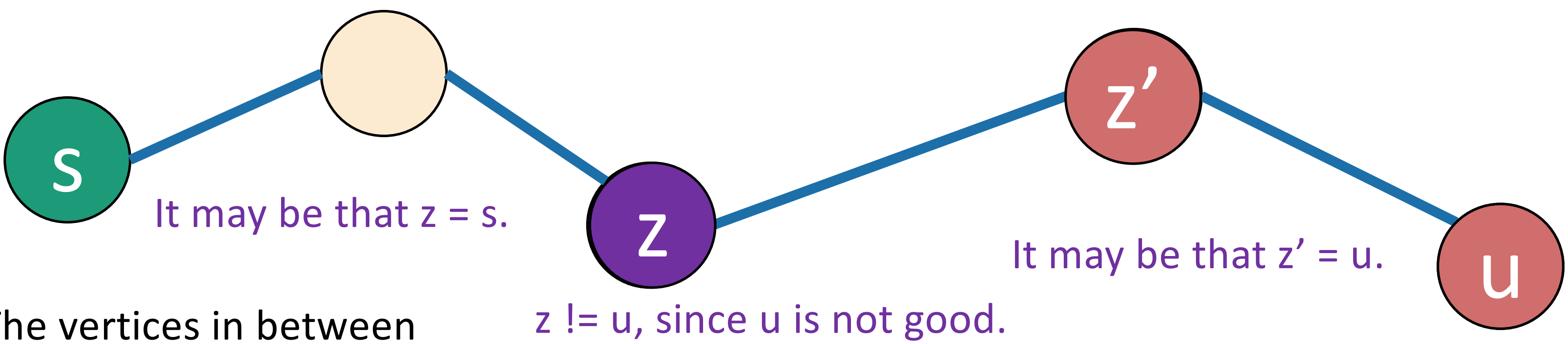
Temporary definition:

v is “good” means that d[v] = d(s,v)

● means good ● means not good

“by way of contradiction”

- Want to show that u is good. BWOC, suppose u isn’t good.
- Say z is the last good vertex before u (on shortest path to u).
- z’ is the vertex after z.



The vertices in between are beige because they may or may not be **sure.**

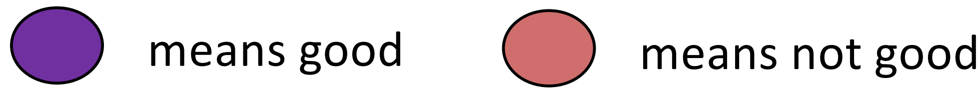
True shortest path.

Claim 2

Inductive step

Temporary definition:

v is "good" means that d[v] = d(s,v)



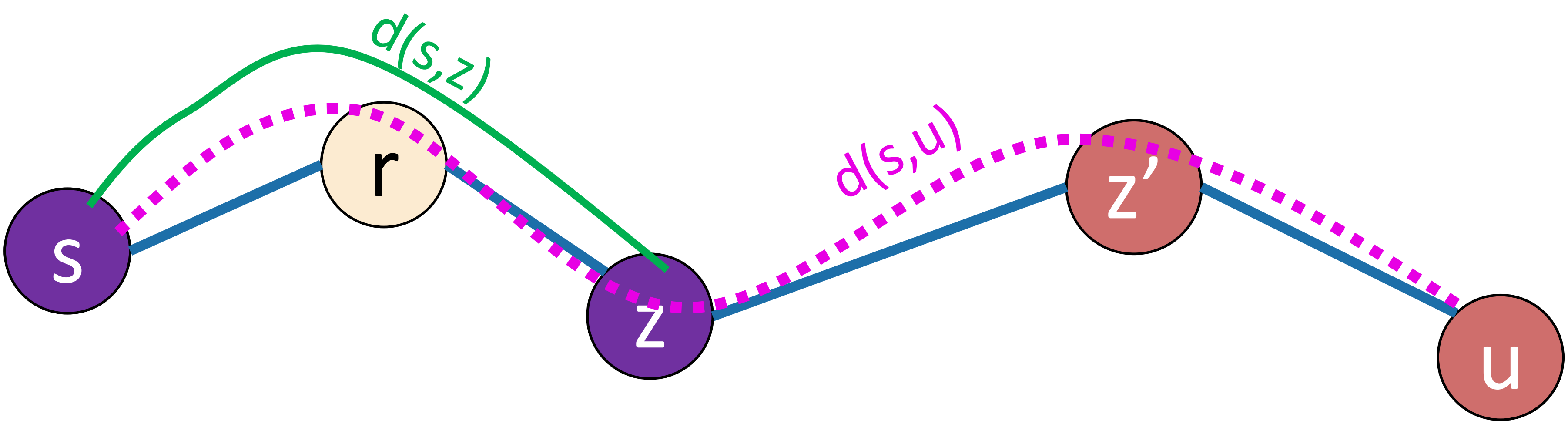
- Want to show that u is good. BWOC, suppose u isn't good.

$$d[z] = d(s,z) \leq d(s,u) \leq d[u]$$

z is good

Subpaths of shortest paths are shortest paths.

(We're also using that the edge weights are non-negative here).



Claim 2

Inductive step

Temporary definition:

v is "good" means that d[v] = d(s,v)

means good   means not good

- Want to show that u is good. BWOC, suppose u isn't good.

$$d[z] = d(s,z) \leq d(s,u) \leq d[u]$$

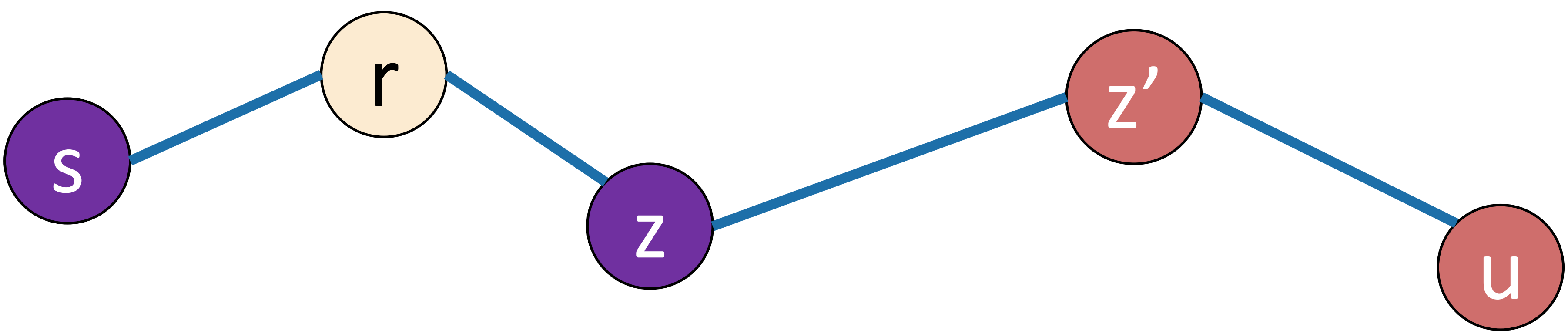
z is good

Subpaths of shortest paths are shortest paths.

Claim 1

- Since u is not good, $d[z] \neq d[u]$.
- So $d[z] < d[u]$, so z is **sure.**

We chose u so that d[u] was smallest of the unsure vertices.



Temporary definition:

v is “good” means that d[v] = d(s,v)

means good

means not good

Claim 2

Inductive step

- Want to show that u is good. BWOC, suppose u isn’t good.

$$d[z] = d(s,z) \leq d(s,u) \leq d[u]$$

z is good

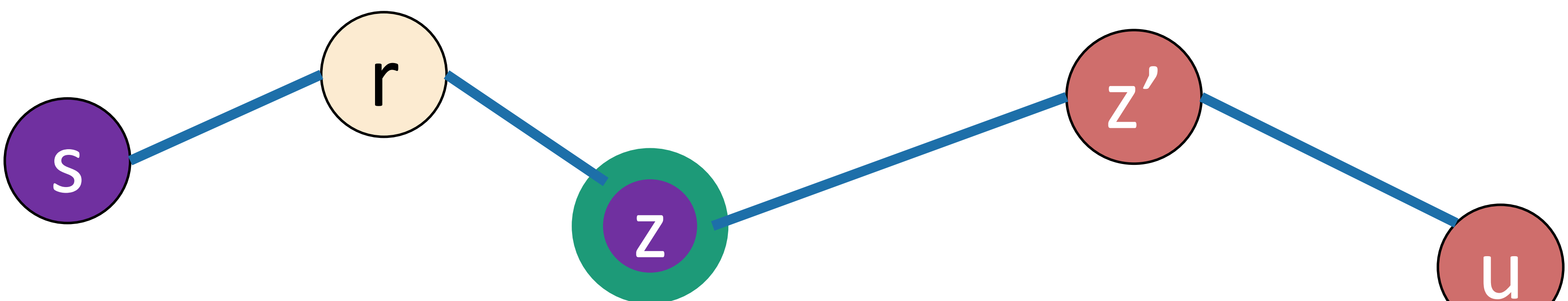
Subpaths of shortest paths are shortest paths.

Claim 1

- If $d[z] = d[u]$, then u is good.
- So $d[z] < d[u]$, so z is **sure.**

But u is not good!

We chose u so that d[u] was smallest of the unsure vertices.



Claim 2

Inductive step

Temporary definition:

v is "good" means that d[v] = d(s,v)

means good   means not good

- Want to show that u is good. BWOC, suppose u isn't good.
- If z is sure then we've already updated z':

$d[z'] \leftarrow min\{\, d[z'], d[z] + w(z,z')\}$

- $d[z'] \leq d[z] + w(z,z')$ def of update

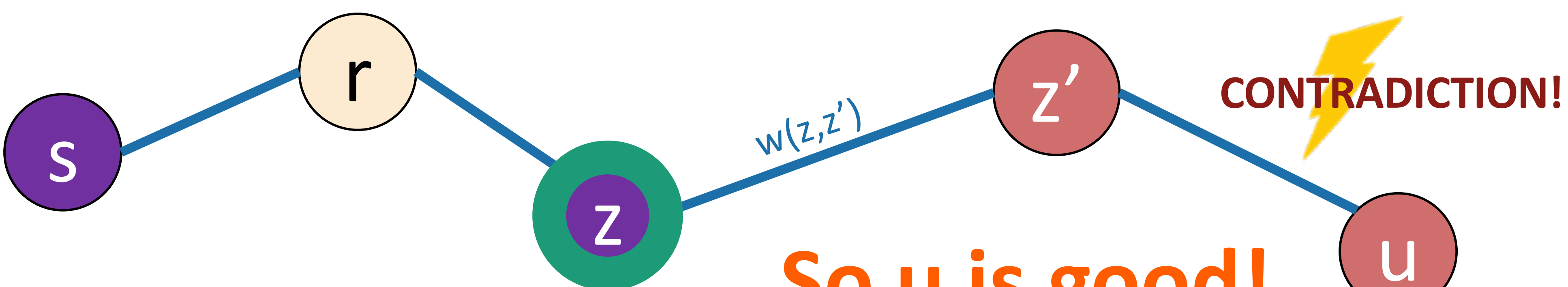
$= \; d(s,z) + w(z,z')$ By induction when z was added to the sure list it had d(s,z) = d[z]

That is, the value of d[z] when z was marked sure…

$= \; d(s,z')$ sub-paths of shortest paths are shortest paths

$\leq d[z']$ Claim 1

So d(s,z') = d[z'] and so z' is good.



So u is good!

Back to this slide

Claim 2

When a vertex u is marked sure, d[u] = d(s,u)

- Inductive Hypothesis:
  - When we mark the t'th vertex v as sure, d[v] = dist(s,v).
- Base case:
  - The first vertex marked **sure** is s, and d[s] = d(s,s) = 0.
- Inductive step:
  - Suppose that we are about to add u to the **sure** list.
  - That is, we picked u in the first line here:

> - Pick the **not-sure** node u with the smallest estimate **d[u].**
> - Update all u's neighbors v:
>   - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
> - Mark u as **sure**.
> - Repeat

  - Assume by induction that every v already marked **sure** has d[v] = d(s,v).
  - Want to show that d[u] = d(s,u).

Conclusion: Claim 2 holds!

Why does this work?

Now back to this slide

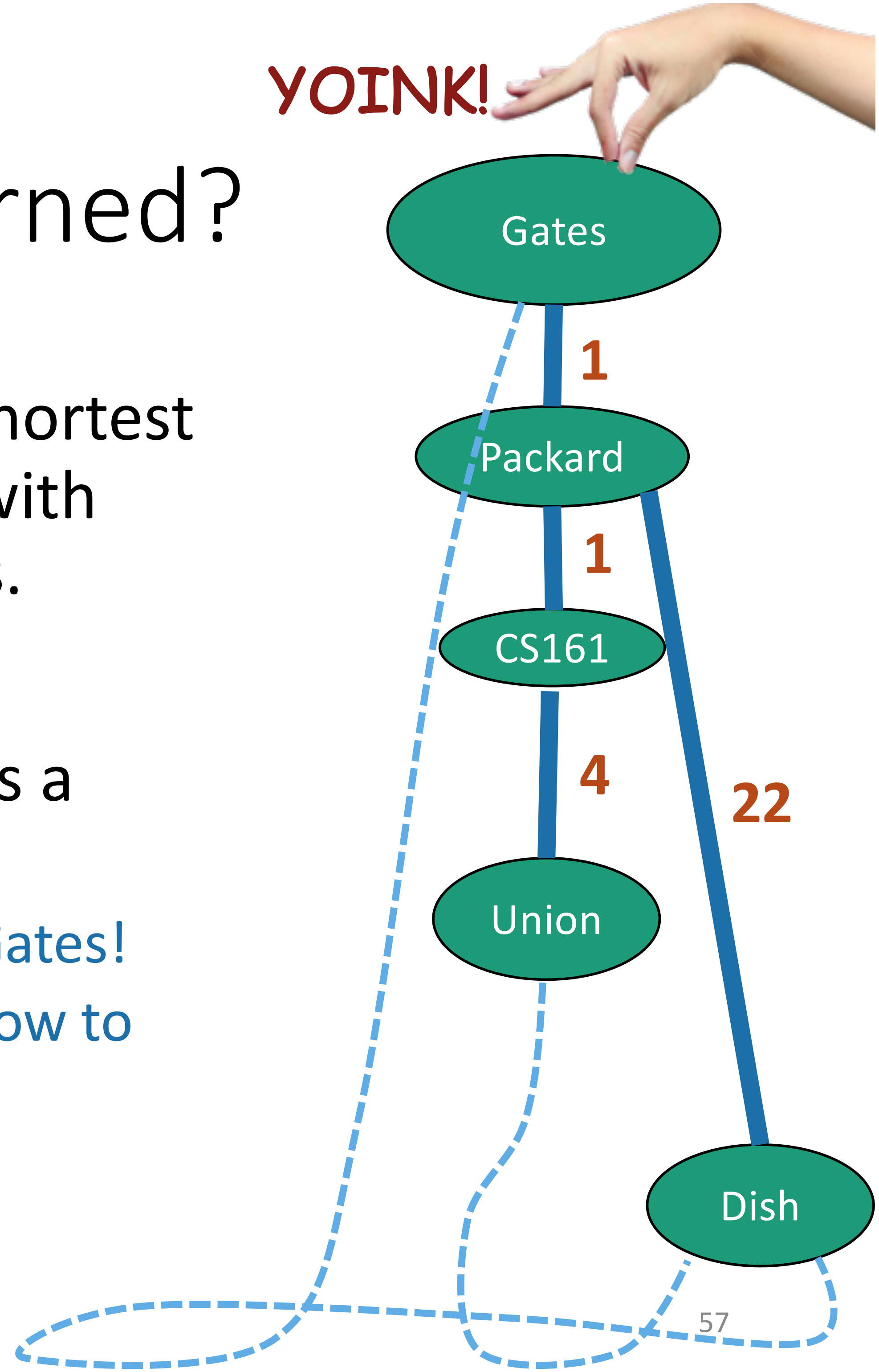
- **Theorem**:
  - Run Dijkstra on G =(V,E) starting from s.
  - At the end of the algorithm, the estimate **d[v]** is the actual distance d(s,v).

- Proof outline:
  - **Claim 1**: For all v, **d[v] $\geq$ d(s,v).**
  - **Claim 2**: When a vertex is marked **sure**, **d[v] = d(s,v).**

- **Claims 1 and 2** imply the **theorem.**

What have we learned?

- Dijkstra's algorithm finds shortest paths in weighted graphs with non-negative edge weights.
- Along the way, it constructs a nice tree.
  - We could post this tree in Gates!
  - Then people would know how to get places quickly.



As usual

- Does it work?
  - Yes.
- Is it fast?
  - Depends on how you implement it.

Running time?

Dijkstra(G,s):

- Set all vertices to **not-sure**
- d[v] = ∞ for all v in V
- d[s] = 0
- **While** there are **not-sure** nodes:
  - Pick the **not-sure** node u with the smallest estimate **d[u].**
  - **For** v in u.neighbors:
    - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
  - Mark u as **sure**.
- Now dist(s, v) = d[v]

- n iterations (one per vertex)
- How long does one iteration take?

Depends on how we implement it…

We need a data structure that:

- Stores unsure vertices v
- Keeps track of d[v]
- Can find u with minimum d[u]
  - `findMin()`
- Can remove that u
  - `removeMin(u)`
- Can update (decrease) d[v]
  - `updateKey(v,d)`

Just the inner loop:

- Pick the **not-sure** node u with the smallest estimate **d[u].**
- Update all u's neighbors v:
  - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
- Mark u as **sure**.

Total running time is big-oh of:

$$\sum_{u \in V} \left(T(\text{findMin}) + \left(\sum_{v \in u.neighbors} T(\text{updateKey}) \right) + T(\text{removeMin}) \right)$$

= n(T(`findMin`) + T(`removeMin`)) + m T(`updateKey`)

If we use an array

- $T(\text{findMin}) = O(n)$
- $T(\text{removeMin}) = O(n)$
- $T(\text{updateKey}) = O(1)$

- Running time of Dijkstra

  $= O(n(T(\texttt{findMin}) + T(\texttt{removeMin})) + m\, T(\texttt{updateKey}))$

  $= O(n^2) + O(m)$

  $= O(n^2)$

If we use a red-black tree

- $T(\text{findMin}) = O(\log(n))$
- $T(\text{removeMin}) = O(\log(n))$
- $T(\text{updateKey}) = O(\log(n))$

- Running time of Dijkstra

$$= O(n(T(\texttt{findMin}) + T(\texttt{removeMin})) + m\, T(\texttt{updateKey}))$$
$$= O(n\log(n)) + O(m\log(n))$$
$$= O((n + m)\log(n))$$

Better than an array if the graph is sparse!

aka if m is much smaller than n^2

O(n(T(findMin) + T(removeMin)) + m T(updateKey))

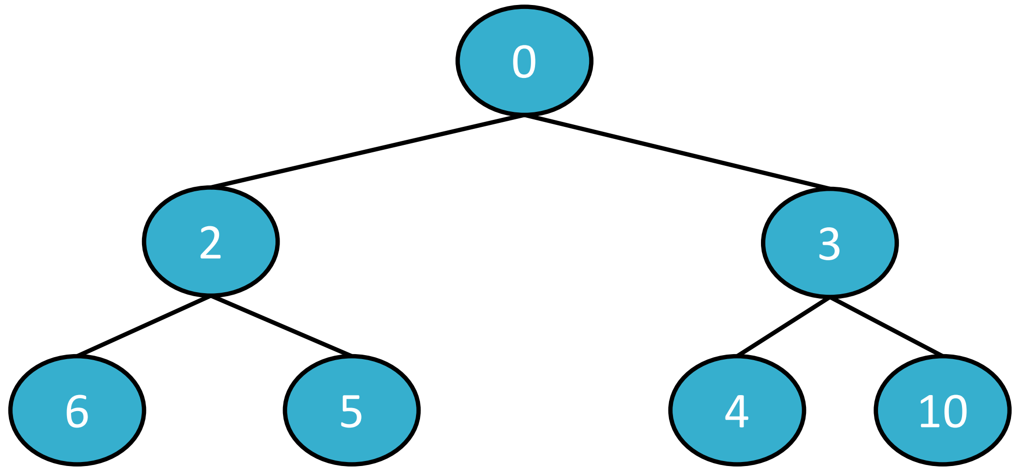
Is a hash table a good idea here?

- **Not really**:
  - Search(v) is fast (in expectation)
  - But findMin() will still take time O(n) without more structure.

Slide skipped in class

Heaps support these operations

- findMin
- removeMin
- updateKey



- A **heap** is a tree-based data structure that has the property that every node has a smaller key than its children.
- Not covered in this class – see CS166
- But! We will use them.

Many heap implementations

Nice chart on Wikipedia:

Operation	Binary[7]	Leftist	Binomial[7]	Fibonacci[7][8]	Pairing[9]	Brodal[10][b]	Rank-pairing[12]	Strict Fibonacci[13]
find-min	$\Theta(1)$	$\Theta(1)$	$\Theta(\log n)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
delete-min	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$	$O(\log n)$[c]	$O(\log n)$[c]	$O(\log n)$	$O(\log n)$[c]	$O(\log n)$
insert	$O(\log n)$	$\Theta(\log n)$	$\Theta(1)$[c]	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
decrease-key	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$	$\Theta(1)$[c]	$o(\log n)$[c][d]	$\Theta(1)$	$\Theta(1)$[c]	$\Theta(1)$
merge	$\Theta(n)$	$\Theta(\log n)$	$O(\log n)$[e]	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$

Say we use a **Fibonacci Heap**

- T(findMin) = O(1) (amortized time*)
- T(removeMin) = O(log(n)) (amortized time*)
- T(updateKey) = O(1) (amortized time*)
- See CS166 for more!
- Running time of Dijkstra

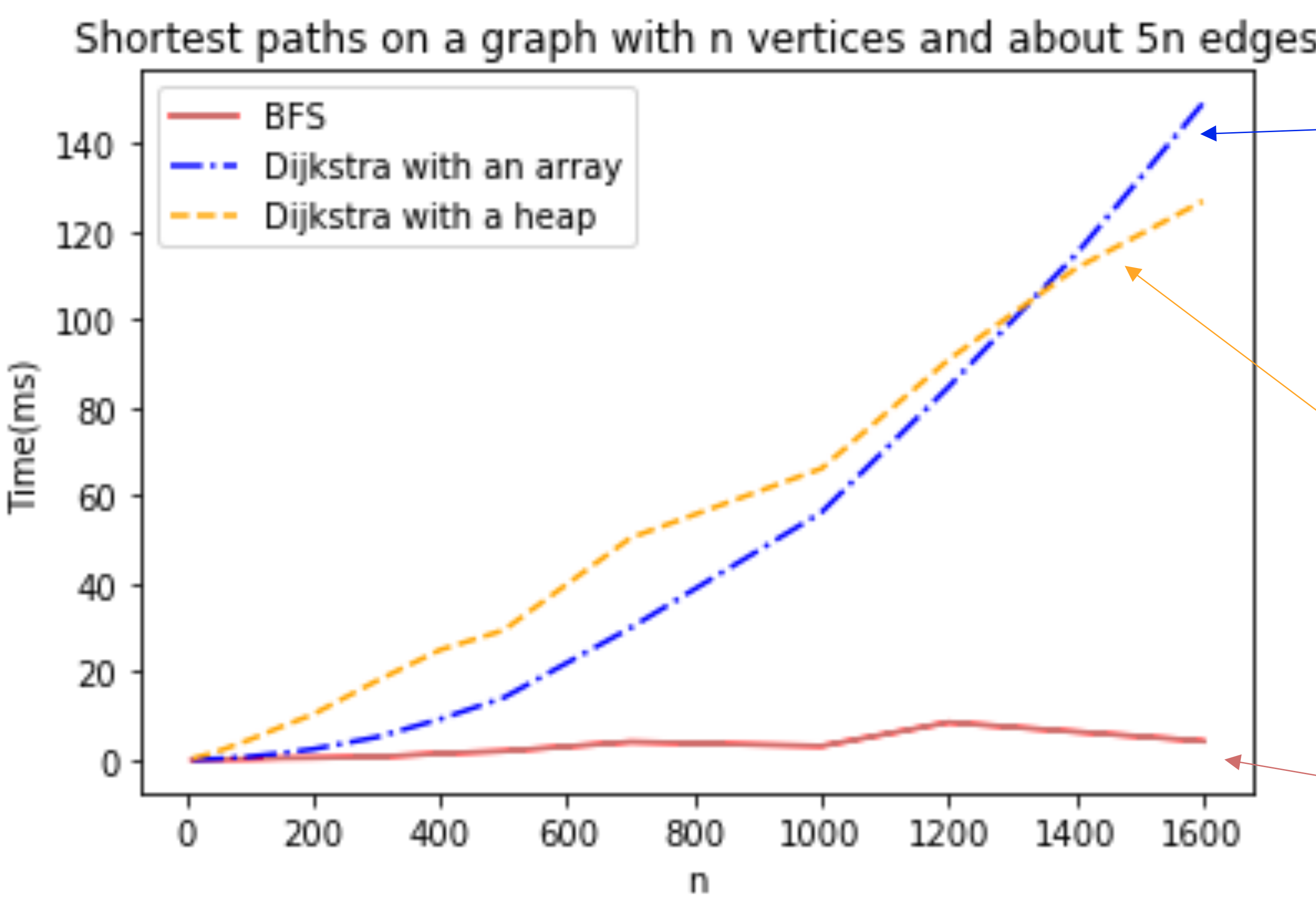
  = O(n(T(`findMin`) + T(`removeMin`)) + m T(`updateKey`))

  = O(nlog(n) + m) (amortized time)

*This means that any sequence of d `removeMin` calls takes time at most O(dlog(n)).
But a few of the d may take longer than O(log(n)) and some may take less time..

See IPython Notebook for Lecture 11
The heap is implemented using `heapdict`

In practice



Dijkstra using a Python list to keep track of vertices has quadratic runtime.

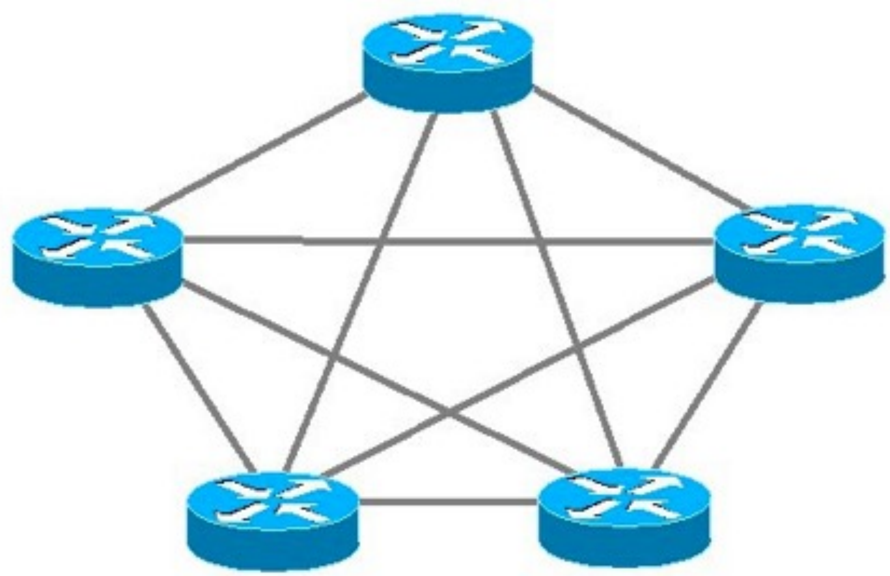
Dijkstra using a heap looks a bit more linear (actually nlog(n))

BFS is really fast by comparison! But it doesn't work on weighted graphs.

Dijkstra is used in practice

- eg, OSPF (Open Shortest Path First), a routing protocol for IP networks, uses Dijkstra.

But there are some things it's not so good at.



Dijkstra Drawbacks

- Needs non-negative edge weights.
- If the weights change, we need to re-run the whole thing.
  - in OSPF, a vertex broadcasts any changes to the network, and then every vertex re-runs Dijkstra's algorithm from scratch.

Bellman-Ford algorithm

- (-) Slower than Dijkstra's algorithm

- (+) Can handle negative edge weights.
  - Can be useful if you want to say that some edges are actively good to take, rather than costly.
  - Can be useful as a building block in other algorithms.

- (+) Allows for some flexibility if the weights change.
  - We'll see what this means later

Today: *intro* to Bellman-Ford

- We'll see a definition by example.
- We'll come back to it next lecture with more rigor.
  - Don't worry if it goes by quickly today.
  - There are some skipped slides with pseudocode, but we'll see them again next lecture.

- Basic idea:
  - Instead of picking the u with the smallest d[u] to update, just update all of the u's simultaneously.

Bellman-Ford algorithm

Bellman-Ford(G,s):

- d[v] = ∞ for all v in V
- d[s] = 0
- **For** i=0,…,n-1:
  - **For** u in V:
    - **For** v in u.neighbors:
      - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))

Instead of picking u cleverly, just update for all of the u's.

Compare to Dijkstra:

- **While** there are **not-sure** nodes:
  - Pick the **not-sure** node u with the smallest estimate **d[u].**
  - **For** v in u.neighbors:
    - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
  - Mark u as **sure**.

For pedagogical reasons

which we will see next lecture

- We are actually going to change this to be less smart.
- Keep n arrays: $d^{(0)}, d^{(1)}, \ldots, d^{(n-1)}$

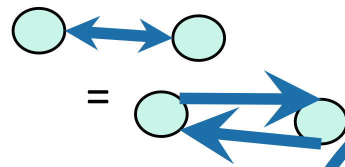
Bellman-Ford*(G,s):

- $d^{(i)}[v] = \infty$ for all v in V, for all i=0,…,n-1
- $d^{(0)}[s] = 0$
- **For** i=0,…,n-2:
  - **For** u in V:
    - **For** v in u.neighbors:
      - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v], d^{(i+1)}[v], d^{(i)}[u] + \text{edgeWeight}(u,v))$
- Then $\text{dist}(s,v) = d^{(n-1)}[v]$

Slightly different than the original Bellman-Ford algorithm, but the analysis is basically the same.

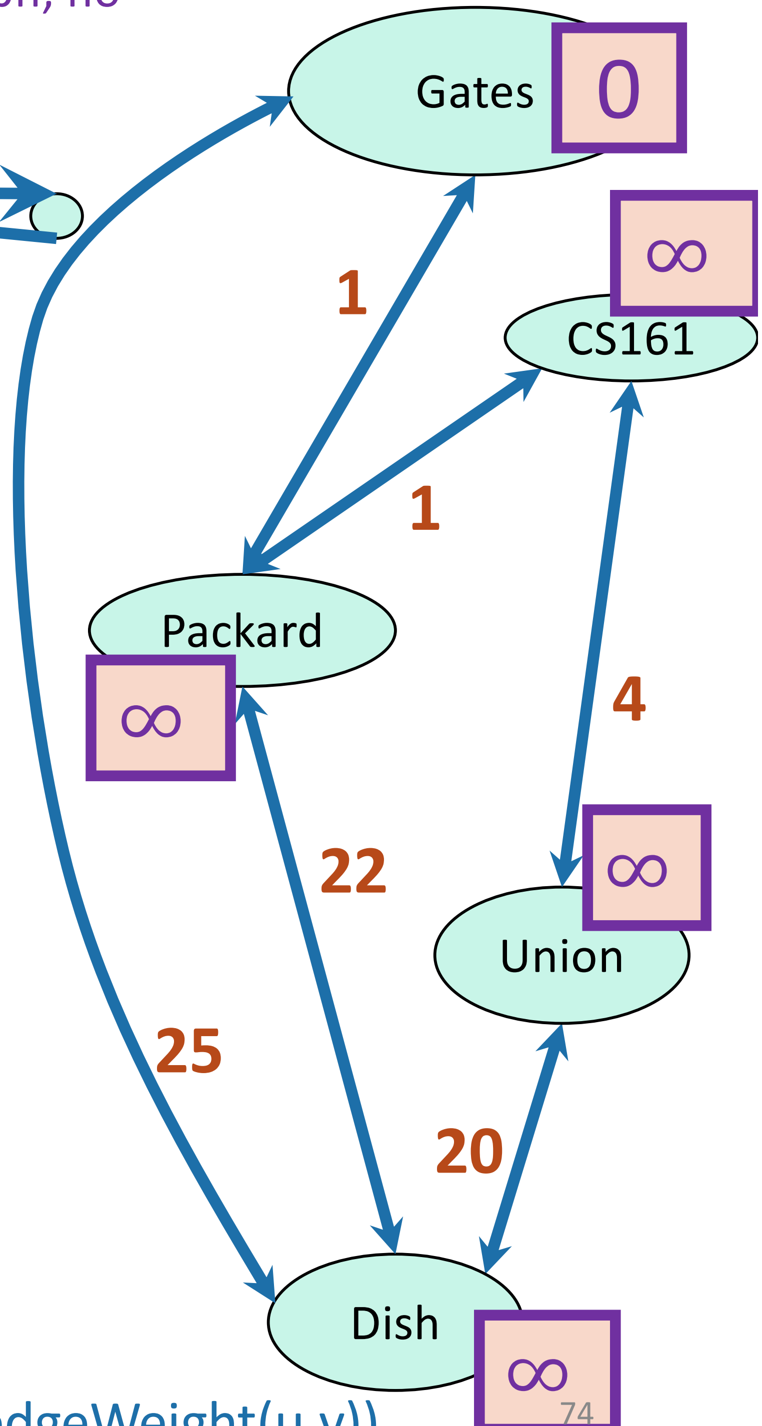
Bellman-Ford

Start with the same graph, no negative weights.



How far is a node from Gates?

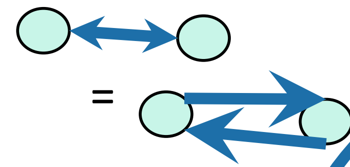
	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$					
$d^{(2)}$					
$d^{(3)}$					
$d^{(4)}$					



- **For** i=0,…,n-2:
  - **For** u in V:
    - **For** v in u.neighbors:
      - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v]\ ,\ d^{(i+1)}[v],\ d^{(i)}[u] + \text{edgeWeight}(u,v))$

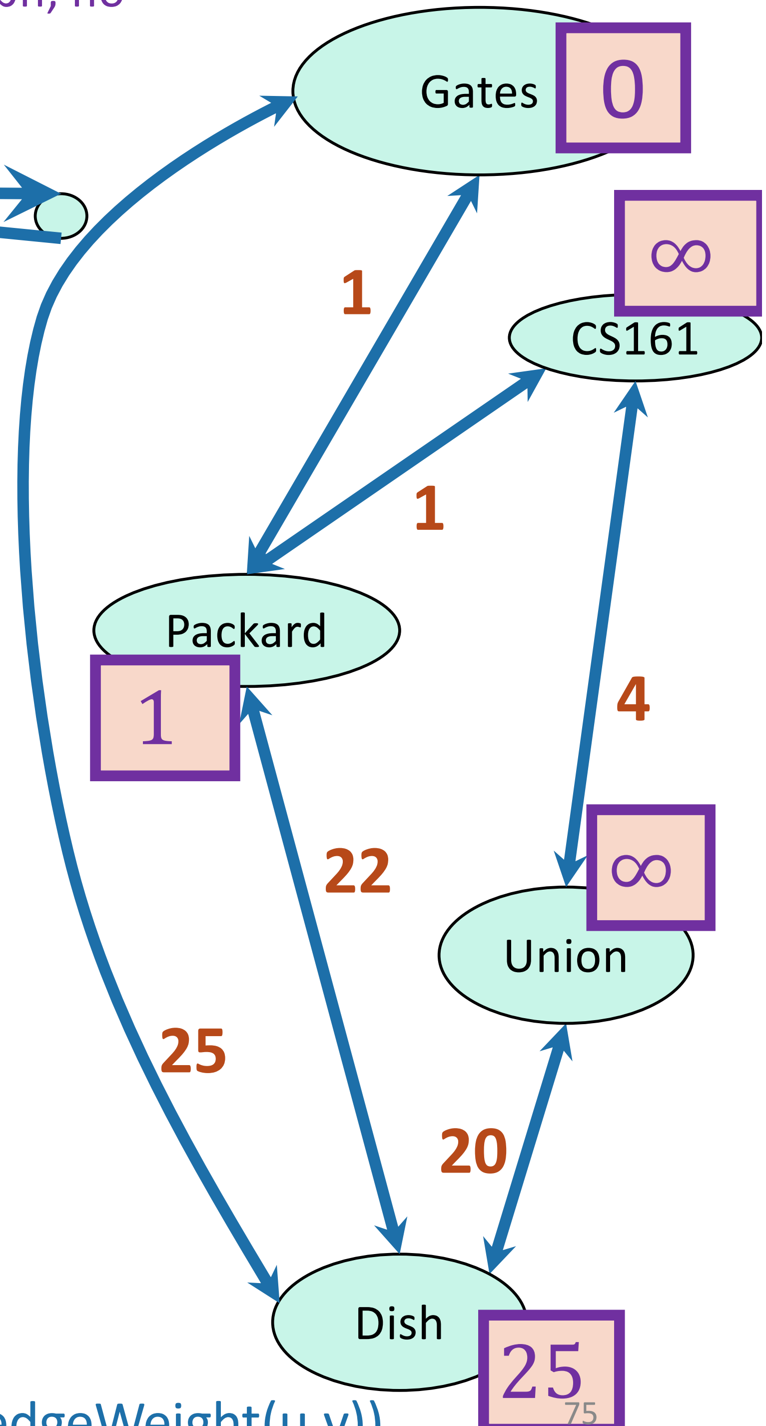
Bellman-Ford

Start with the same graph, no negative weights.



How far is a node from Gates?

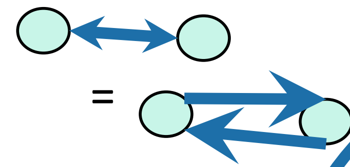
	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	**1**	∞	∞	**25**
$d^{(2)}$					
$d^{(3)}$					
$d^{(4)}$					

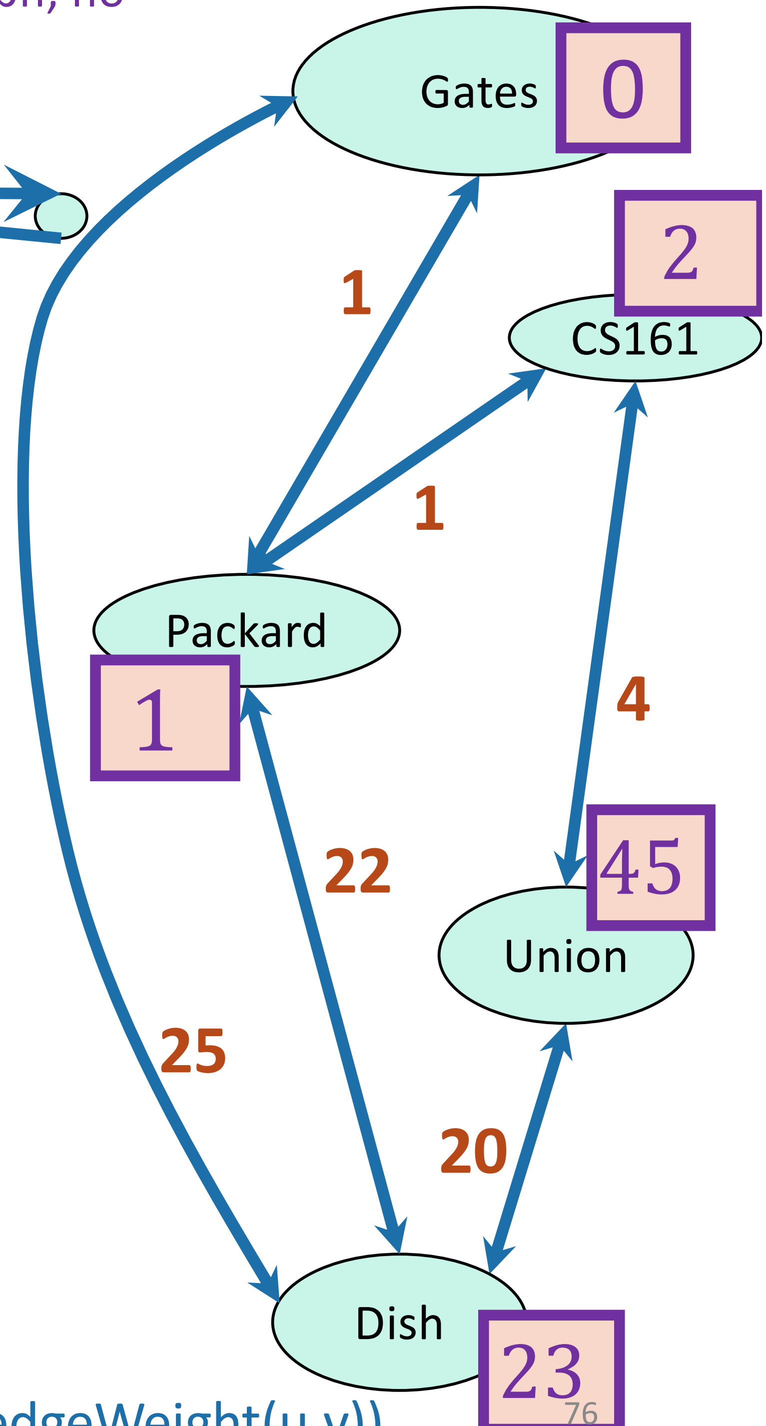


- **For** i=0,…,n-2:
  - **For** u in V:
    - **For** v in u.neighbors:
      - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v],\ d^{(i+1)}[v],\ d^{(i)}[u] + \text{edgeWeight}(u,v))$

Bellman-Ford

Start with the same graph, no negative weights.





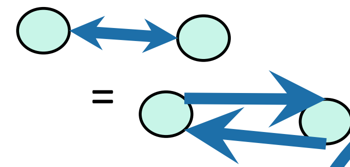
How far is a node from Gates?

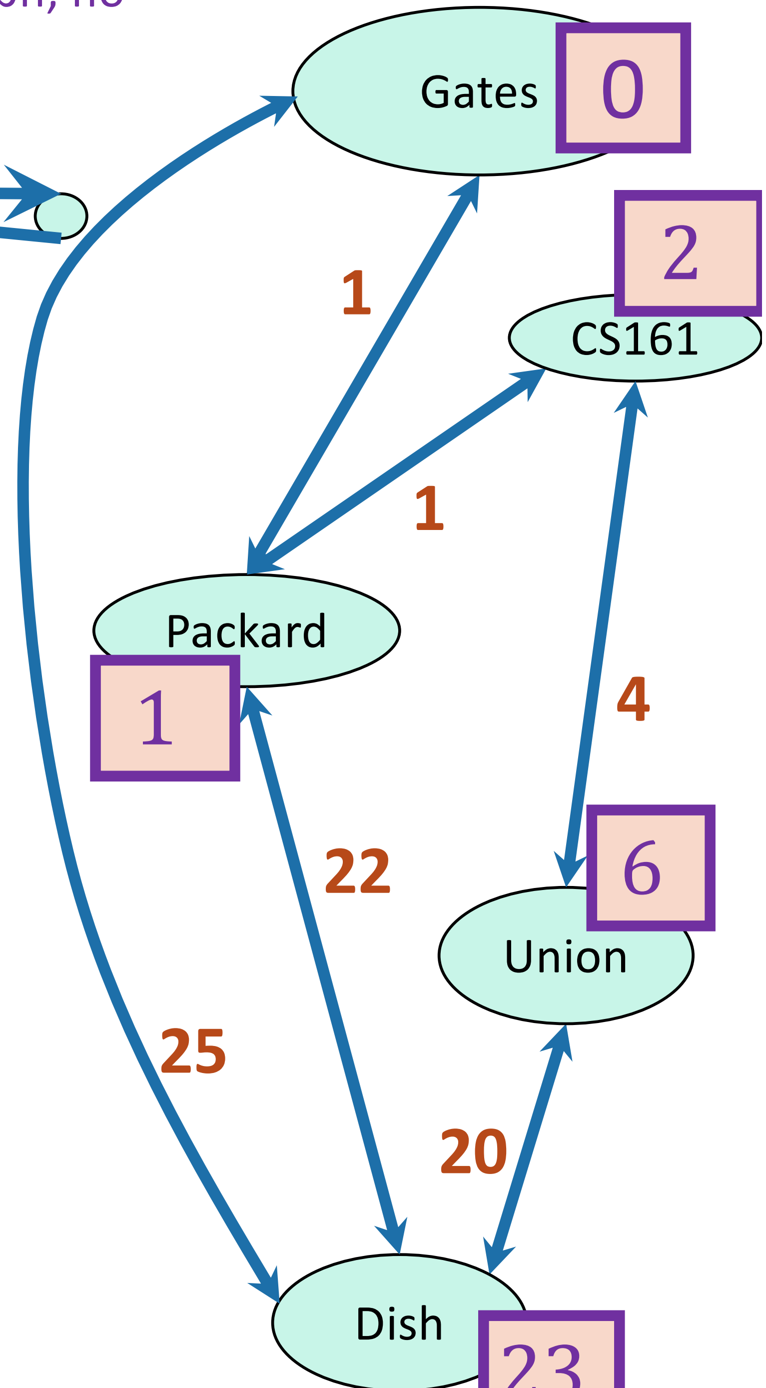
	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	**1**	∞	∞	**25**
$d^{(2)}$	0	1	**2**	**45**	**23**
$d^{(3)}$					
$d^{(4)}$					

- **For** i=0,…,n-2:
    - **For** u in V:
        - **For** v in u.neighbors:
            - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v] , d^{(i+1)}[v], d^{(i)}[u] + \text{edgeWeight}(u,v))$

Bellman-Ford

Start with the same graph, no negative weights.





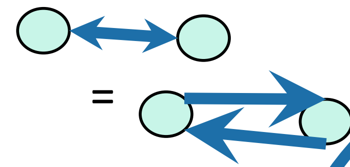
How far is a node from Gates?

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	**1**	∞	∞	**25**
$d^{(2)}$	0	1	**2**	**45**	**23**
$d^{(3)}$	0	1	2	**6**	23
$d^{(4)}$					

- **For** i=0,…,n-2:
  - **For** u in V:
    - **For** v in u.neighbors:
      - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v]\ ,\ d^{(i+1)}[v],\ d^{(i)}[u] + \text{edgeWeight}(u,v))$

Bellman-Ford

Start with the same graph, no negative weights.



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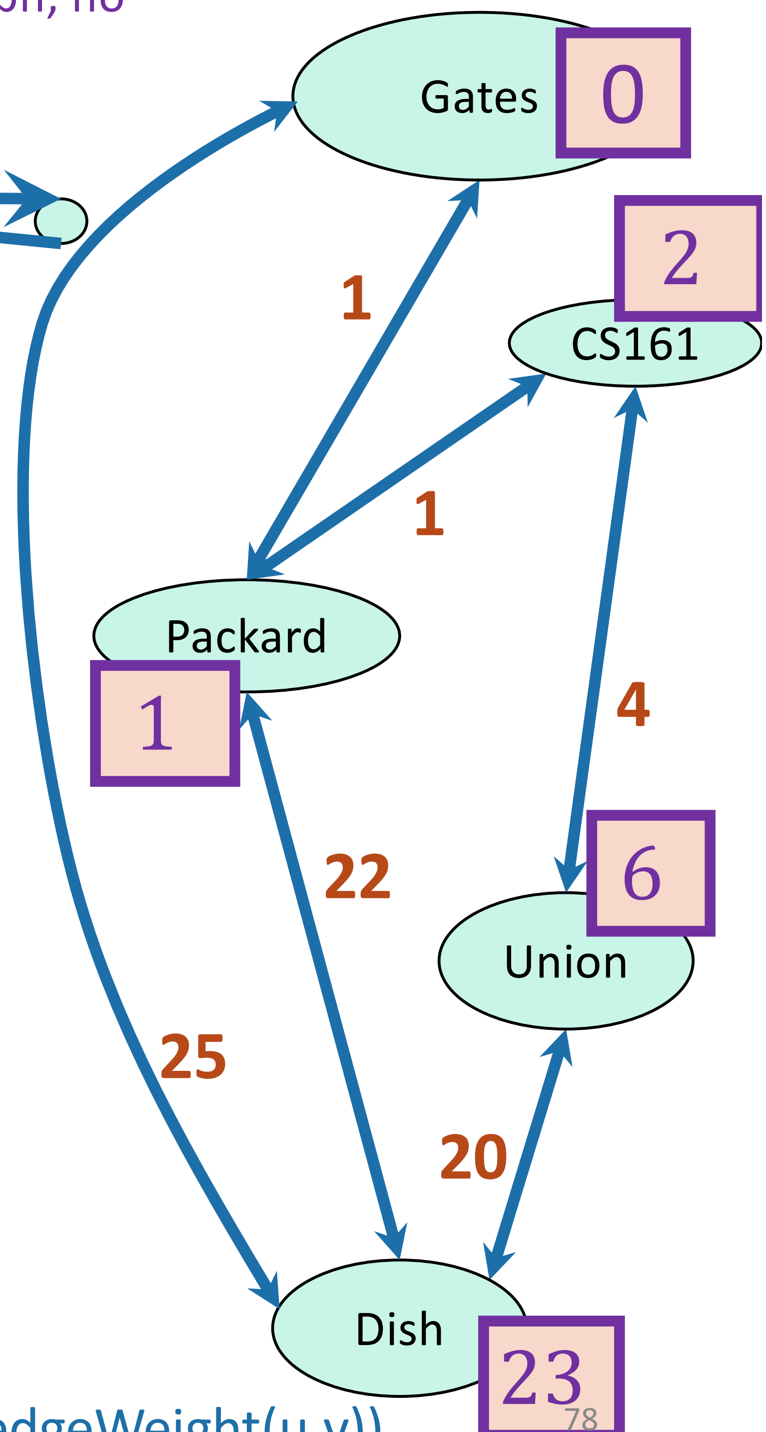
$d^{(3)}$	0	1	2	**6**	23
$d^{(4)}$	0	1	2	6	23

These are the final distances!



- **For** i=0,…,n-2:
  - **For** u in V:
    - **For** v in u.neighbors:
      - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v], d^{(i+1)}[v], d^{(i)}[u] + \text{edgeWeight}(u,v))$

As usual

- Does it work?
  - Yes
  - Idea to the right.
  - (See hidden slides for details)
- Is it fast?
  - Not really…

A **simple path** is a path with no cycles.

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	**1**	∞	∞	**25**
$d^{(2)}$	0	1	**2**	**45**	**23**
$d^{(3)}$	0	1	2	**6**	23
$d^{(4)}$	0	1	2	6	23

Idea: proof by induction.
Inductive Hypothesis:
$d^{(i)}[v]$ is equal to the cost of the shortest path between s and v **with at most i edges**.
Conclusion:
$d^{(n-1)}[v]$ is equal to the cost of the shortest simple path between s and v. **(Since all simple paths have at most n-1 edges).**

Skipped in class

Proof by induction

- **Inductive Hypothesis:**
  - After iteration i, for each v, $d^{(i)}[v]$ is equal to the cost of the shortest path between s and v with at most i edges.
- **Base case:**
  - After iteration 0…
- **Inductive step:**

Skipped in class

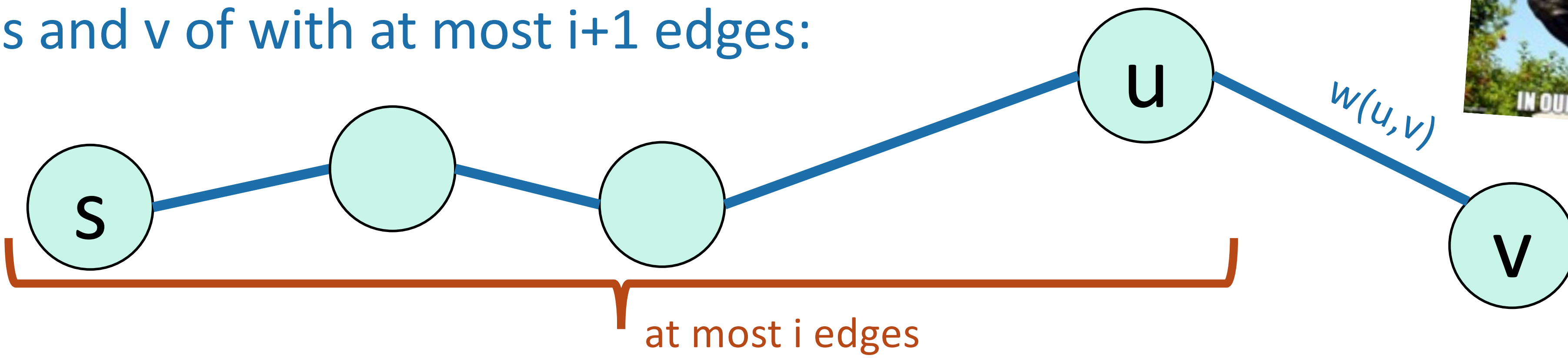
Inductive step

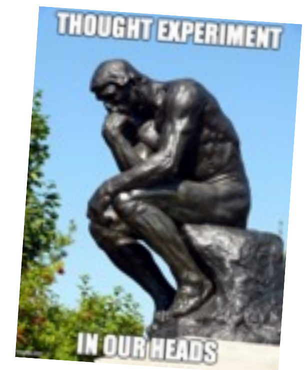
Hypothesis: After iteration i, for each v, $d^{(i)}[v]$ is equal to the cost of the shortest path between s and v with at most i edges.

- Suppose the inductive hypothesis holds for i.
- We want to establish it for i+1.

Say this is the shortest path between s and v of with at most i+1 edges:

Let u be the vertex right before v in this path.





- By induction, $d^{(i)}[u]$ is the cost of a shortest path between s and u of i edges.
- By setup, $d^{(i)}[u] + w(u,v)$ is the cost of a shortest path between s and v of i+1 edges.
- In the i+1'st iteration, we ensure $\mathbf{d^{(i+1)}[v] <= d^{(i)}[u] + w(u,v)}$.
- So $d^{(i+1)}[v]$ <= cost of shortest path between s and v with i+1 edges.
- But $d^{(i+1)}[v]$ = cost of a particular path of at most i+1 edges >= cost of shortest path.
- So d[v] = cost of shortest path with at most i+1 edges.

Skipped in class

Proof by induction

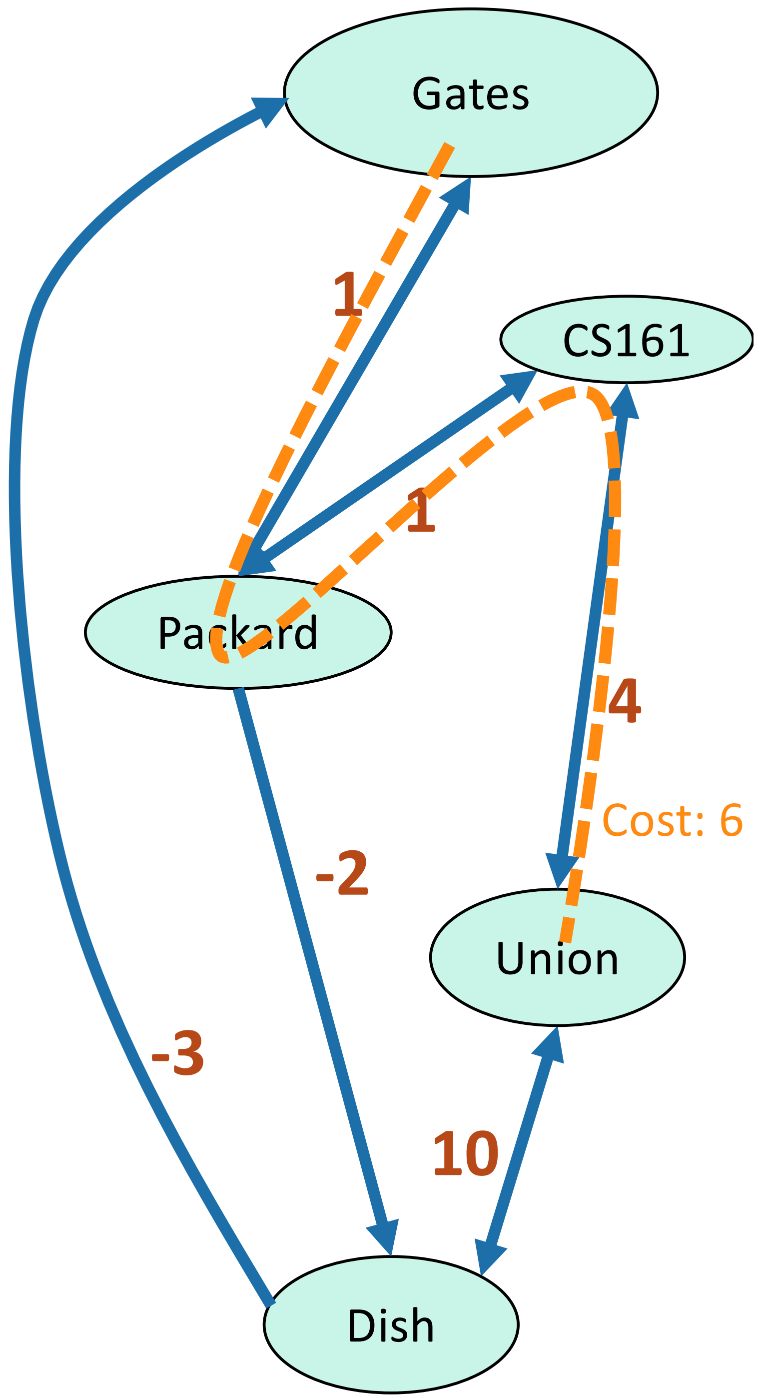
- **Inductive Hypothesis:**
  - After iteration i, for each v, $d^{(i)}[v]$ is equal to the cost of the shortest path between s and v of length at most i edges.
- **Base case:**
  - After iteration 0…
- **Inductive step:**
- **Conclusion:**
  - After iteration n-1, for each v, d[v] is equal to the cost of the shortest path between s and v of length at most n-1 edges.
  - **Aka, d[v] = d(s,v) for all v** as long as there are no negative cycles!

Pros and cons of Bellman-Ford

- Running time: O(mn) running time
  - For each of n steps we update m edges
  - Slower than Dijkstra
- However, it's also more flexible in a few ways.
  - Can handle negative edges
  - If we constantly do these iterations, any changes in the network will eventually propagate through.

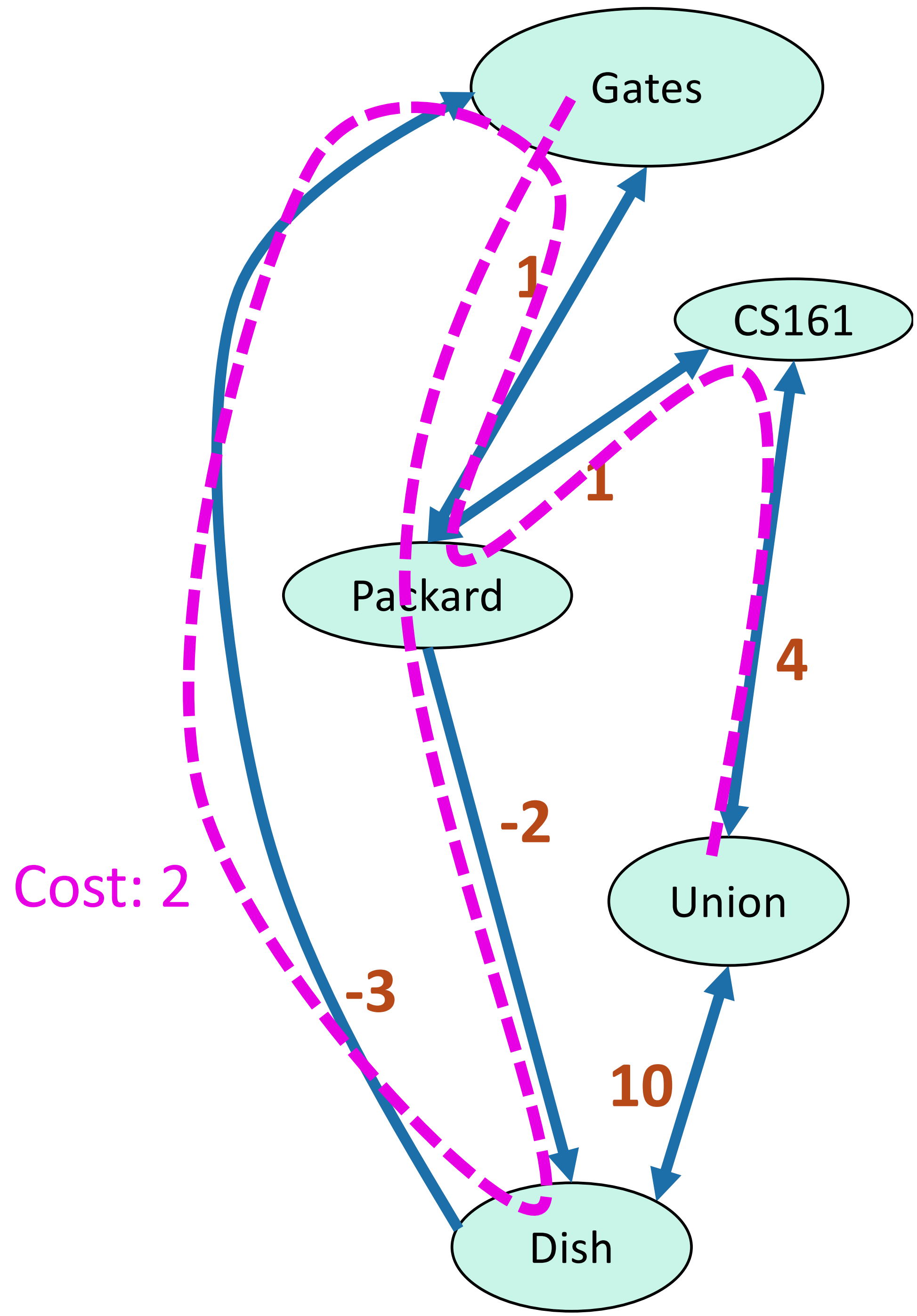
Wait a second…

- What is the shortest path from Gates to the Union?



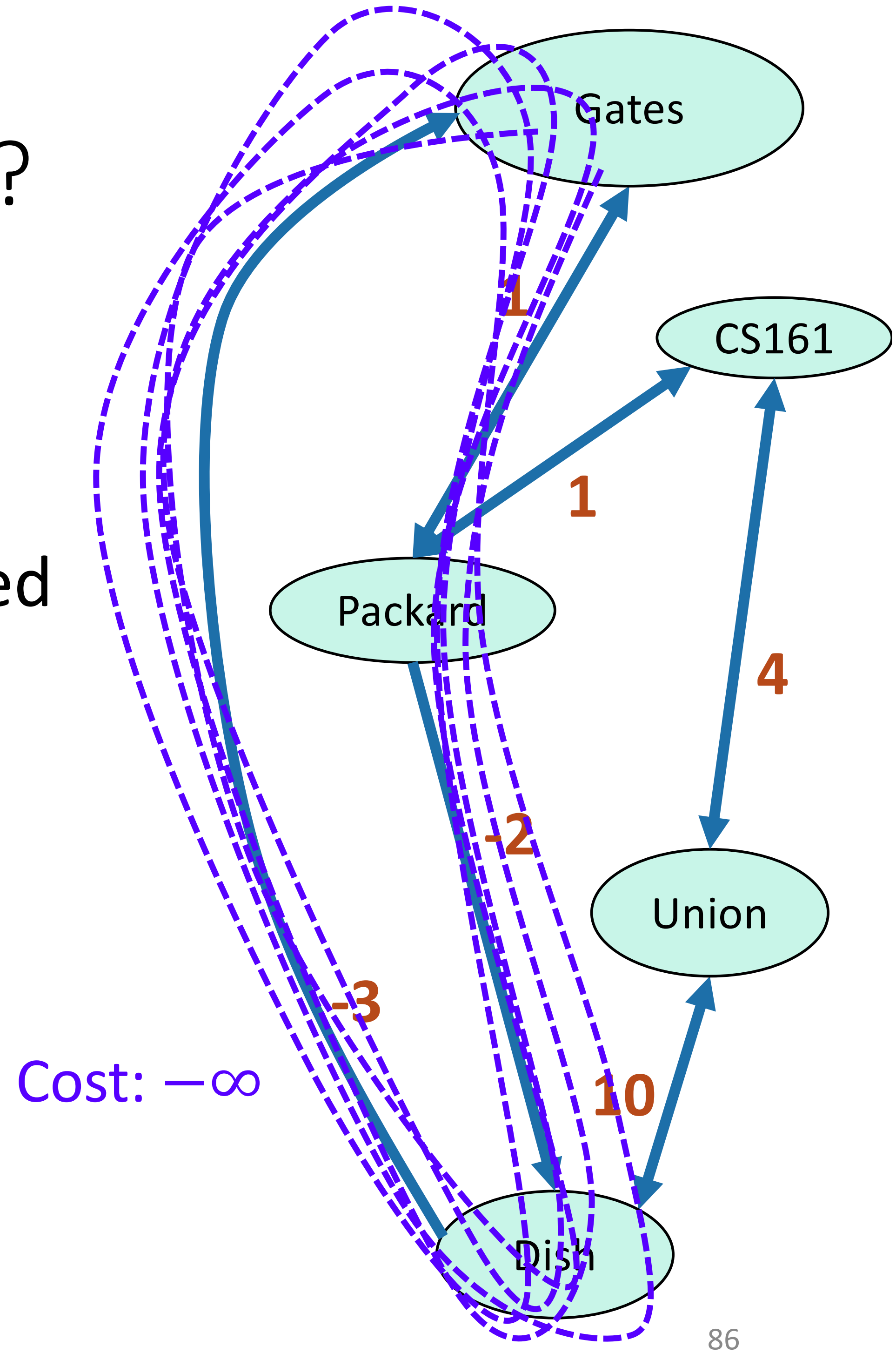
Wait a second…

- What is the shortest path from Gates to the Union?



Negative edge weights?

- What is the shortest path from Gates to the Union?
- Shortest paths aren't defined if there are negative cycles!



Bellman-Ford and negative edge weights

- B-F works with negative edge weights…as long as there are no negative cycles.
  - A negative cycle is a path with the same start and end vertex whose cost is negative.
- However, B-F can detect negative cycles.

Back to the correctness

- Does it work?
  - Yes
  - Idea to the right.

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	**1**	∞	∞	**25**
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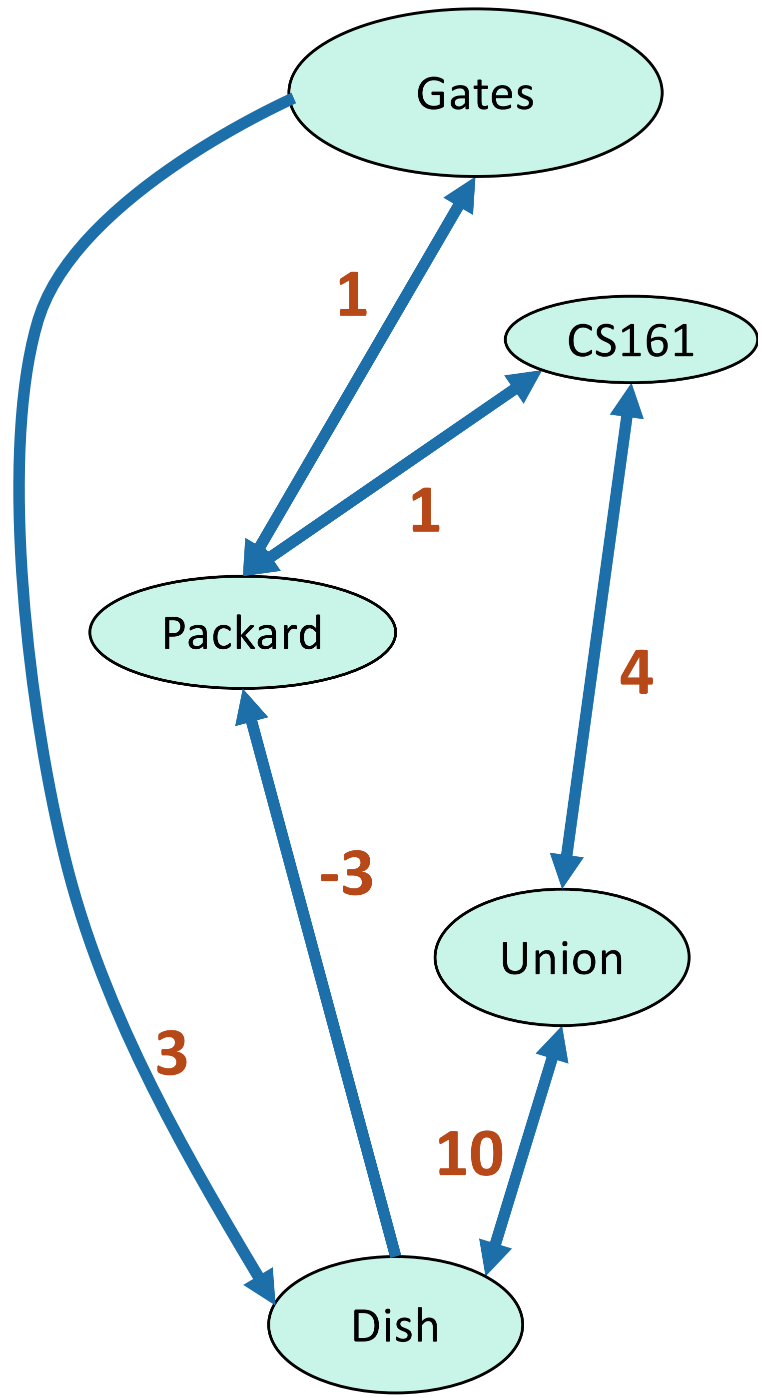
Idea: proof by induction.
Inductive Hypothesis:
$d^{(i)}[v]$ is equal to the cost of the shortest path between s and v **with at most i edges**.
Conclusion:
$d^{(n-1)}[v]$ is equal to the cost of the shortest simple path between s and v. **(Since all simple paths have at most n-1 edges).**

If there are negative cycles, then non-simple paths matter!
So the proof breaks for negative cycles.

Negative edge weights

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	**1**	∞	∞	**3**
$d^{(2)}$	0	**0**	**2**	**13**	3
$d^{(3)}$	0	0	**1**	**6**	3
$d^{(4)}$	0	0	1	**5**	3

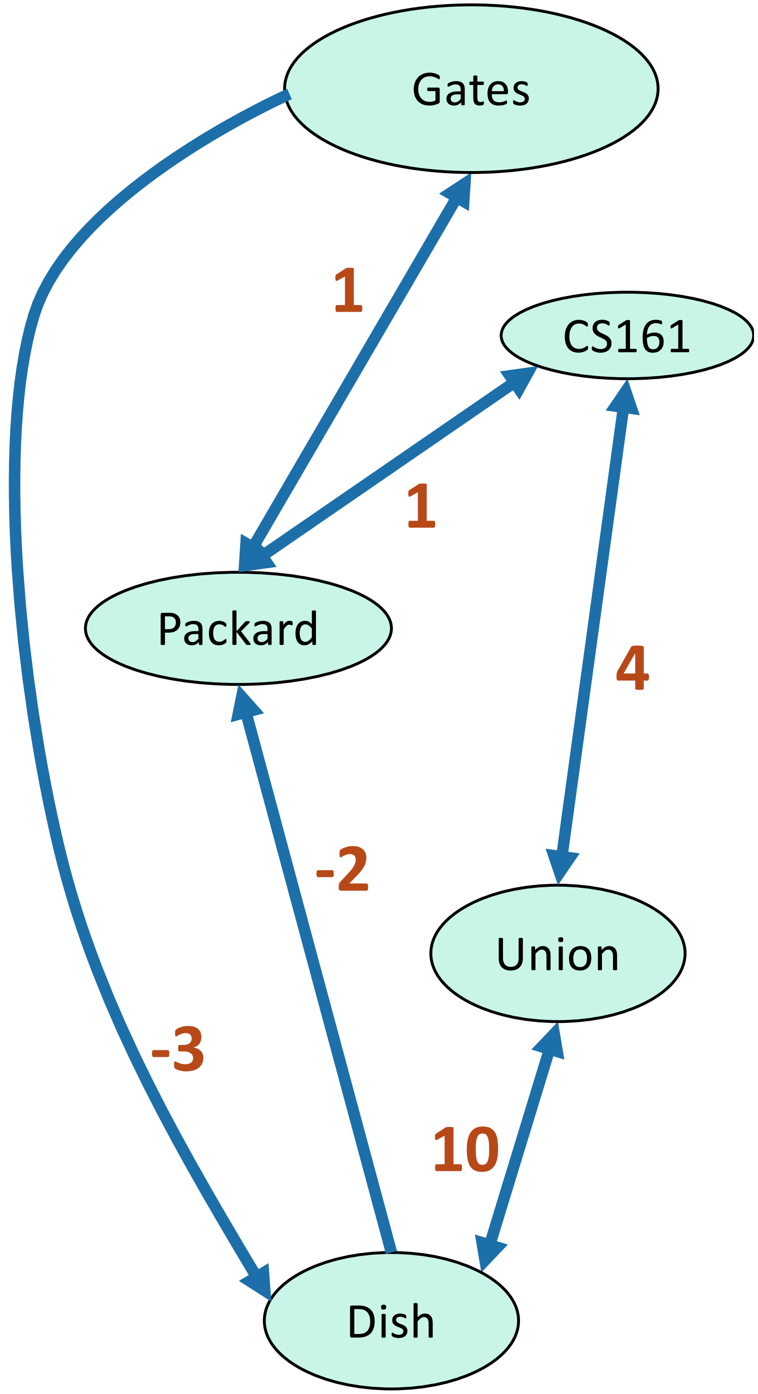


- **For** i=0,…,n-2:
  - **For** u in V:
    - **For** v in u.neighbors:
      - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v]\ ,\ d^{(i+1)}[v],\ d^{(i)}[u] + \text{edgeWeight}(u,v))$

B-F with negative cycles

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	**1**	∞	∞	**-3**
$d^{(2)}$	0	**-5**	**2**	**7**	**-3**
$d^{(3)}$	**-4**	-5	**-4**	**6**	**-3**

This is not looking good!



- **For** i=0,…,n-2:
  - **For** u in V:
    - **For** v in u.neighbors:
      - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v]\ ,\ d^{(i+1)}[v],\ d^{(i)}[u] + \text{edgeWeight}(u,v))$

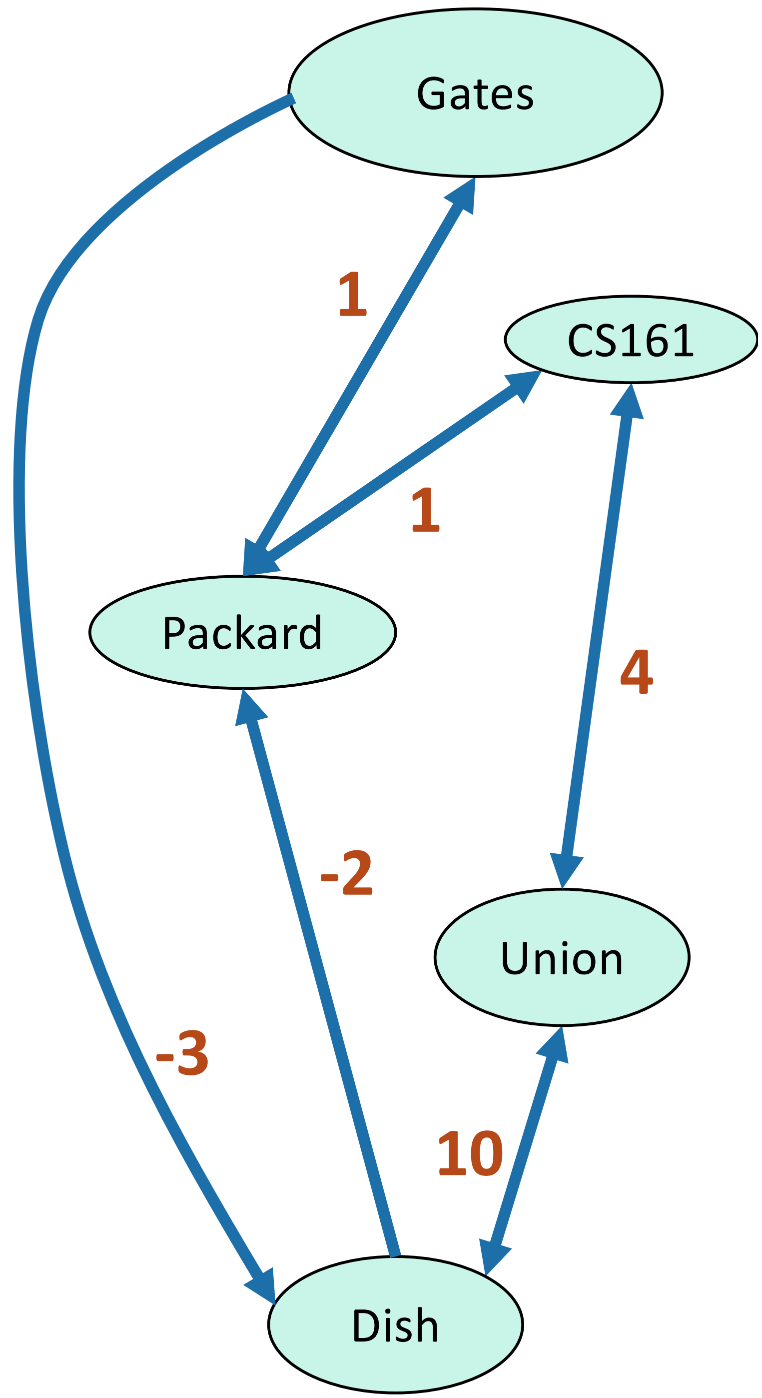
B-F with negative cycles

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	**1**	∞	∞	**-3**
$d^{(2)}$	0	**-5**	**2**	**7**	**-3**
$d^{(3)}$	**-4**	-5	**-4**	**6**	**-3**
$d^{(4)}$	-4	-5	-4	6	**-7**

But **we can tell** that it's not looking good:

	Gates	Packard	CS161	Union	Dish
$d^{(5)}$	-4	**-9**	-4	**3**	-7

Some stuff changed!



- **For** i=0,…,n-1:
  - **For** u in V:
    - **For** v in u.neighbors:
      - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v]\ ,\ d^{(i+1)}[v],\ d^{(i)}[u] + \text{edgeWeight}(u,v))$

How Bellman-Ford deals with negative cycles

- If there are no negative cycles:
  - Everything works as it should.
  - The algorithm stabilizes after n-1 rounds.
  - Note: Negative ***edges*** are okay!!
- If there are negative cycles:
  - Not everything works as it should…
    - it couldn't possibly work, since shortest paths aren't well-defined if there are negative cycles.
  - The d[v] values will keep changing.
- Solution:
  - Go one round more and see if things change.
    - If so, return NEGATIVE CYCLE ☹
  - (Pseudocode on skipped slide)

Bellman-Ford algorithm

SLIDE SKIPPED IN CLASS

Bellman-Ford*(G,s):

- $d^{(0)}[v] = \infty$ for all v in V
- $d^{(0)}[s] = 0$
- **For** i=0,…,n-1:
  - **For** u in V:
    - **For** v in u.neighbors:
      - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v]\ ,\ d^{(i+1)}[v],\ d^{(i)}[u] + \text{edgeWeight}(u,v))$
- If $d^{(n-1)} \mathrel{!}= d^{(n)}$:
  - **Return** NEGATIVE CYCLE ☹
- Otherwise, $\text{dist}(s,v) = d^{(n-1)}[v]$

Summary

It's okay if that went by fast, we'll come back to Bellman-Ford

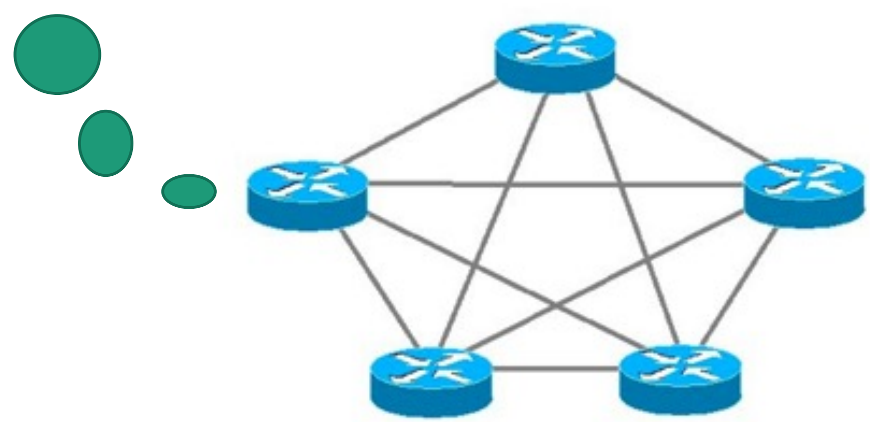
- The Bellman-Ford algorithm:
  - Finds shortest paths in weighted graphs with negative edge weights
  - runs in time O(nm) on a graph G with n vertices and m edges.
- If there are no negative cycles in G:
  - the BF algorithm terminates with $d^{(n-1)}[v] = d(s,v)$.
- If there are negative cycles in G:
  - the BF algorithm returns `negative cycle`.

Bellman-Ford is also used in practice.

- eg, Routing Information Protocol (RIP) uses something like Bellman-Ford.
  - Older protocol, not used as much anymore.

- Each router keeps a **table** of distances to every other router.
- Periodically we do a Bellman-Ford update.
- This means that if there are changes in the network, this will propagate. (maybe slowly…)

Destination	Cost to get there	Send to whom?
172.16.1.0	34	172.16.1.1
10.20.40.1	10	192.168.1.2
10.155.120.1	9	10.13.50.0



Recap: shortest paths

- BFS:
  - (+) O(n+m)
  - (-) only unweighted graphs
- Dijkstra’s algorithm:
  - (+) weighted graphs
  - (+) O(nlog(n) + m) if you implement it right.
  - (-) no negative edge weights
  - (-) very “centralized” (need to keep track of all the vertices to know which to update).
- The Bellman-Ford algorithm:
  - (+) weighted graphs, even with negative weights
  - (+) can be done in a distributed fashion, every vertex using only information from its neighbors.
  - (-) O(nm)

Next Time

- Dynamic Programming!!!

Before next time

- Pre-lecture exercise for Lecture 12
    - Remember the Fibonacci numbers from HW1?

Mini-topic (bonus slides; not on exam)

Amortized analysis!

- We mentioned this when we talked about implementing Dijkstra.

*Any sequence of d `deleteMin` calls takes time at most O(d log(n)). But some of the d may take longer and some may take less time.

- What's the difference between this notion and expected runtime?

Example

- Incrementing a binary counter n times.

0 1 10 11 100 101 110 111 1000 1001 1010 1011 1100 1101 1110 1111

1 2 1 3 1 2 1 4 1 2 1 3 1 2 1

- Say that flipping a bit is costly.
  - Above, we've noted the cost in terms of bit-flips.

Example

- Incrementing a binary counter n times.

0 1 10 11 100 101 110 111 1000 1001 1010 1011 1100 1101 1110 1111

1 2 1 3 1 2 1 4 1 2 1 3 1 2 1

- Say that flipping a bit is costly.
  - Some steps are very expensive.
  - Many are very cheap.
- ***Amortized*** over all the inputs, it turns out to be pretty cheap.
  - O(n) for all n increments.

This is different from expected runtime.

- The statement is deterministic, no randomness here.

- But it is still weaker than worst-case runtime.
  - We may need to wait for a while to start making it worth it.