Lecture 18

what we’ve done and what’s to come

Announcements

- HW8 (last one) due today

- Don’t forget about the final exam on March 16 (from 3:30pm – 6:30pm).

- Two pages of handwritten notes (front and back) allowed for the final exam.

Today

- What just happened?
  - A whirlwind tour of CS161

- What's next?
  - A few gems from future algorithms classes

It's been a fun ride…

Sorting and friends!

Data structures: BSTs and Hashing!

Graphs!

Greedy algorithms!

Dynamic Programming!

Scheduling and etc.

MinCuts and MaxFlows

Stable Matchings

Divide-and-conquer and recurrence relations

LCS, Knapsack(s)

BFS, DFS, SCCs

MSTs: Prim and Kruskal

O() and worst-case analysis

Randomized algorithms

Bellman-Ford, Floyd-Warshall

Ford-Fulkerson

Dijkstra's algorithm

What have we learned?

17 lectures in 12 slides.

General approach
to algorithm design and analysis

Can I do better?

To answer this question we need both **rigor** and **intuition**:

Algorithm designer

Plucky the Pedantic Penguin

Detail-oriented
Precise
Rigorous

Lucky the Lackadaisical Lemur

Big-picture
Intuitive
Hand-wavey

We needed more details

Does it work?
Is it fast?

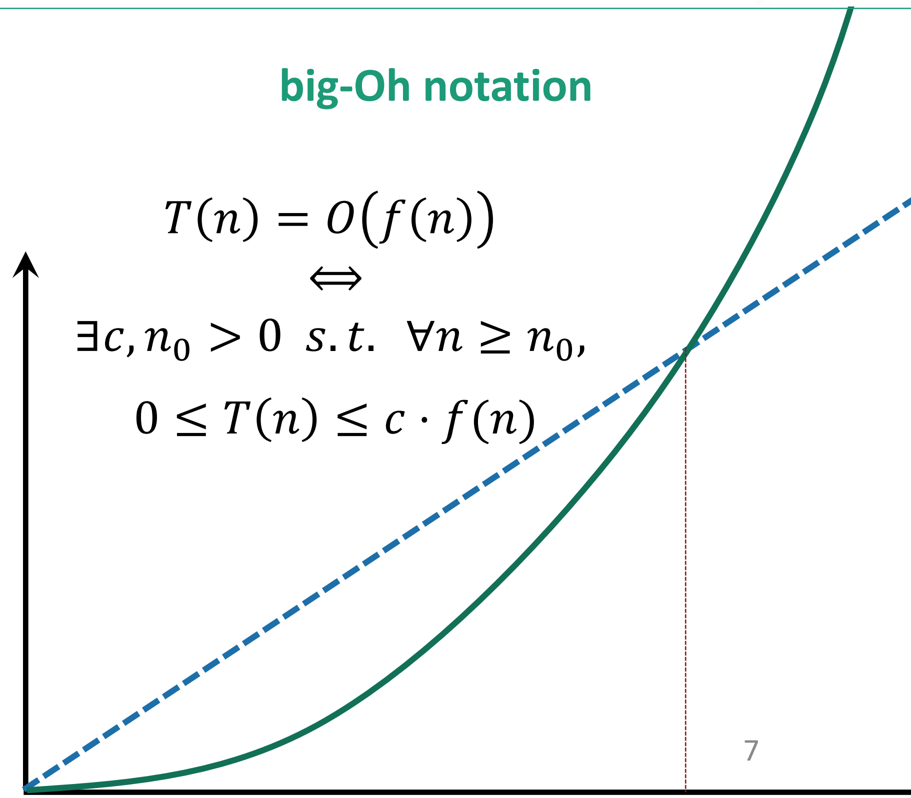
What does that mean??

Worst-case analysis

HERE IS AN INPUT!

big-Oh notation

$$T(n) = O\big(f(n)\big)$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq T(n) \leq c \cdot f(n)$$



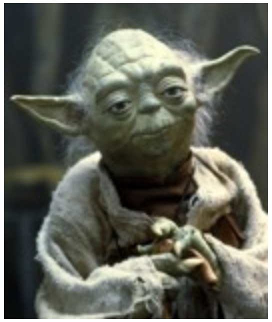
Algorithm design paradigm: divide and conquer

- Like MergeSort!
- Or Karatsuba's algorithm!
- Or SELECT!
- How do we analyze these?

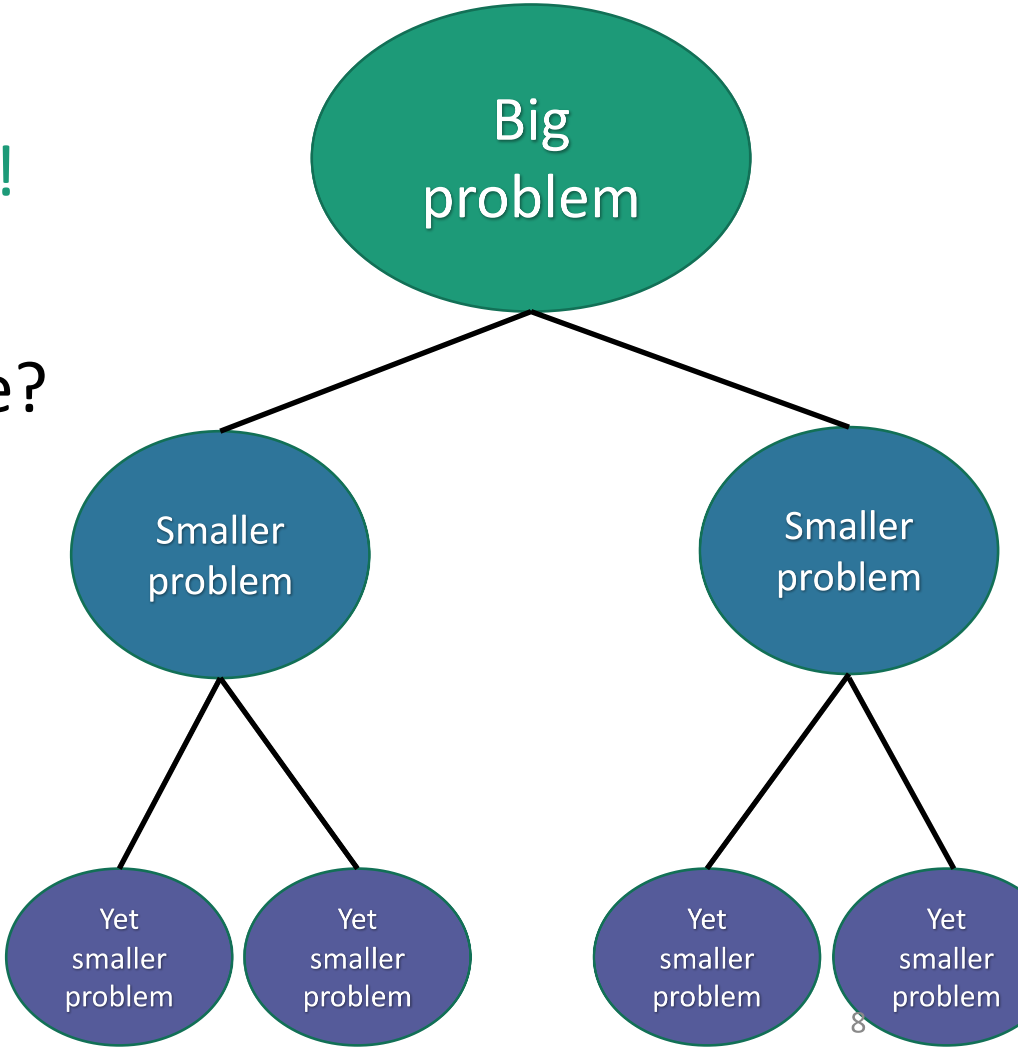
By careful analysis!

Plucky the Pedantic Penguin

Useful shortcut, the **master method** is.



Jedi master Yoda



While we're on the topic of sorting
Why not use randomness?

- We analyzed **QuickSort!**
- Still worst-case input, but we use randomness after the input is chosen.
- Always correct, usually fast.
  - This is a Las Vegas algorithm

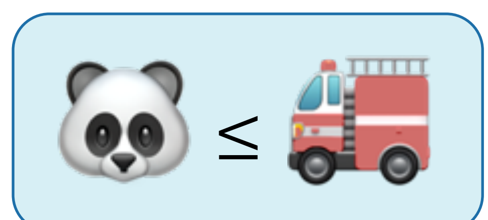
All this sorting is making me wonder…

Can we do better?

- Depends on who you ask:

- **RadixSort** takes time O(n) if the objects are, for example, small integers!

- Can't do better in a **comparison-based** model.



beyond sorted arrays/linked lists:
Binary Search Trees!

- Useful data structure!
- Especially the self-balancing ones!

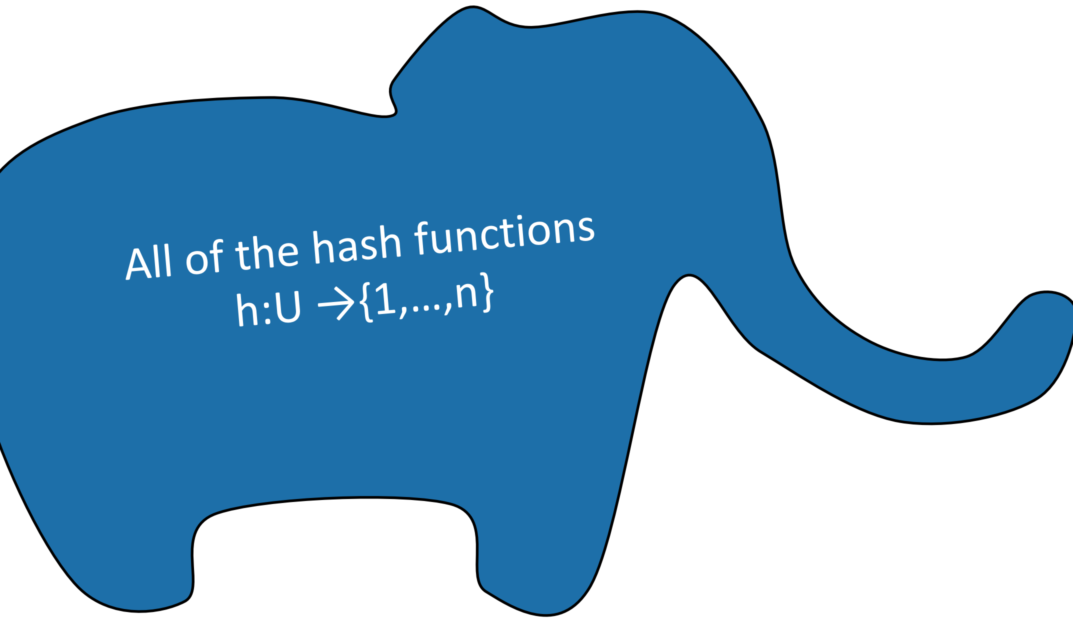
Red-Black tree!

Maintain balance by stipulating that **black nodes** are balanced, and that there *aren't too many* **red nodes.**

It's just good sense!

Another way to store things

Hash tables!



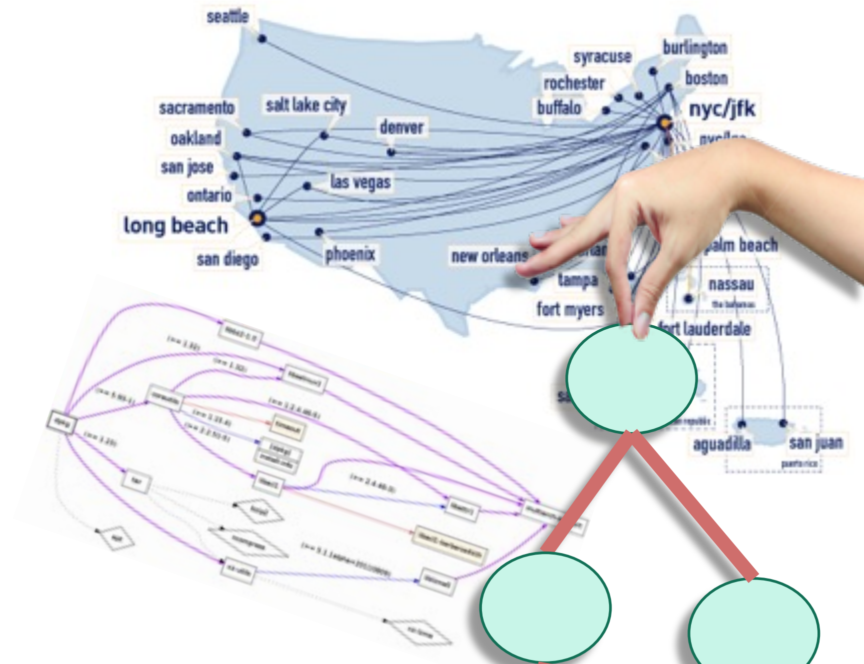
Some buckets

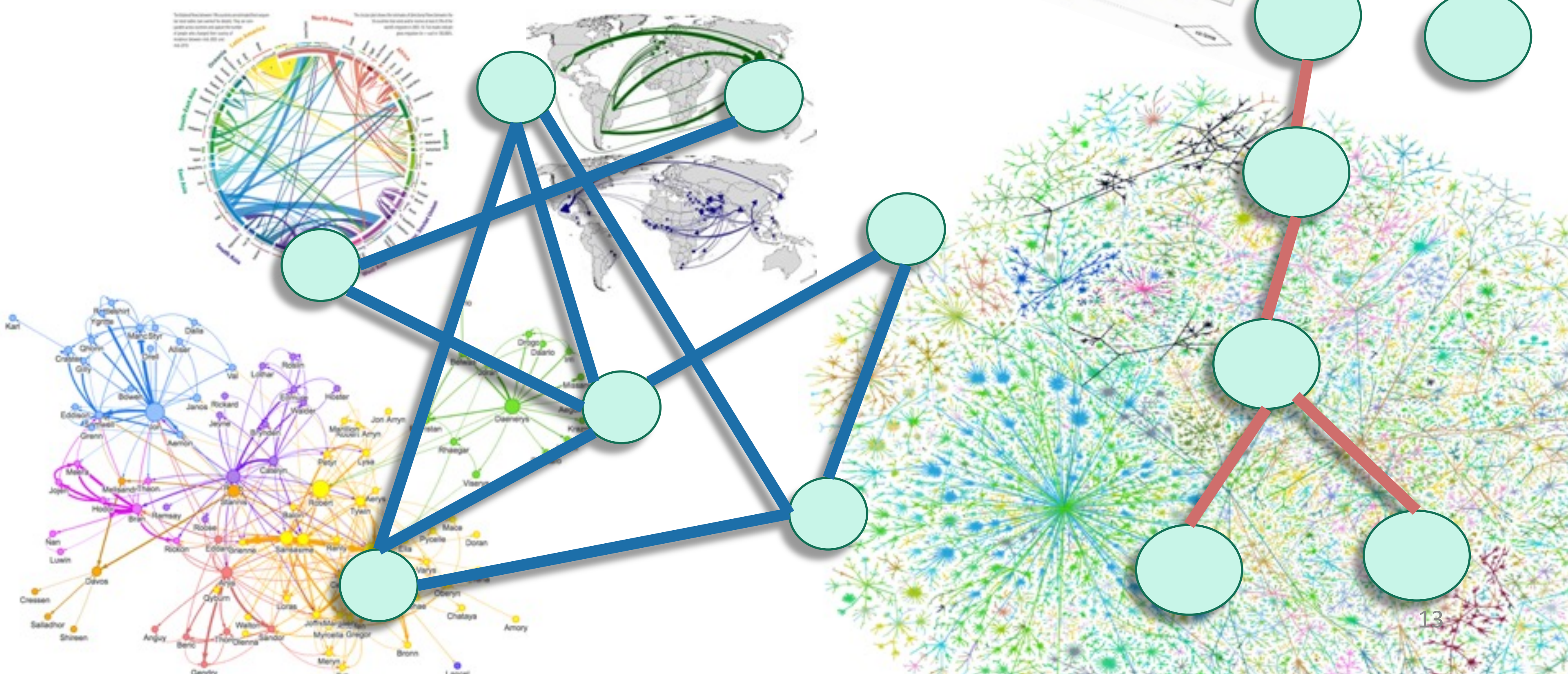
Choose h randomly from a universal hash family.

It’s better if the hash family is small! Then it takes less space to store h.

OMG GRAPHS

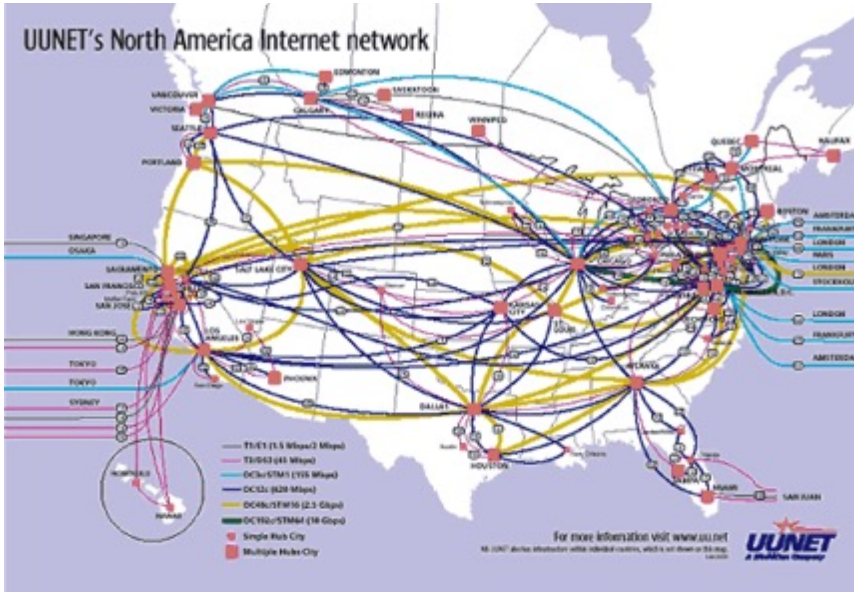
- BFS, DFS, and applications!
- SCCs, Topological sorting, …





A fundamental graph problem: shortest paths

- E.g., transit planning, packet routing, …
- Dijkstra!
- Bellman-Ford!
- Floyd-Warshall!



```
DN0a22a0e3:~ mary$ traceroute -a www.ethz.ch
traceroute to www.ethz.ch (129.132.19.216), 64 hops max, 52 byte packets
 1  [AS0] 10.34.160.2 (10.34.160.2)  38.168 ms  31.272 ms  28.841 ms
 2  [AS0] cwa-vrtr.sunet (10.21.196.28)  33.769 ms  28.245 ms  24.373 ms
 3  [AS32] 171.66.2.229 (171.66.2.229)  24.468 ms  20.115 ms  23.223 ms
 4  [AS32] hpr-svl-rtr-vlan8.sunet (171.64.255.235)  24.644 ms  24.962 ms  17.453 ms
 5  [AS2152] hpr-svl-hpr2--stan-ge.cenic.net (137.164.27.161)  22.129 ms  4.902 ms  3.642 ms
 6  [AS2152] hpr-lax-hpr3--svl-hpr3-100ge.cenic.net (137.164.25.73)  12.125 ms  43.361 ms  32.3
 7  [AS2152] hpr-i2--lax-hpr2-r&e.cenic.net (137.164.26.201)  40.174 ms  38.399 ms  34.499 ms
 8  [AS0] et-4-0-0.4079.sdn-sw.lasv.net.internet2.edu (162.252.70.28)  46.573 ms  23.926 ms  17
 9  [AS0] et-5-1-0.4079.rtsw.salt.net.internet2.edu (162.252.70.31)  30.424 ms  25.770 ms  23.1
10  [AS0] et-4-0-0.4079.sdn-sw.denv.net.internet2.edu (162.252.70.8)  47.454 ms  57.273 ms  73.
11  [AS0] et-4-1-0.4079.rtsw.kans.net.internet2.edu (162.252.70.11)  70.825 ms  67.809 ms  62.1
12  [AS0] et-4-1-0.4070.rtsw.chic.net.internet2.edu (198.71.47.206)  77.937 ms  57.421 ms  63.6
13  [AS0] et-0-1-0.4079.sdn-sw.ashb.net.internet2.edu (162.252.70.60)  77.682 ms  71.993 ms  73
14  [AS0] et-4-1-0.4079.rtsw.wash.net.internet2.edu (162.252.70.65)  71.565 ms  74.988 ms  71.0
15  [AS21320] internet2-gw.mx1.lon.uk.geant.net (62.40.124.44)  154.926 ms  145.606 ms  145.872
16  [AS21320] ae0.mx1.lon2.uk.geant.net (62.40.98.79)  146.565 ms  146.604 ms  146.801 ms
17  [AS21320] ae0.mx1.par.fr.geant.net (62.40.98.77)  153.289 ms  184.995 ms  152.682 ms
18  [AS21320] ae2.mx1.gen.ch.geant.net (62.40.98.153)  160.283 ms  160.104 ms  164.147 ms
19  [AS21320] swice1-100ge-0-3-0-1.switch.ch (62.40.124.22)  162.068 ms  160.595 ms  163.095 ms
20  [AS559] swizh1-100ge-0-1-0-1.switch.ch (130.59.36.94)  165.824 ms  164.216 ms  163.983 ms
21  [AS559] swiez3-100ge-0-1-0-4.switch.ch (130.59.38.109)  164.269 ms  164.370 ms  163.929 ms
22  [AS559] rou-gw-lee-tengig-to-switch.ethz.ch (192.33.92.1)  164.082 ms  170.645 ms  165.372
23  [AS559] rou-fw-rz-rz-gw.ethz.ch (192.33.92.169)  164.773 ms  165.193 ms  172.158 ms
```

Bellman-Ford and Floyd-Warshall were examples of…

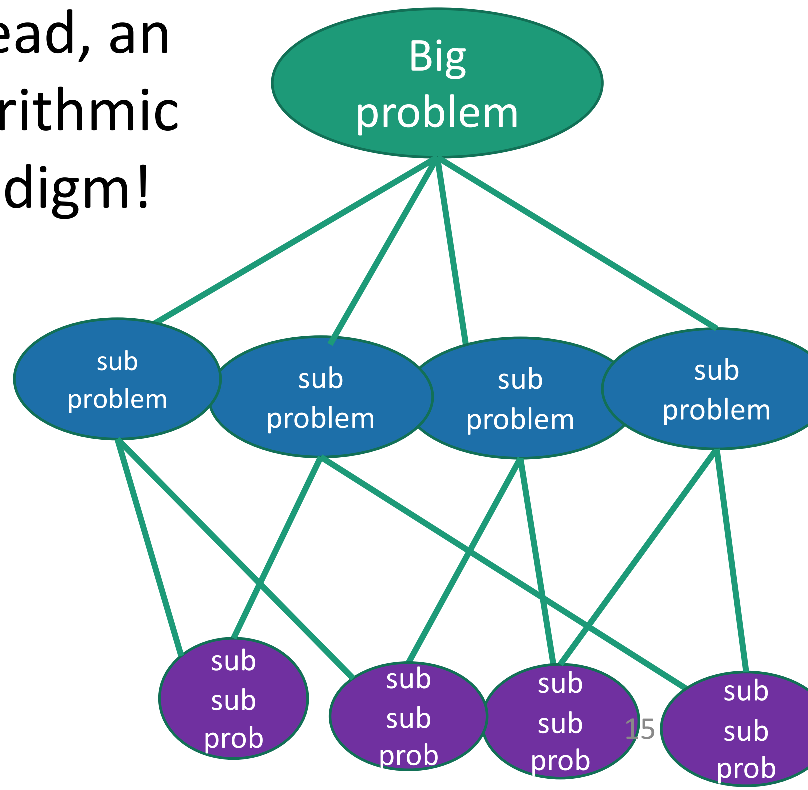
Dynamic Programming!

We saw many other examples, including Longest Common Subsequence and Knapsack Problems.

- Not programming in an action movie.

Instead, an algorithmic paradigm!

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Steps 3-5:** Use dynamic programming: fill in a table to find the answer!



Sometimes we can take even better advantage of optimal substructure…with

Greedy algorithms

- Make a series of choices, and commit!

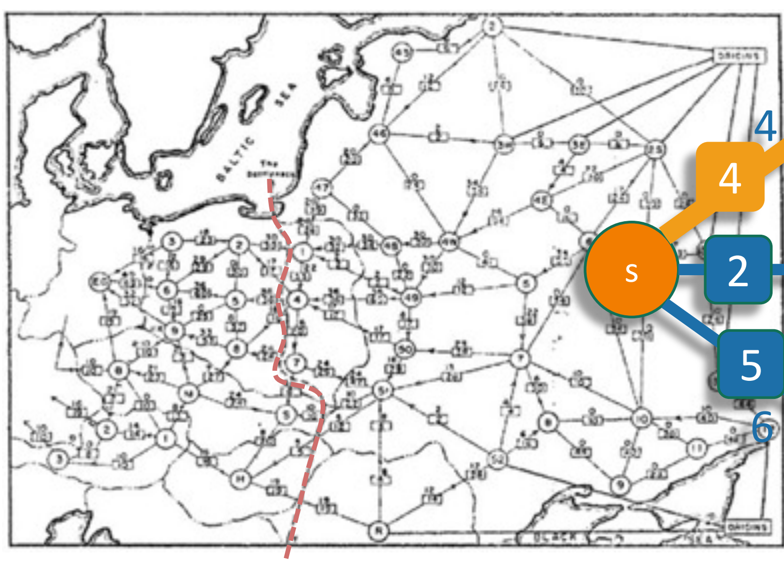
- Intuitively we want to show that our greedy choices never rule out success.
- Rigorously, we usually analyzed these by induction.
- Examples!
  - Activity Selection
  - Job Scheduling
  - Huffman Coding
  - **Minimum Spanning Trees**

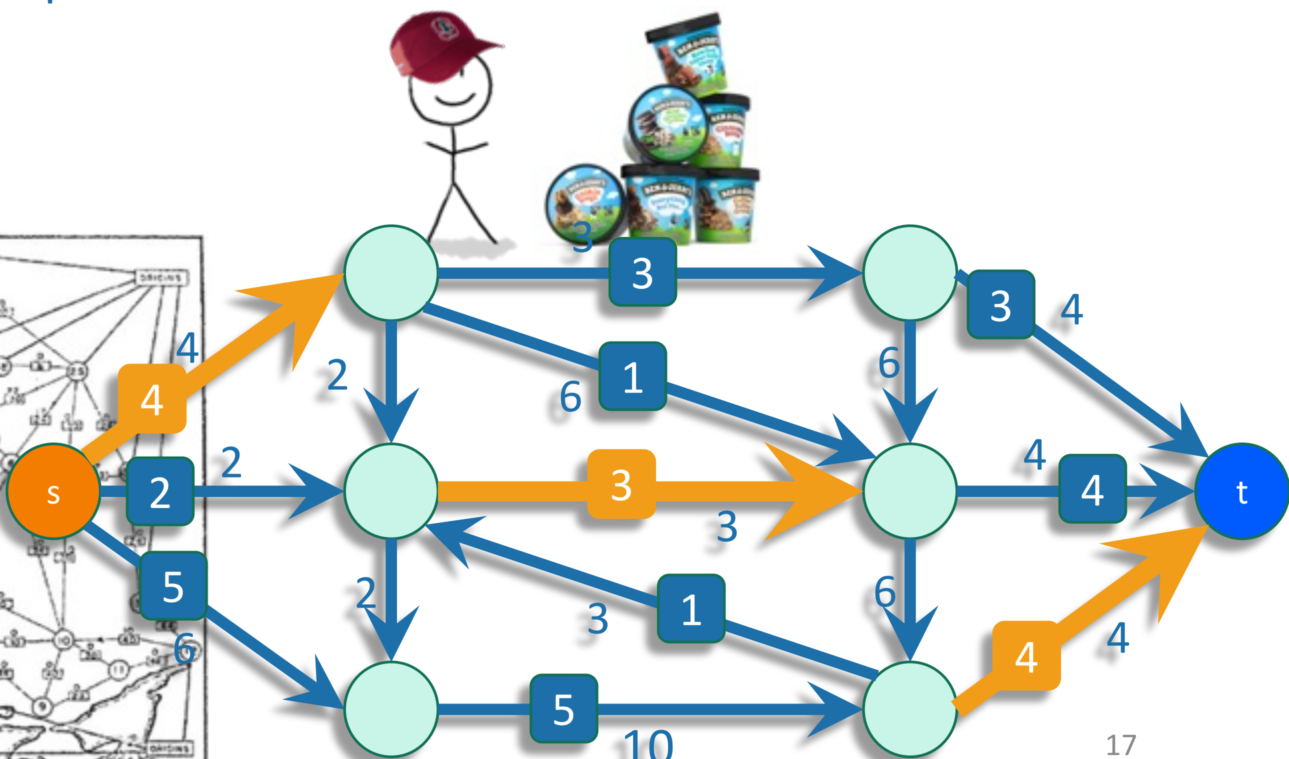
Prim's algorithm: greedily grow a tree

Kruskal's algorithm: greedily grow a forest

Cuts and flows

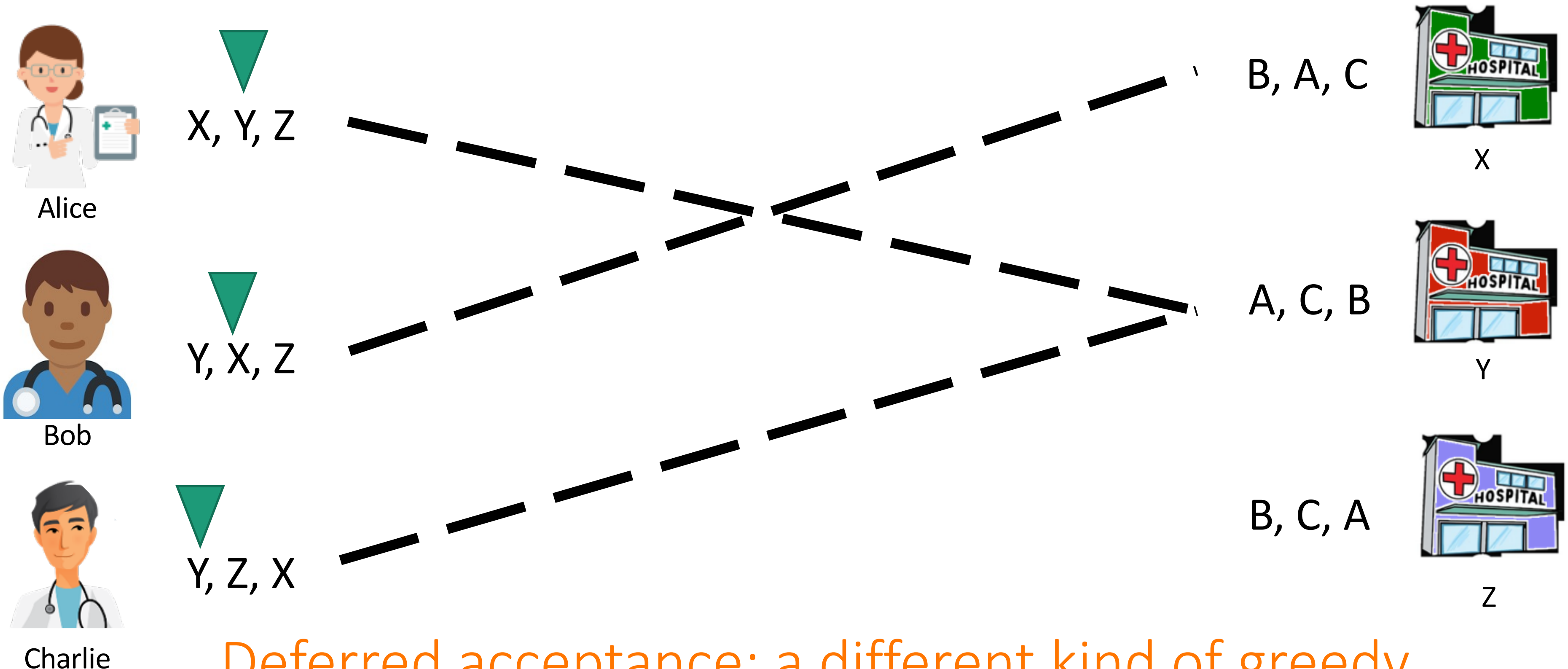
- Minimum s-t cut:
  - is the same as maximum s-t flow!
  - Ford-Fulkerson can find them!
    - useful for routing
    - also assignment problems





Stable matching

How to convince actors to use our matching?
Where do preferences come from?
Are the incentives set correctly?



Deferred acceptance: a different kind of greedy algorithm, this time with recourse.

And now we're here

What have we learned?

- A few algorithm design paradigms:
  - Divide and conquer, dynamic programming, greedy
- A few analysis tools:
  - Worst-case analysis, asymptotic analysis, recurrence relations, probability tricks, proofs by induction
- A few common objects:
  - Graphs, arrays, trees, hash functions
- A LOT of examples!

What have we learned?

We've filled out a toolbox

- Tons of examples give us intuition about what algorithmic techniques might work when.
- The technical skills make sure our intuition works out.

But there's lots more out there

- What's next???

A taste of what's to come

- CS154 – Introduction to Automata and Complexity
- CS163 – The Practice of Theory Research
- CS166 – Data Structures
- CS168 – The Modern Algorithmic Toolbox
- MS&E 212 – Combinatorial Optimization
- CS250 – Error Correcting Codes
- CS252 – Analysis of Boolean Functions
- CS254 – Computational Complexity
- CS255 – Introduction to Cryptography
- CS259Q – Quantum Computing
- CS260 – Geometry of Polynomials in Algorithm Design
- CS261 – Optimization and Algorithmic Paradigms
- CS263 – Counting and Sampling
- CS265 – Randomized Algorithms
- CS269O – Introduction to Optimization Theory
- MS&E 316 – Discrete Mathematics and Algorithms
- CS352 – Pseudorandomness
- CS366 – Computational Social Choice
- CS368 – Algorithmic Techniques for Big Data
- EE364A/B – Convex Optimization I and II

findSomeTheoryCourses():

- go to theory.stanford.edu
- Click on "People"
- Look at what we're teaching!

...and many many more!

Today

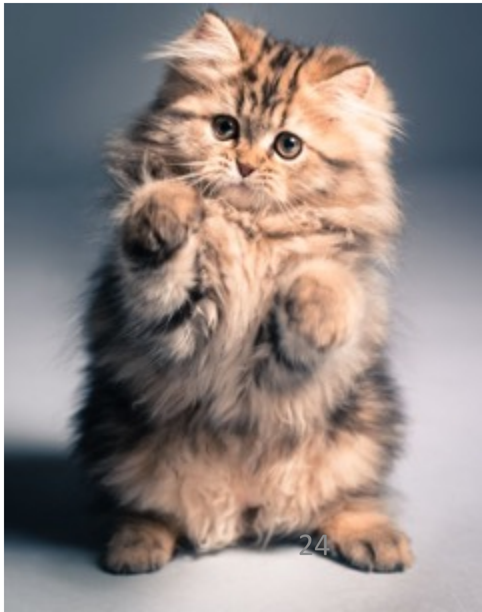
A few gems

- Linear programming

- Random projections

- Low-degree polynomials

This will be fluffy, without much detail – take more CS theory classes for more detail!



Linear Programming

- This is a fancy name for optimizing a linear function subject to linear constraints.
- For example:

Maximize $x + y$ subject to

$$x \geq 0$$
$$y \geq 0$$
$$4x + y \leq 2$$
$$x + 2y \leq 1$$

- It turns out the be an extremely general problem.

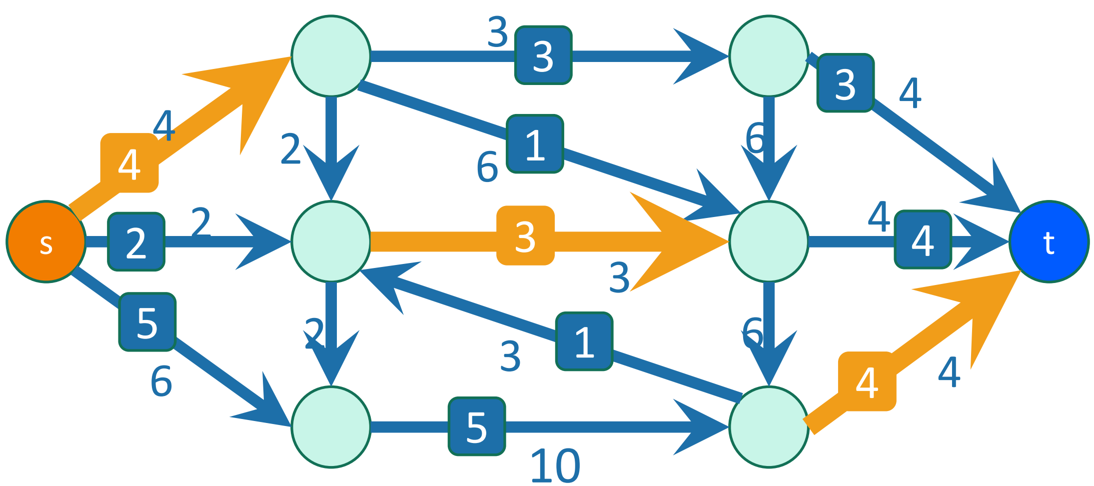
We've already seen an example!

Maximize

the sum of the flows leaving s

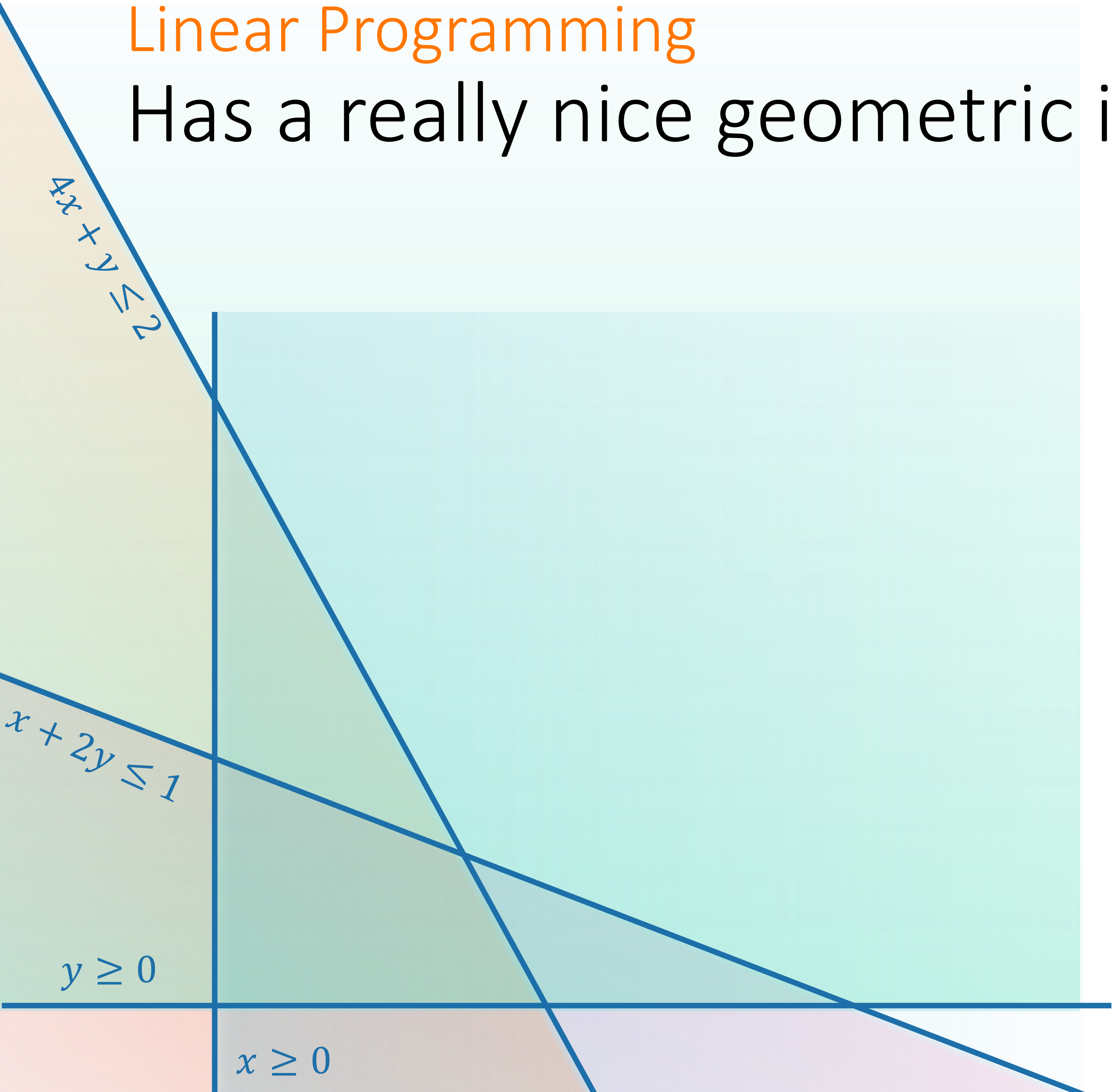
subject to

- None of the flows are bigger than the edge capacities
- At every vertex, stuff going in = stuff going out.



Linear Programming

Has a really nice geometric intuition



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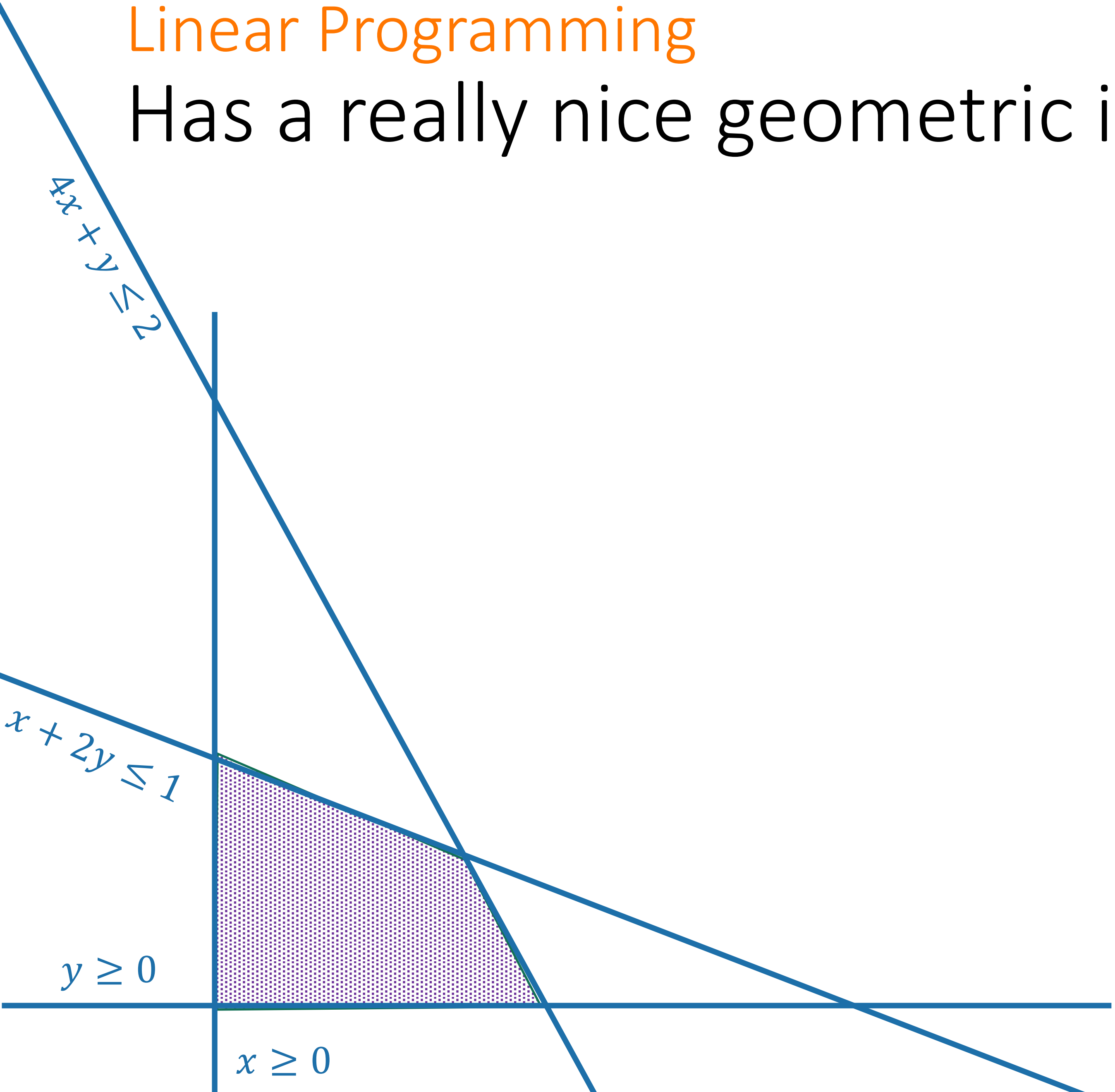

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Linear Programming

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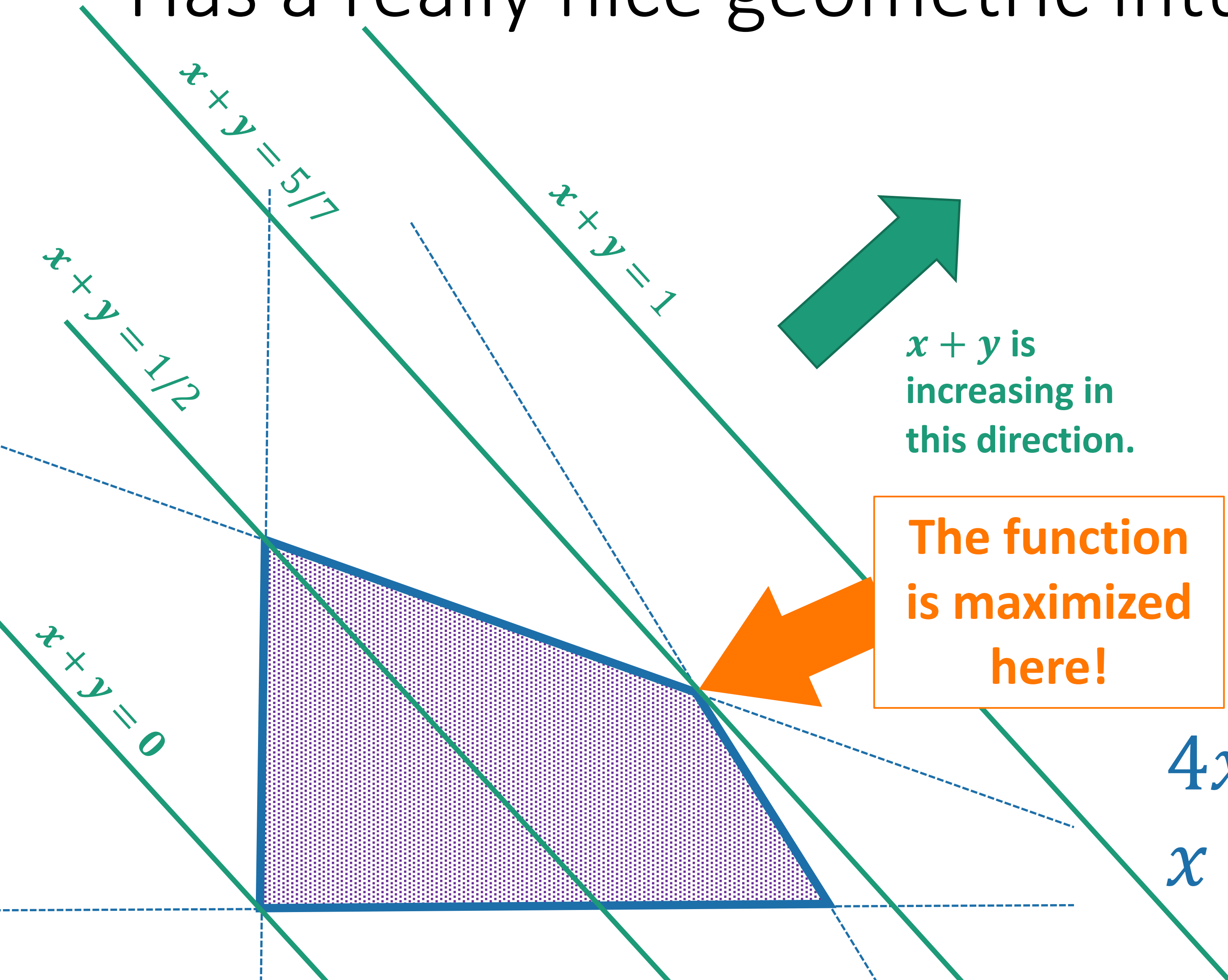


Maximize

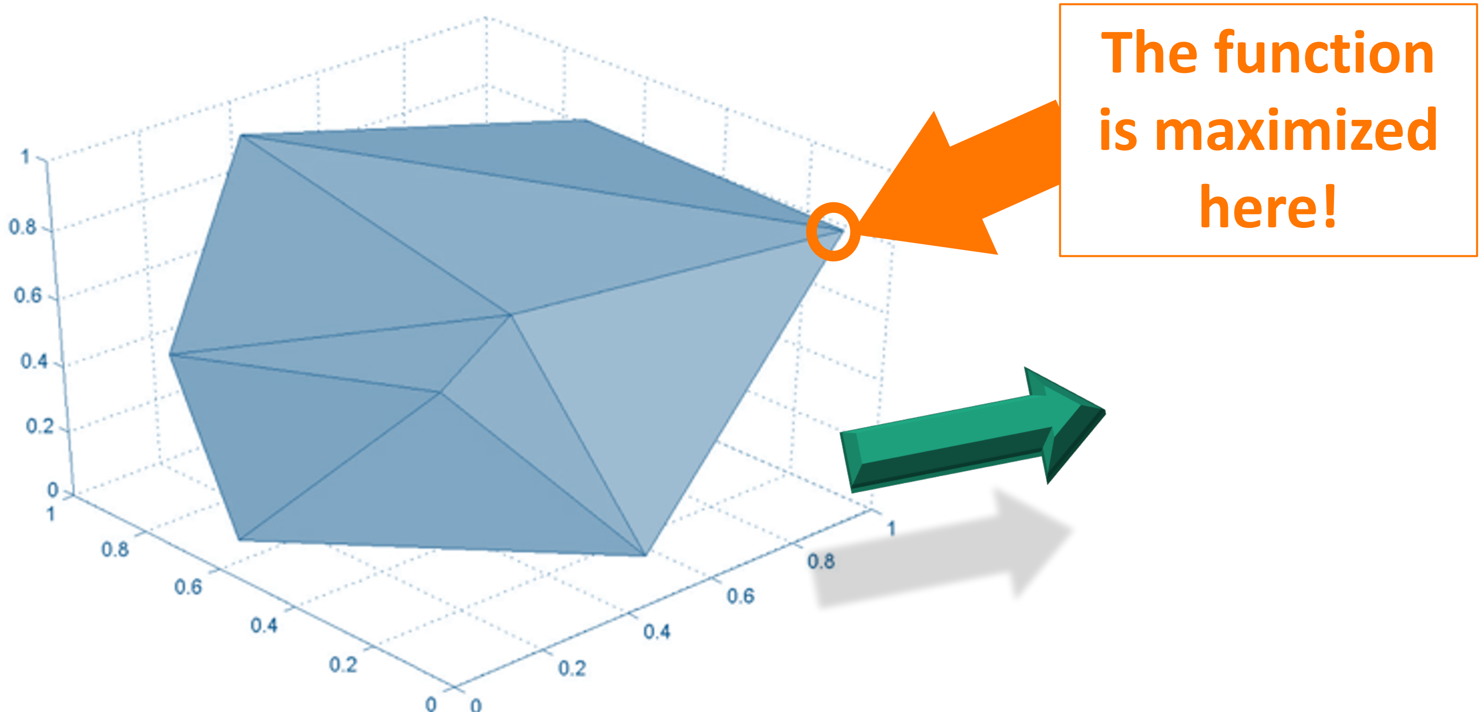
$$x+y$$

subject to

$$x \geq 0$$
$$y \geq 0$$
$$4x+y \leq 2$$
$$x+2y \leq 1$$

In general

- The constraints define a **polytope**
- The function defines a **direction**
- We just want to find the vertex that is furthest in that direction.



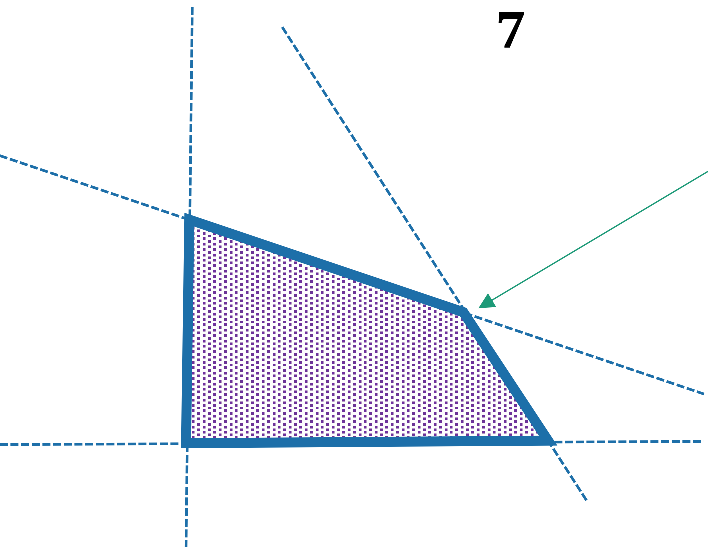
Duality

How do we know we have an optimal solution?

I claim that the optimum is 5/7.

Proof: say x and y satisfy the constraints.

- $x + y = \frac{1}{7}(4x + y) + \frac{3}{7}(x + 2y)$
- $\leq \frac{1}{7} \cdot 2 + \frac{3}{7} \cdot 1$
- $= \frac{5}{7}$



You can check this point has value 5/7...but how would we prove it's optimal other than by eyeballing it?

Maximize

$$x + y$$

subject to

$$x \geq 0$$
$$y \geq 0$$
$$4x + y \leq 2$$
$$x + 2y \leq 1$$

cute, but

How did you come up with 1/7, 3/7?

I claim that the optimum is 5/7.

Proof: say x and y satisfy the constraints.

- $\boldsymbol{x + y \leq \square(4x + y) + \square(x + 2y)}$
- $\boldsymbol{\leq \square \cdot 2 + \square\, 1}$
- $\boldsymbol{= \square}$

- I want to choose things to put here
- So that I minimize this
- Subject to these things

Maximize

$$x + y$$

subject to

$$x \geq 0$$
$$y \geq 0$$
$$4x + y \leq 2$$
$$x + 2y \leq 1$$

Note: it's not immediately obvious how to turn that into a linear program, this is just meant to convince you that it's plausible.

That's a linear program!

In this case the dual is:
min $2w + z$ s.t. $w, z \geq 0$,
$4w + z \geq 1$ and $w + 2z \geq 1$

- How did I find those special values 1/7, 3/7?
- I solved some linear program.
- It's called the dual program.

Minimize the upper bound you get, subject to the proof working.

Maximize stuff subject to stuff

Primal

The optimal values are the same!

Minimize other stuff subject to other stuff

Dual

We've actually already seen this too

The Min-Cut Max-Flow Theorem!

Maximize the sum of the flows leaving s

s.t

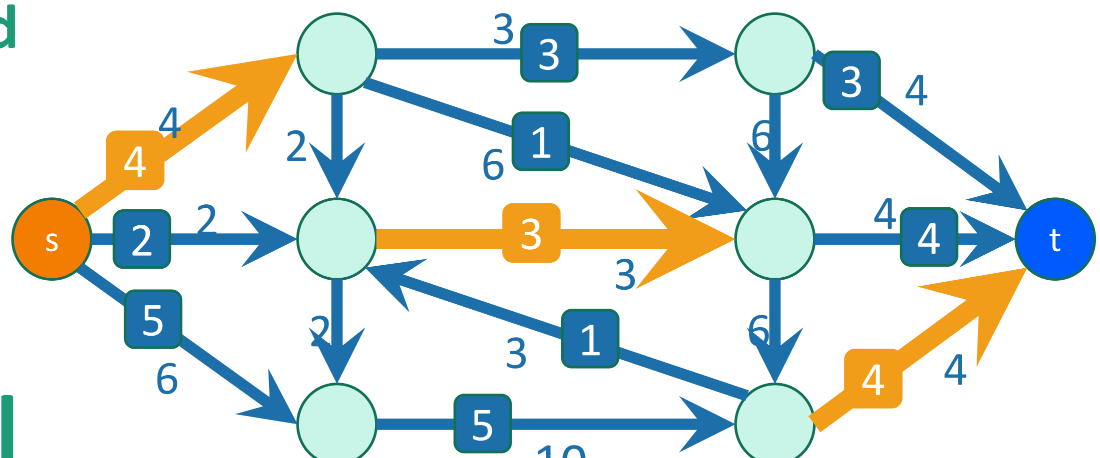
All the flow constraints are satisfied

The optimal values are the same!

Minimize the sum of the capacities on a cut

s.t.

it's a legit cut



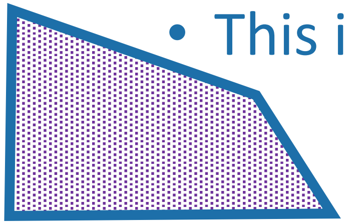
Primal

Dual

LPs and Duality are really powerful

- This general phenomenon shows up all over the place
  - Min-Cut Max-Flow is a special case.

- Duality helps us reason about an optimization problem
  - The dual provides a **certificate** that we've solved the primal.
  - E.g., if you have a cut and a flow with the same value, you must have found a max flow and a min cut.
- We can solve LPs quickly!
  - For example, by intelligently bouncing around the vertices of the feasible region.
  - This is an **extremely powerful algorithmic primitive**.



Today

A few gems

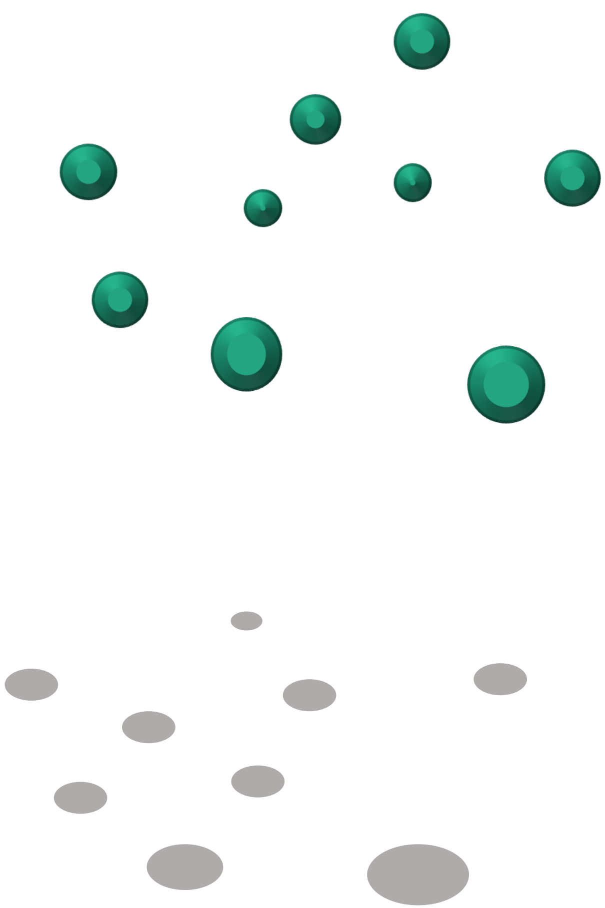
- Linear programming

- Random projections

- Low-degree polynomials

A very useful trick

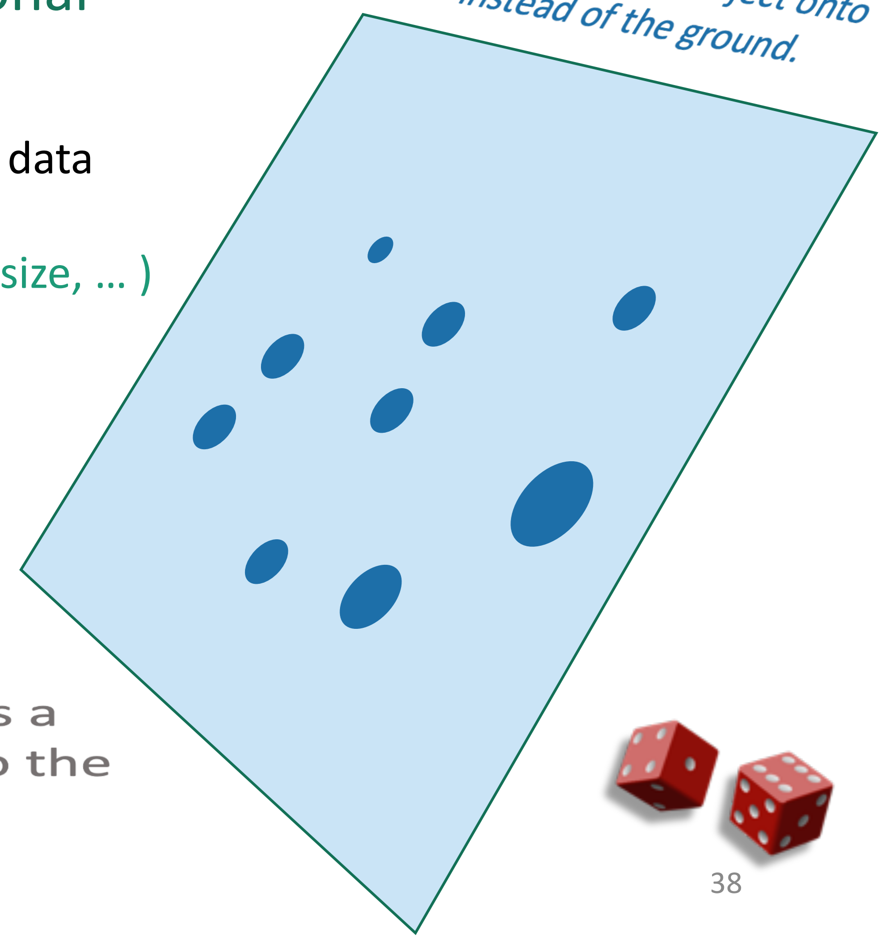
Take a random projection and hope for the best.



High-dimensional set of points

For example, each data point is a vector
(age, height, shoe size, …)

Choose a random subspace to project onto instead of the ground.



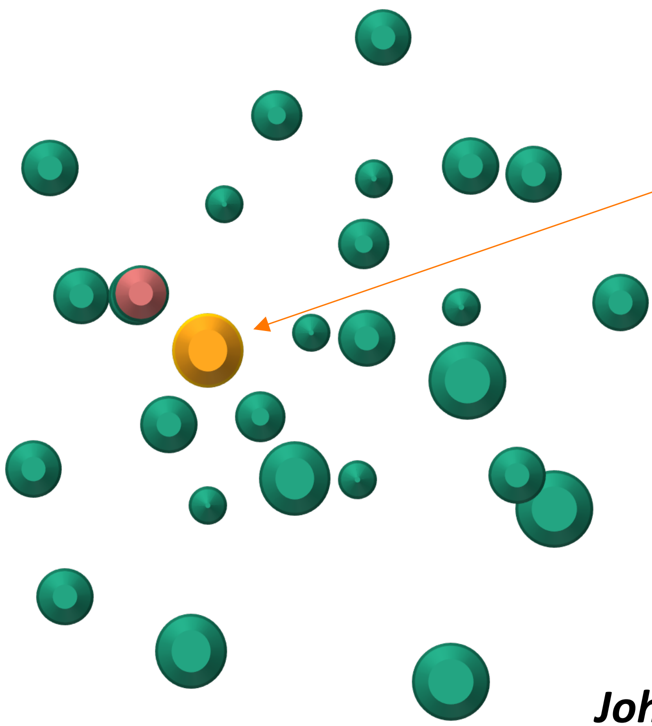
Their shadow is a projection onto the ground.

Why would we do this?

- High dimensional data takes a long time to process.
- Low dimensional data can be processed quickly.
- **“THEOREM”:** Random projections approximately preserve properties of data that you care about.

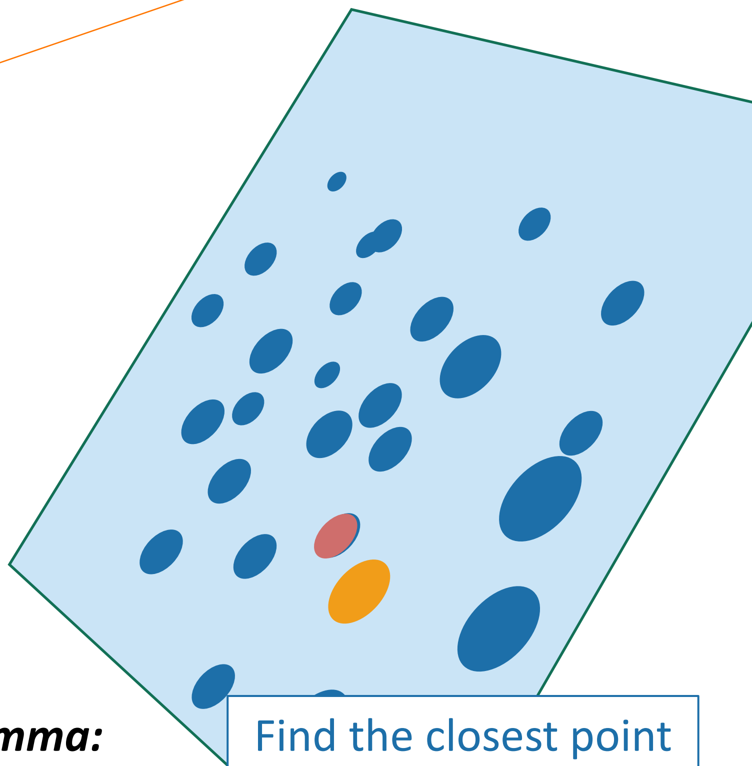
Example: nearest neighbors

- I want to find which point is closest to **this one**.



That takes a really long time in high dimensions.

Johnson-Lindenstrauss Lemma:
Euclidean distance is approximately preserved by random projections.



Find the closest point down here, you're probably pretty correct.

Another example:

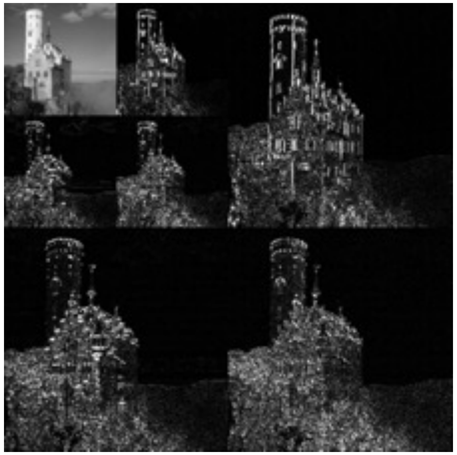
Compressed Sensing

- Start with a sparse vector
  - Mostly zero or close to zero

(**5** , 0 , 0 , 0 ,0 , 0.01 , 0.01 , **5.8** , **32** , **14** , 0 , 0 , 0 , **12** , 0 , 0 , **5** , 0 , .03)

- For example:

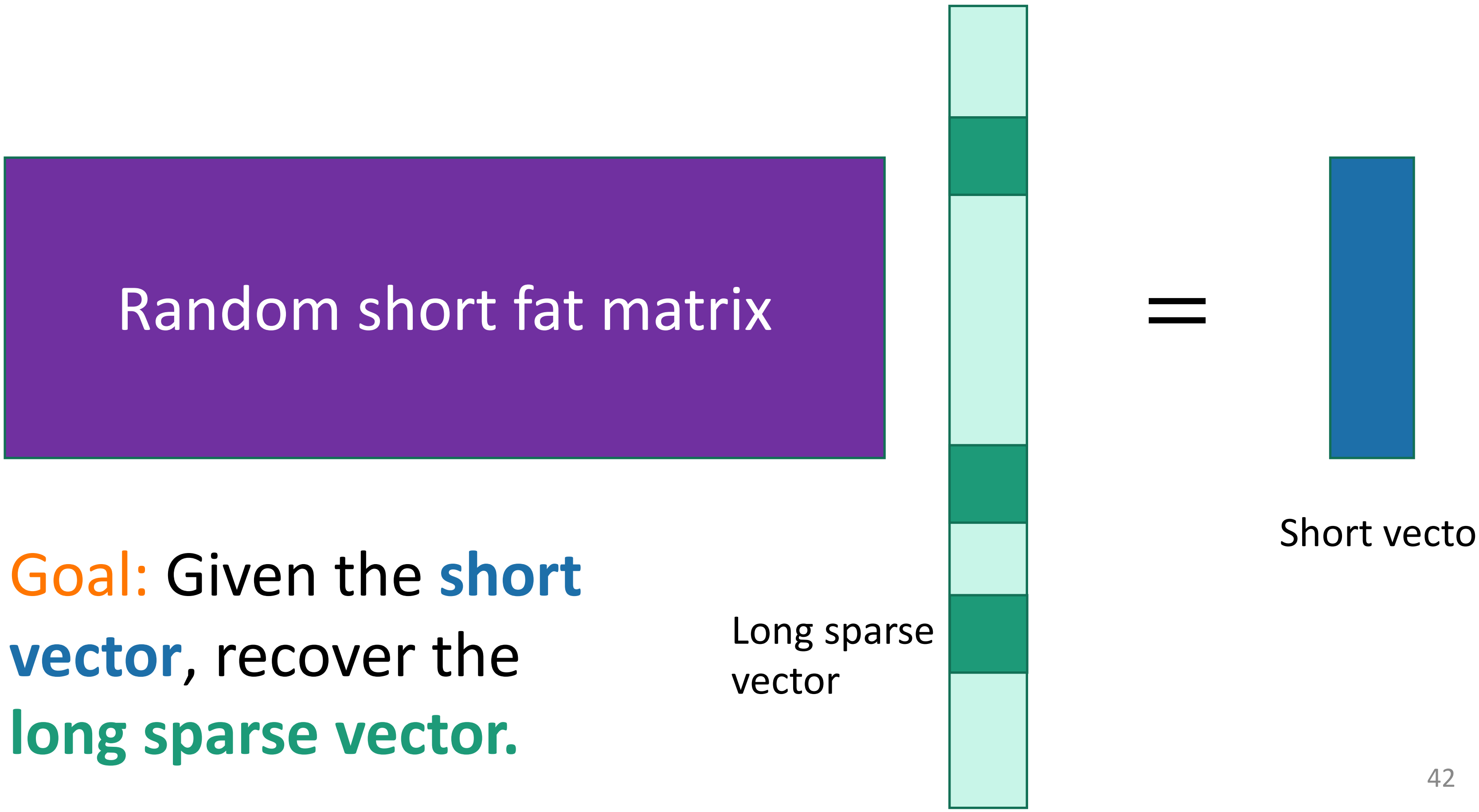
This image is sparse



This image is sparse after I take a wavelet transform.

Compressed sensing continued

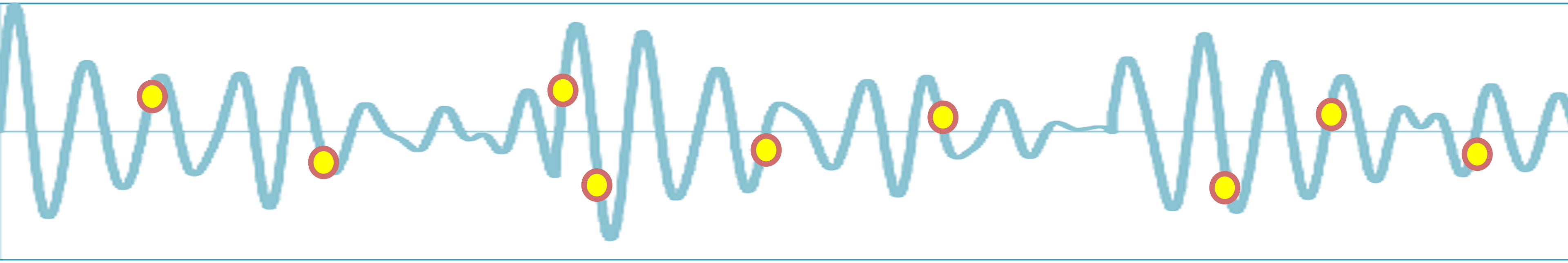
- Take a random projection of that sparse vector:



Goal: Given the **short vector**, recover the **long sparse vector.**

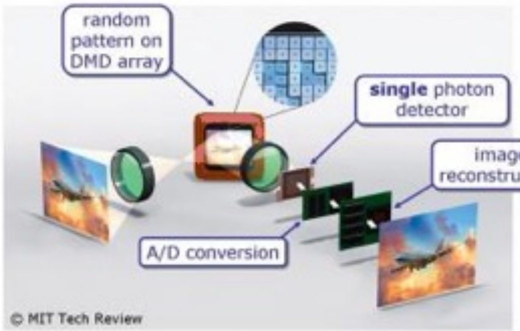
Why would I want to do that?

- Image compression and signal processing
- Especially when you **never have space to store the whole sparse vector to begin with**.

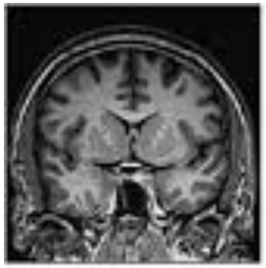


Randomly sampling (in the time domain) a signal that is sparse in the Fourier domain.

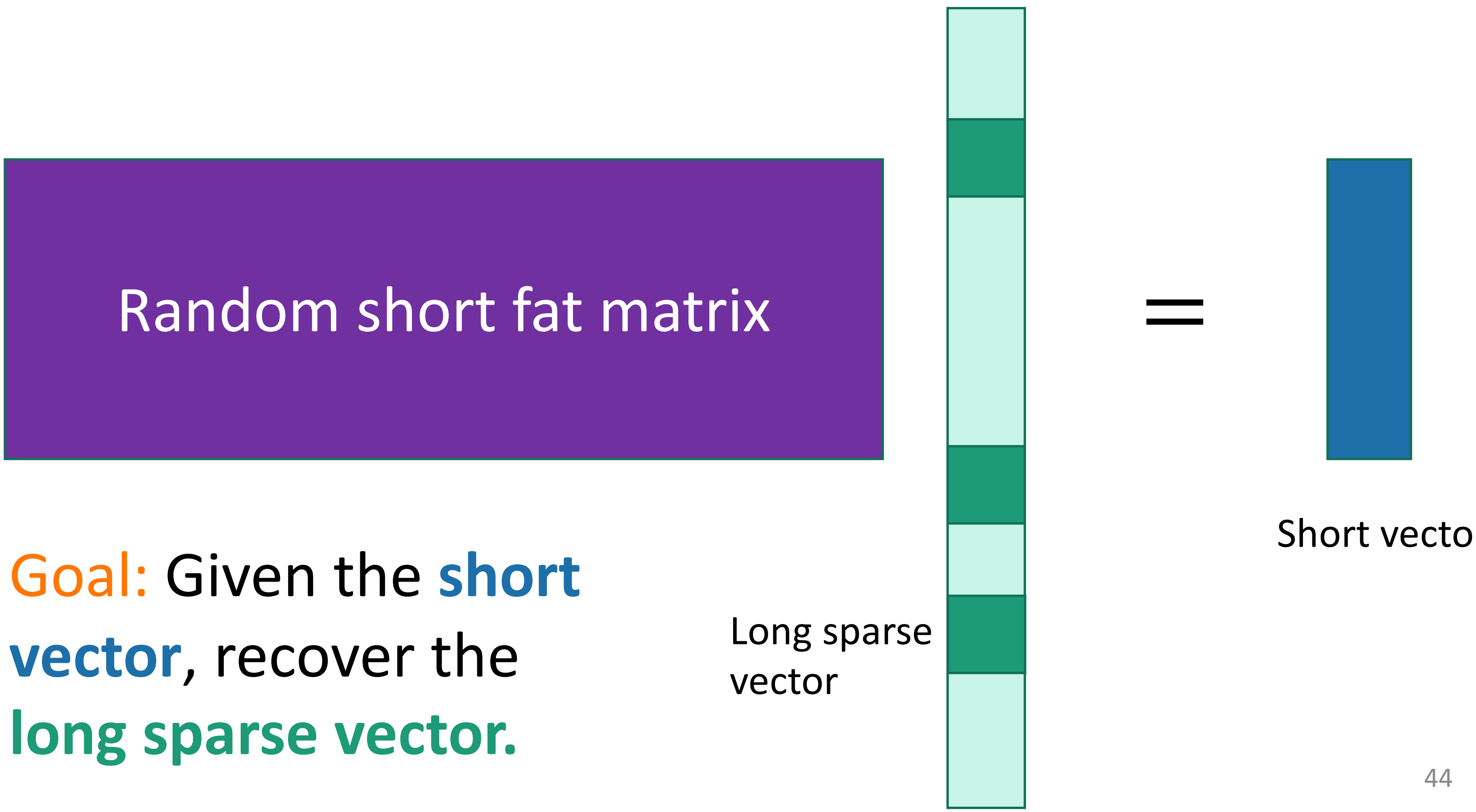
Random measurements in an fMRI means you spend less time inside an fMRI



A “single pixel camera” is a thing.



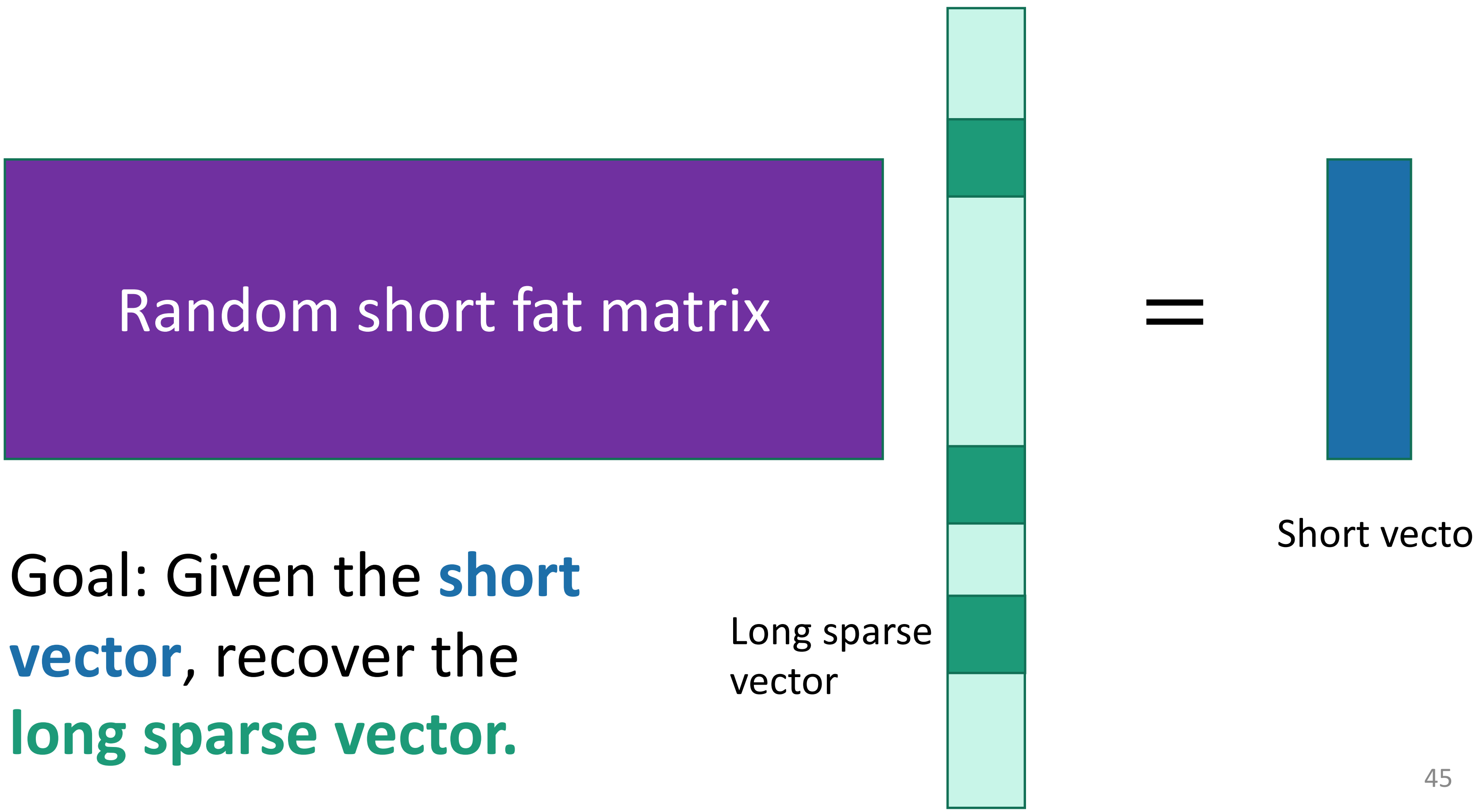
All examples of this:



Goal: Given the **short vector**, recover the **long sparse vector.**

But why should this be possible?

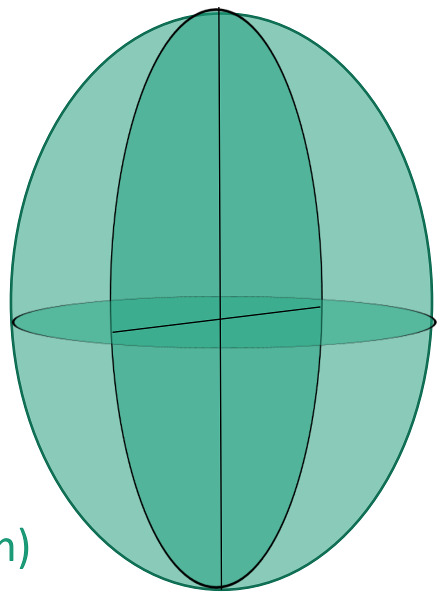
- There are tons of long vectors that map to the short vector!



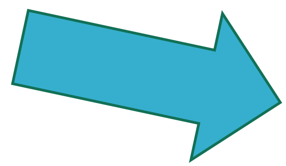
Goal: Given the **short vector**, recover the **long sparse vector.**

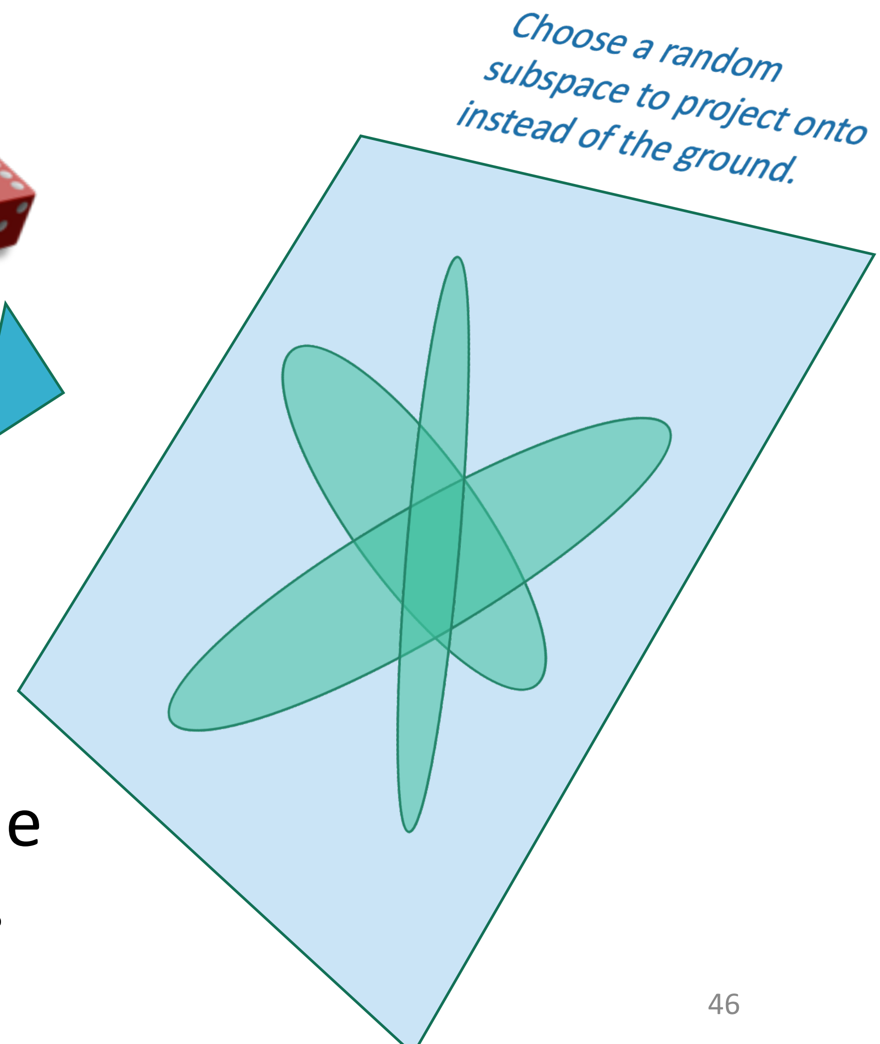
Back to the geometry

All of the sparse vectors



(Infinitely many of them)

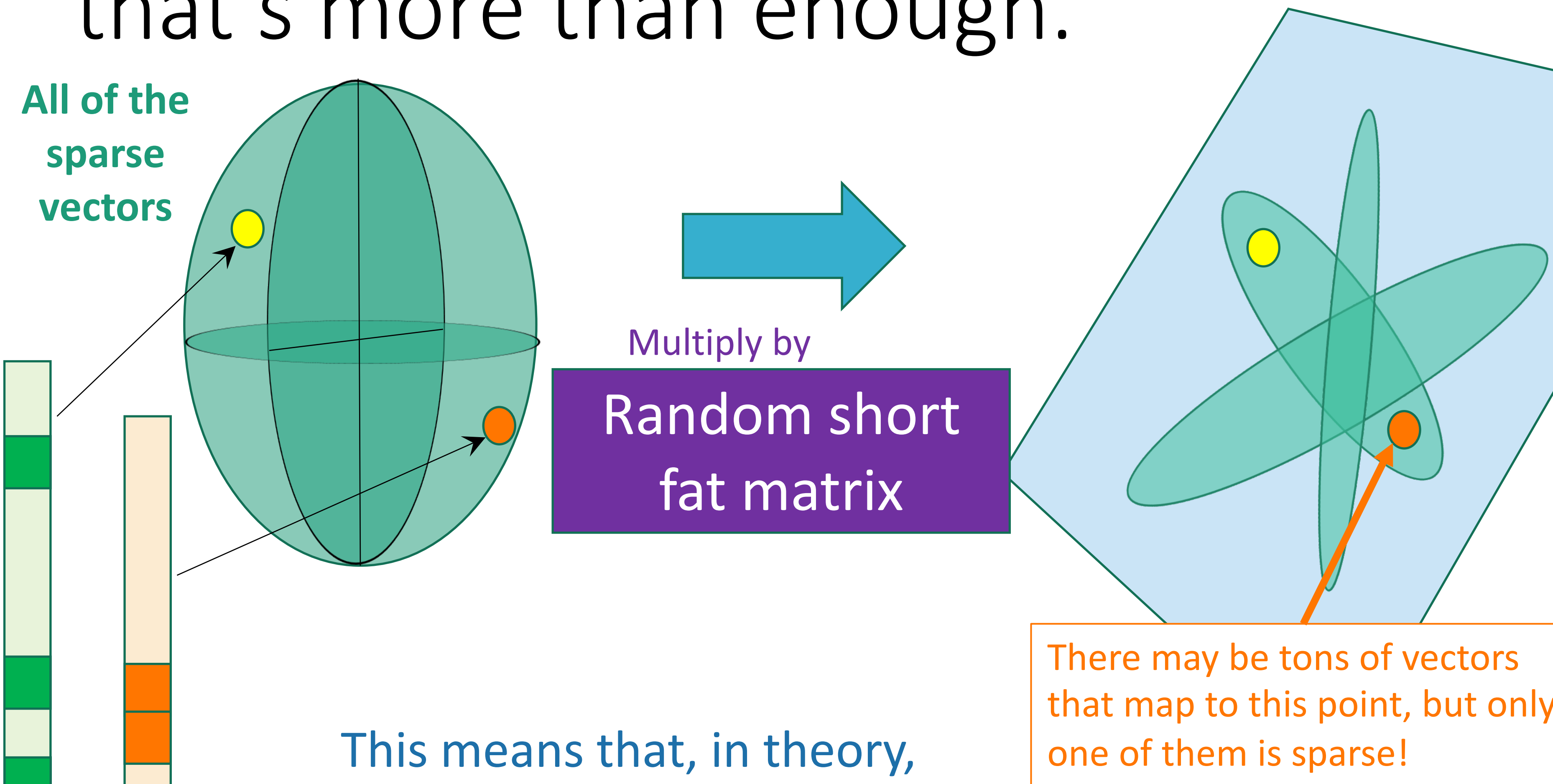




Theorem:

random projections preserve the geometry of sparse vectors too.

If we don’t care about algorithms, that’s more than enough.

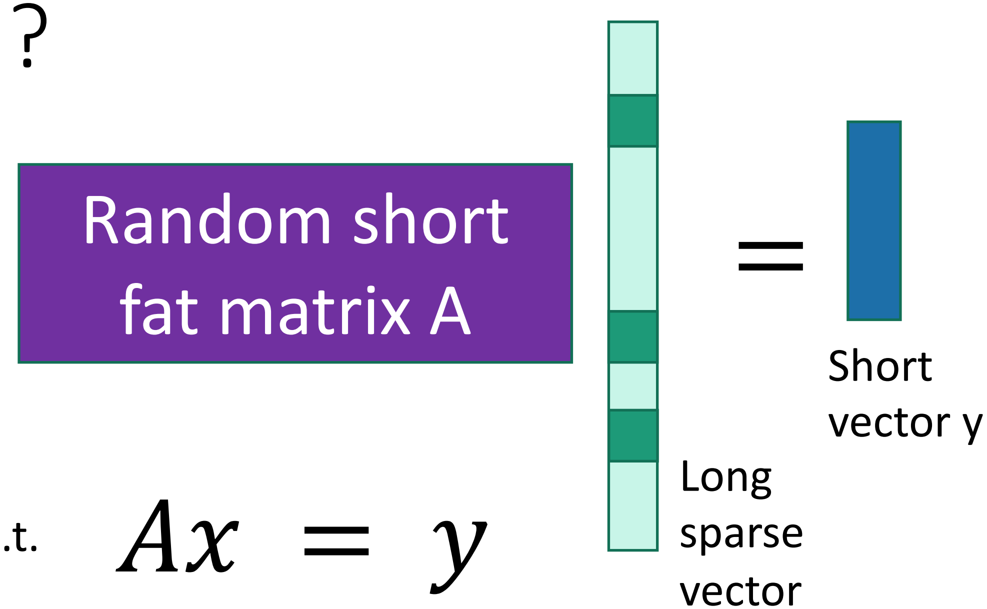


This means that, in theory, we can invert that arrow.

How do we do this efficiently??

Goal: Given the **short vector**, recover the **long sparse vector.**

An efficient algorithm?



What we'd like to do is:

Minimize number of nonzero entries in x s.t. $Ax = y$

This isn't a nice function

Problem: I don't know how to do that efficiently!

This norm is the sum of the absolute values of the entries of x

Instead:

Minimize $||x||_1$ s.t. $Ax = y$

- It turns out that because the geometry of sparse vectors is preserved, this optimization problem **gives the same answer**.
- We can use **linear programming** to solve this quickly!

Today

A few gems

- Linear programming

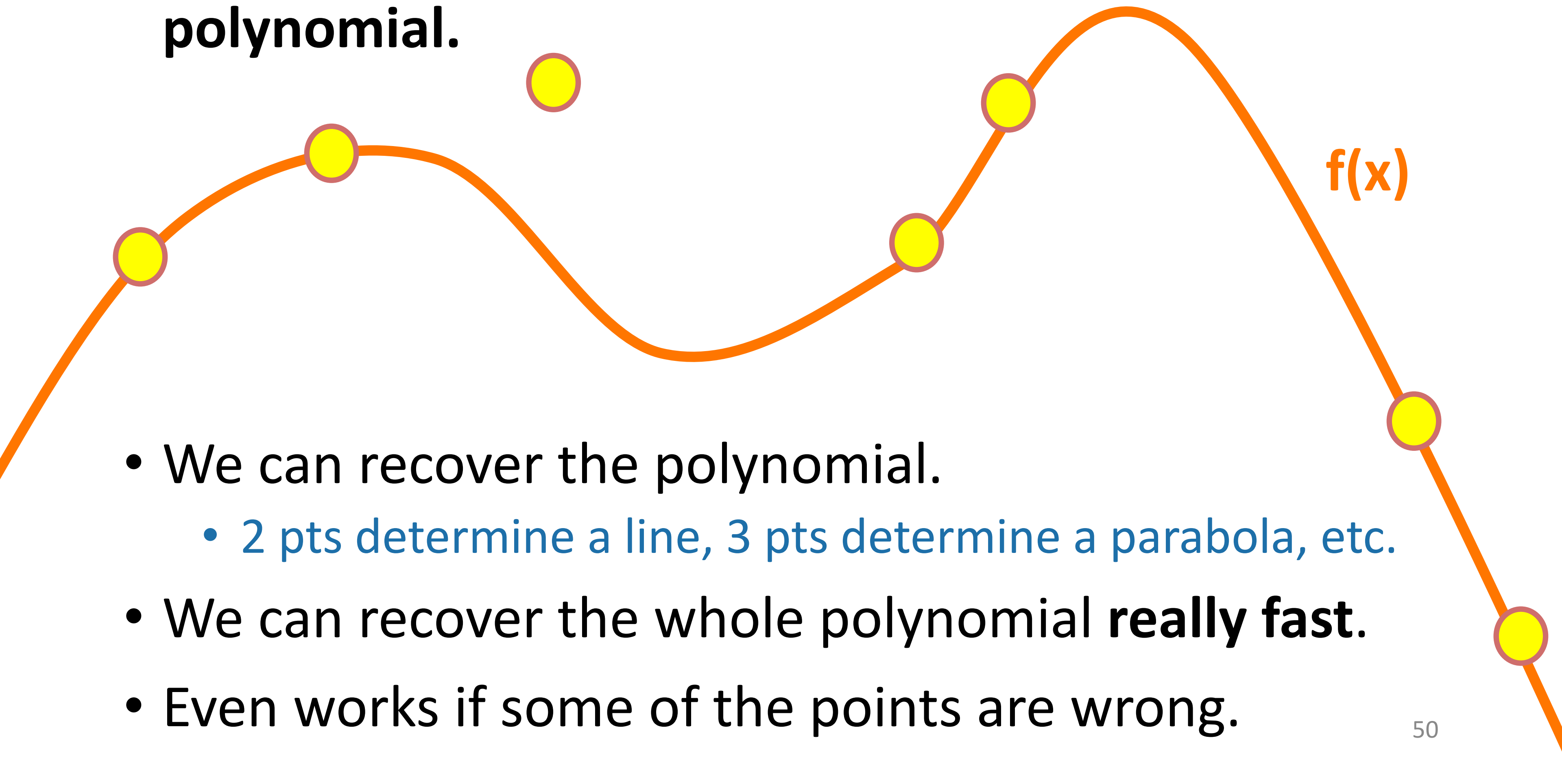
- Random projections

- Low-degree polynomials

Another very useful trick

Polynomial interpolation

- Say we have a few evaluation points of a **low-degree polynomial.**



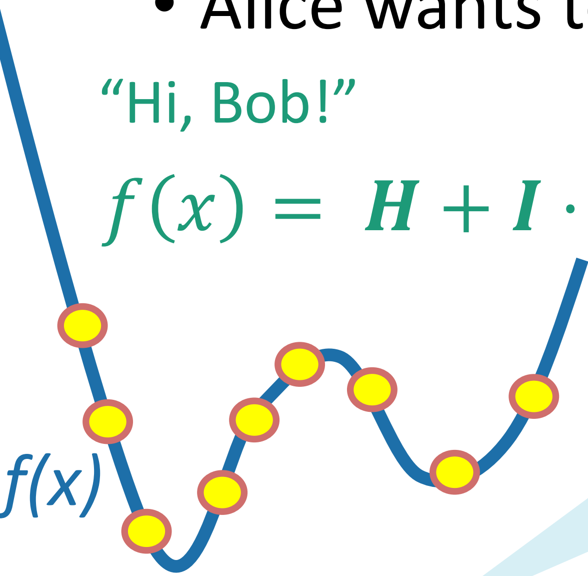
- We can recover the polynomial.
  - 2 pts determine a line, 3 pts determine a parabola, etc.
- We can recover the whole polynomial **really fast**.
- Even works if some of the points are wrong.

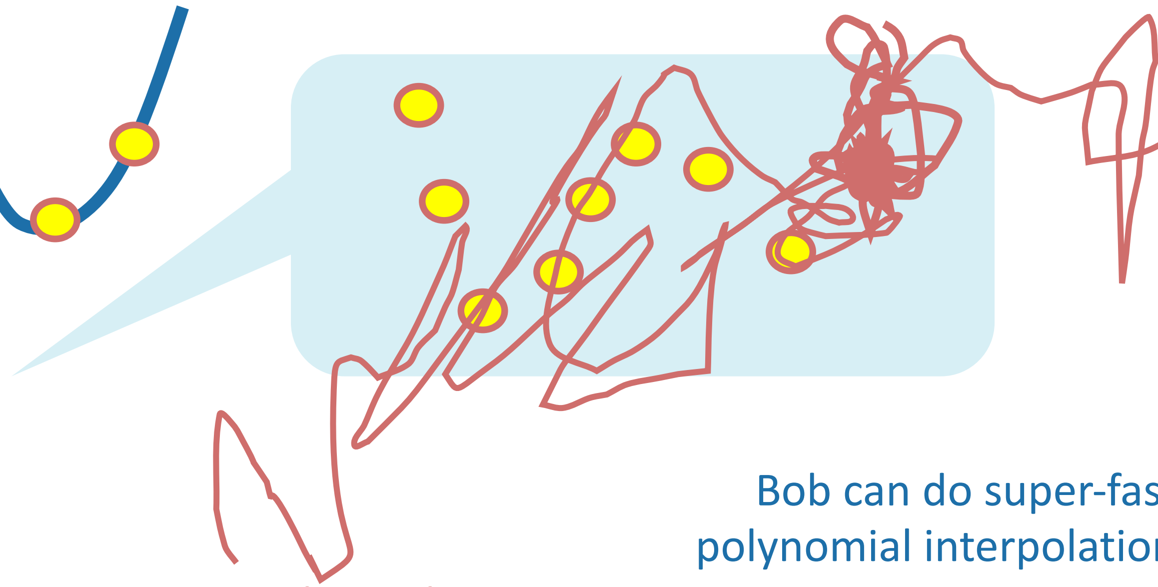
One application:
Communication and Storage

- Alice wants to send a message to Bob

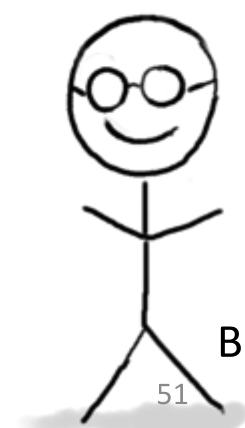
“Hi, Bob!”

$$f(x) = \boldsymbol{H} + \boldsymbol{I} \cdot x + \boldsymbol{B} \cdot x^2 + \boldsymbol{O} \cdot x^3 + \boldsymbol{B} \cdot x^4$$



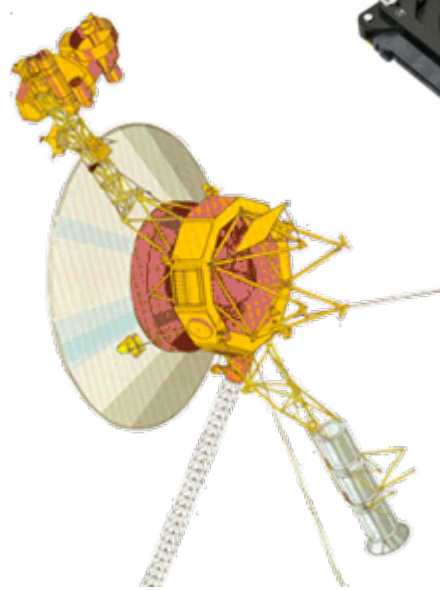


Bob can do super-fast polynomial interpolation and figure out what Alice meant to say!



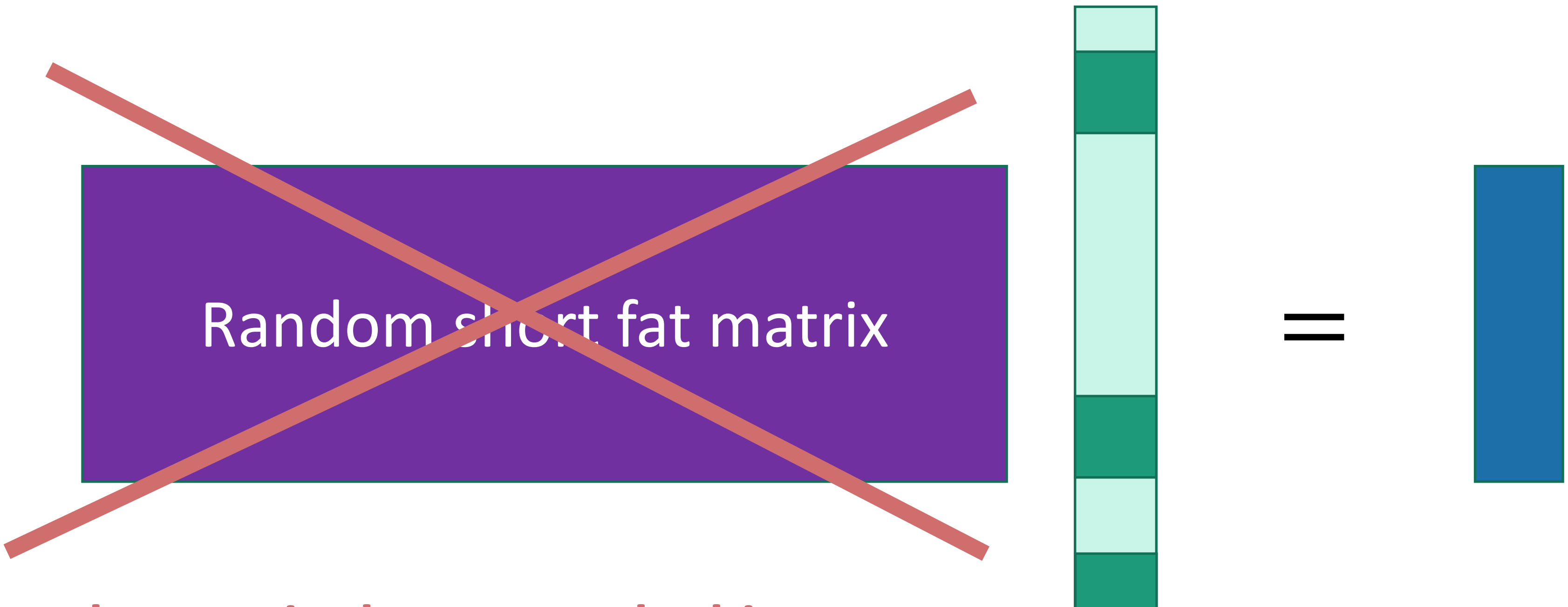
This is used in practice

- It's called "Reed-Solomon Encoding"



Another application:

Designing "random" projections that are better than random



The matrix that treats the big long vector as Alice's message polynomial and evaluates it REALLY FAST at random points.

- This is still "random enough" to make the LP solution work.
- It is much more efficient to manipulate and store!

Today

A few gems

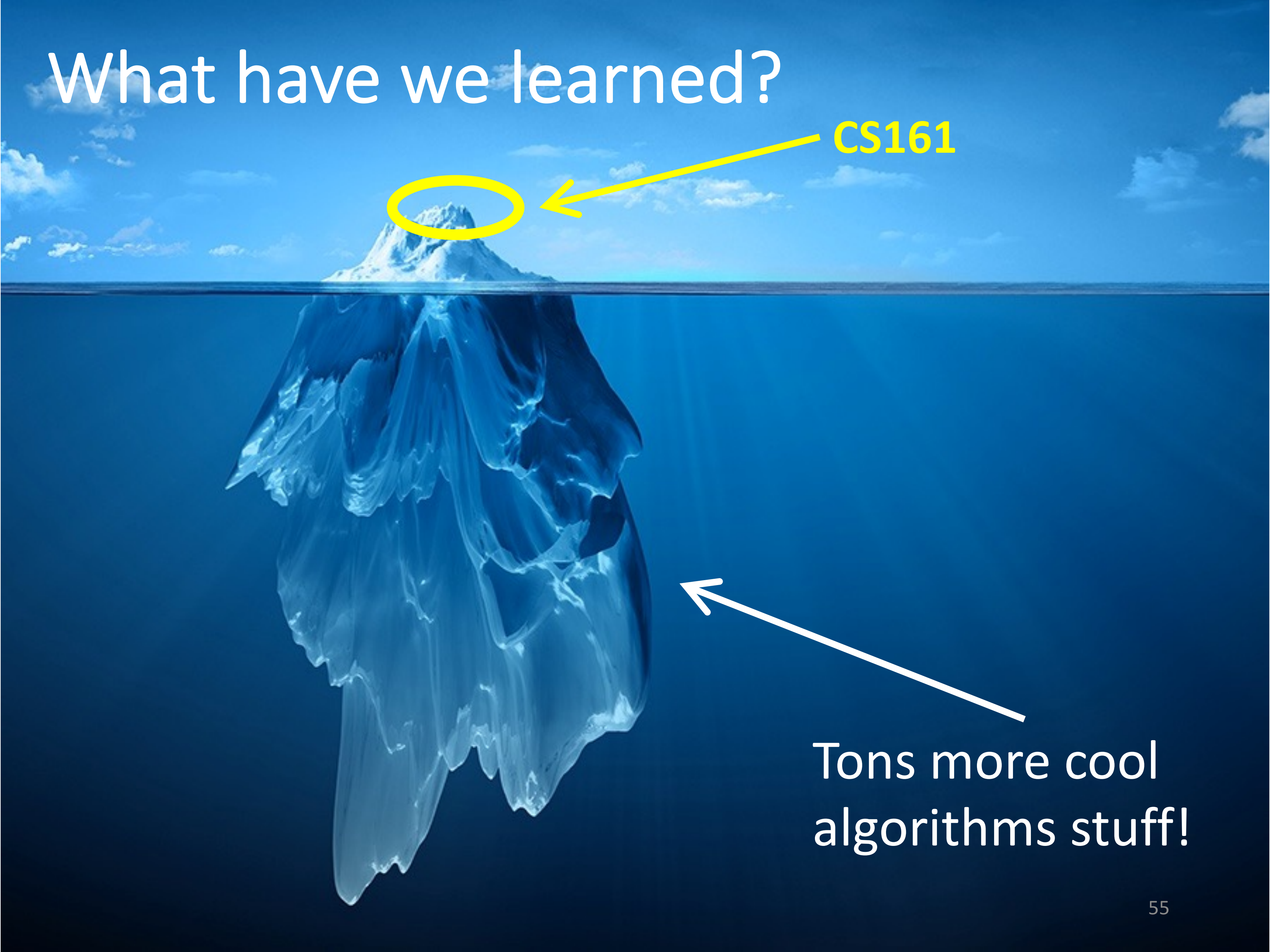
To learn more:

- Linear programming — CS168, CS261, …

- Random projections — CS168, CS261, CS265, …

- Low-degree polynomials — CS168, CS250, …

What have we learned?



To see more…

- Take more classes!
- Come hang out with the theory group!
  - Theory lunch, most Thursdays at noon.
  - Join the theory-seminar mailing list for updates.

theory.stanford.edu

Stanford theory group (circa 2017):
We are very friendly.

A few final messages…

Thanks to our course coordinator Amelie Byun!

- Amelie has been making all the logistics work behind the scenes.

Thanks to Diana Acosta-Navas!

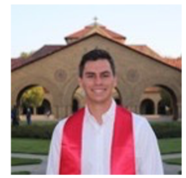
- Diana has been helping integrate EthiCS components into the course.

Thanks to our superstar CAs!!!

tell them you appreciate them!





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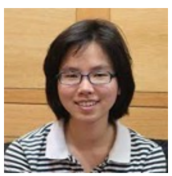
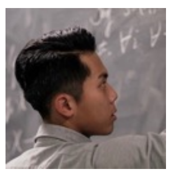
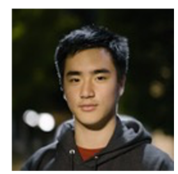
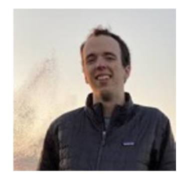
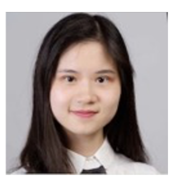



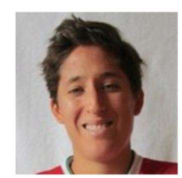
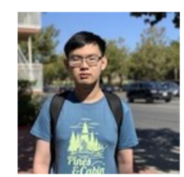




Carmen June Andrew Jose Manda Nash Peter Sam Samar







Aditya Emily Yuchen Ziang Seiji Shubham Teresa Tim

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THANKS

to you!!!!!!