Lecture 3

Recurrence Relations and how to solve them!

Announcements

- Online for next week
  - Same lecture Zoom links and office hours on Nooks
  - We will follow university guidance on in-person classes.
  - For CS 161, this means the latest plan is to start in-person teaching from Jan 24.
- Homework 1 is due this Wednesday at midnight
- Students with OAE accommodations send your letter to the staff mailing list ASAP:

cs161-win2122-staff@lists.stanford.edu

Last time….

- Sorting: InsertionSort and MergeSort
- What does it mean to work and be fast?
  - Worst-Case Analysis
  - Big-Oh Notation
- Analyzing correctness of iterative + recursive algs
  - Induction!
- Analyzing running time of recursive algorithms
  - By drawing out a tree and adding up all the work done.

Today

- Recurrence Relations!
  - How do we calculate the runtime of a recursive algorithm?
- The Master Theorem
  - A useful theorem so we don't have to answer this question from scratch each time.
- The Substitution Method
  - A different way to solve recurrence relations, more generally applicable than the Master Method.

Running time of MergeSort

- Let T(n) be the running time of MergeSort on a length n array.
- We know that $T(n) = O(n\log(n))$.
- We also know that T(n) satisfies:

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + O(n)$$

```
MERGESORT(A):
   n = length(A)
   if n ≤ 1:
      return A
   L = MERGESORT(A[:n/2])
   R = MERGESORT(A[n/2:])
   return MERGE(L,R)
```

Running time of MergeSort

- Let T(n) be the running time of MergeSort on a length n array.
- We know that $T(n) = O(n\log(n))$.
- We also know that T(n) satisfies:

$$T(n) \le 2 \cdot T\left(\frac{n}{2}\right) + 11 \cdot n$$

Last time we showed that the time to run MERGE on a problem of size n is O(n). For concreteness, let's say that it's at most 11n operations.

```
MERGESORT(A):
   n = length(A)
   if n ≤ 1:
      return A
   L = MERGESORT(A[:n/2])
   R = MERGESORT(A[n/2:])
   return MERGE(L,R)
```

Note (read after class):
$T(n) \leq 2 \cdot T\left(\frac{n}{2}\right) + 11 \cdot n$ (with a $\leq$) is also a recurrence relation. A recurrence relation with an "=" exactly defines a function; a recurrence relation with an inequality only bounds it.

Recurrence Relations

- $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 11 \cdot n$ is a **recurrence relation.**
- It gives us a formula for T(n) in terms of T(less than n)
- The challenge:

Given a recurrence relation for T(n), find a closed form expression for T(n).

- For example, $T(n) = O(n\log(n))$

Technicalities I
Base Cases

Plucky the Pedantic Penguin

- Formally, we should always have base cases with recurrence relations.
- $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 11 \cdot n$ with $T(1) = 1$

  is not the same function as
- $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 11 \cdot n$ with $T(1) = 1000000000$
- However, no matter what T is, T(1) = O(1), so sometimes we'll just omit it.

Why does T(1) = O(1)?

Siggi the Studious Stork

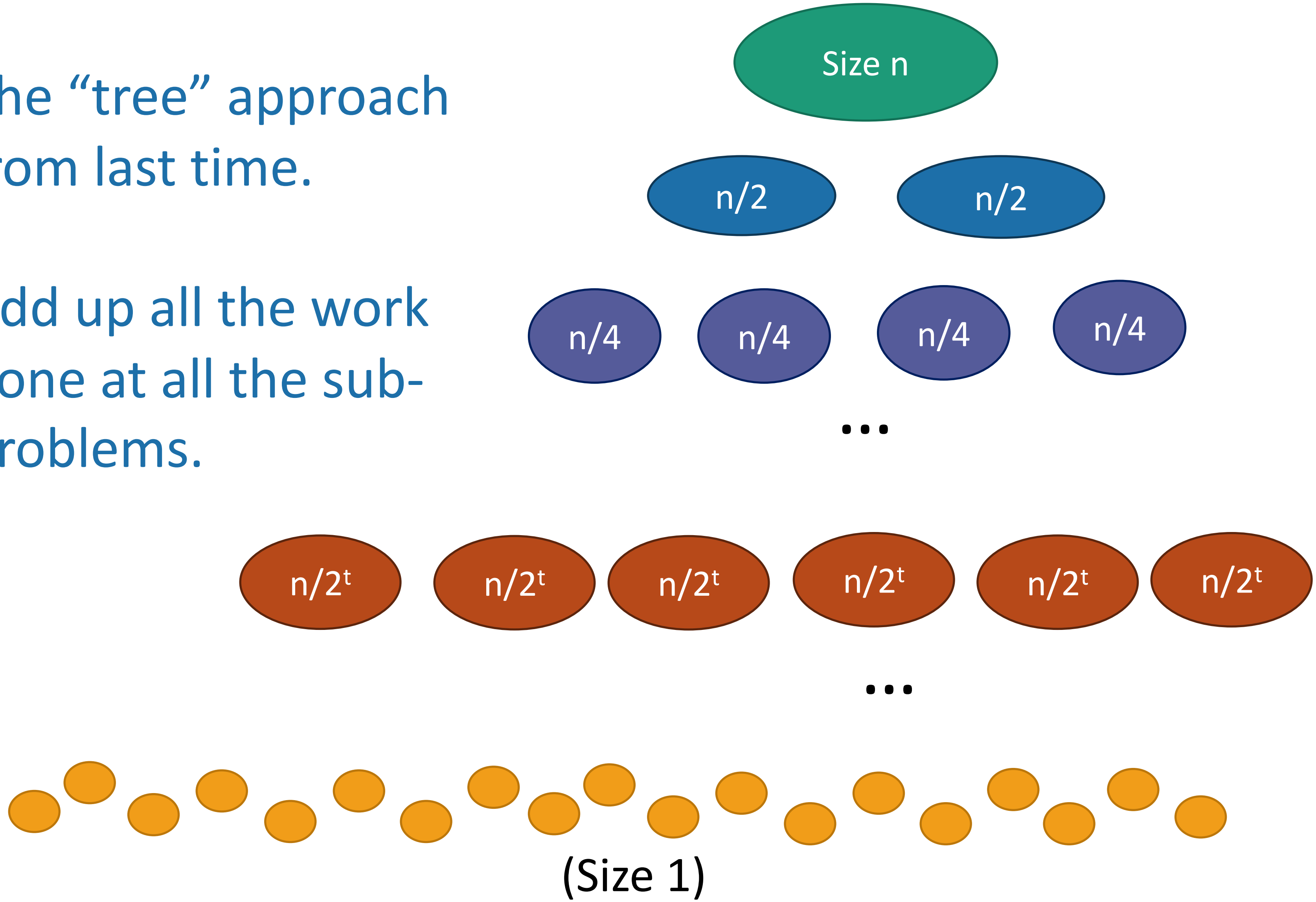
On your pre-lecture exercise

- You played around with these examples (when n is a power of 2):

  1. $T(n) = T\left(\frac{n}{2}\right) + n, \qquad T(1) = 1$
  2. $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n, \qquad T(1) = 1$
  3. $T(n) = 4 \cdot T\left(\frac{n}{2}\right) + n, \qquad T(1) = 1$

One approach for all of these

- The "tree" approach from last time.
- Add up all the work done at all the sub-problems.



Pre-lecture exercise

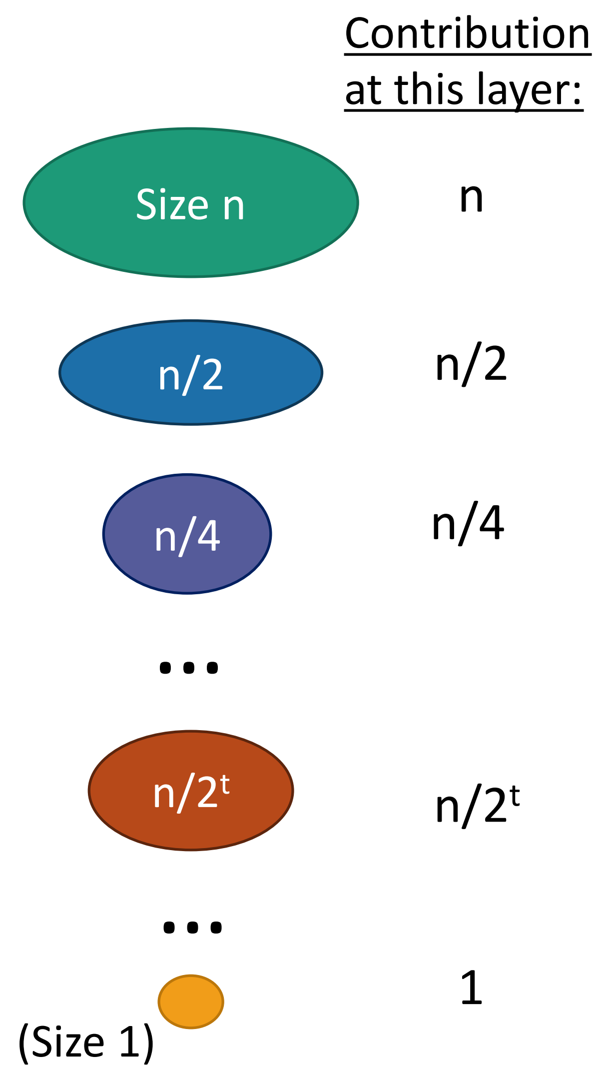
- (when n is a power of 2):
  1. $T(n) = T\left(\frac{n}{2}\right) + n,$ $\qquad T(1) = 1$
  2. $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n,$ $\qquad T(1) = 1$
  3. $T(n) = 4 \cdot T\left(\frac{n}{2}\right) + n,$ $\qquad T(1) = 1$

Solutions to pre-lecture exercise (1)

- $T_1(n) = T_1\left(\frac{n}{2}\right) + n, \quad T_1(1) = 1.$
- Adding up over all layers:

$$\sum_{i=0}^{\log(n)} \frac{n}{2^i} = 2n - 1$$

- So $T_1(n) = O(n).$



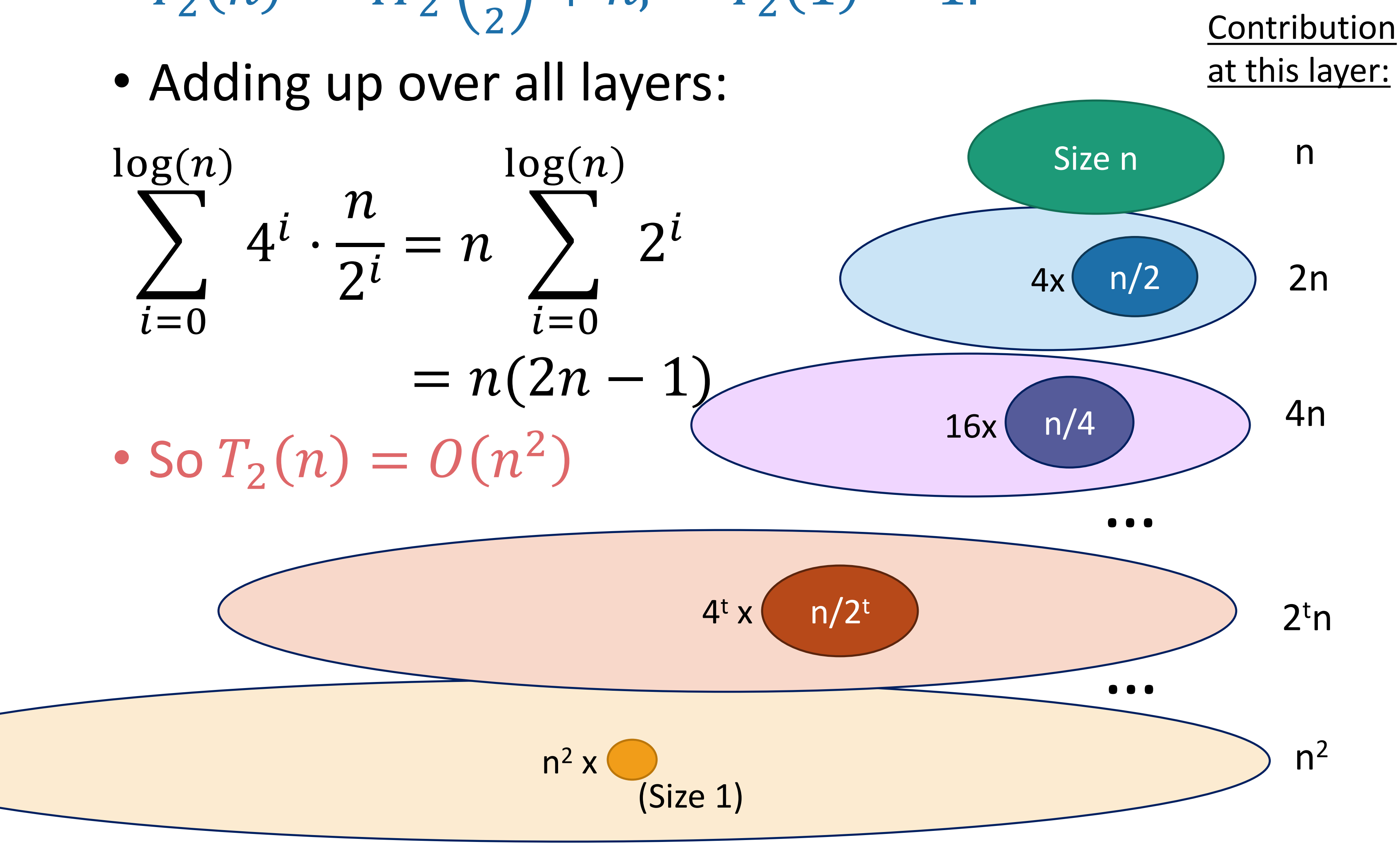
Solutions to pre-lecture exercise (2)

- $T_2(n) = 4T_2\left(\frac{n}{2}\right) + n, \quad T_2(1) = 1.$
- Adding up over all layers:

$$\sum_{i=0}^{\log(n)} 4^i \cdot \frac{n}{2^i} = n \sum_{i=0}^{\log(n)} 2^i$$

$$= n(2n - 1)$$

- So $T_2(n) = O(n^2)$



More examples

T(n) = time to solve a problem of size n.

- Needlessly recursive integer multiplication
- $T(n) = 4T(n/2) + O(n)$
- $T(n) = O(n^2)$

This is similar to T_2 from the pre-lecture exercise.

- Karatsuba integer multiplication
- $T(n) = 3T(n/2) + O(n)$
- $T(n) = O(n^{\log_2(3)} \approx n^{1.6})$

- MergeSort
- $T(n) = 2T(n/2) + O(n)$
- $T(n) = O(n\log(n))$

What's the pattern?!?!?!?!

The master theorem

- A formula for many recurrence relations.
    - We'll see an example Wednesday where it won't work.
- Proof: "Generalized" tree method.

A useful
formula it is.
Know why it works
you should.

Jedi master Yoda

We can also take n/b to mean either $\left\lfloor \frac{n}{b} \right\rfloor$ or $\left\lceil \frac{n}{b} \right\rceil$ and the theorem is still true.

The master theorem

- Suppose that $a \geq 1, b > 1,$ and d are constants (independent of n).
- Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O\left(n^d\right)$. Then

$$T(n) = \begin{cases} O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

Three parameters:

a : number of subproblems

b : factor by which input size shrinks

d : need to do n^d work to create all the subproblems and combine their solutions.

Many symbols those are….

Technicalities II

Integer division

Plucky the Pedantic Penguin

- If n is odd, I can't break it up into two problems of size n/2.

$$T(n) = T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + T\left(\left\lceil\frac{n}{2}\right\rceil\right) + O(n)$$

- However (see CLRS, Section 4.6.2 for details), one can show that the Master theorem works fine if you pretend that what you have is:

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + O(n)$$

- From now on we'll mostly **ignore floors and ceilings** in recurrence relations.

Examples

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d).$$

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

- Needlessly recursive integer mult.
  - T(n) = 4 T(n/2) + O(n)
  - T(n) = O(n^2)

a = 4
b = 2
d = 1

$a > b^d$ ✓

- Karatsuba integer multiplication
  - T(n) = 3 T(n/2) + O(n)
  - T(n) = O($n^{\text{log\_2(3)}} \approx n^{1.6}$)

a = 3
b = 2
d = 1

$a > b^d$ ✓

- MergeSort
  - T(n) = 2T(n/2) + O(n)
  - T(n) = O(nlog(n))

a = 2
b = 2
d = 1

$a = b^d$ ✓

- That other one
  - T(n) = T(n/2) + O(n)
  - T(n) = O(n)

a = 1
b = 2
d = 1

$a < b^d$ ✓

Proof of the master theorem

- We'll do the same recursion tree thing we did for MergeSort, but be more careful.
- Suppose that $T(n) \leq a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$.

Hang on! The hypothesis of the Master Theorem was that the extra work at each level was $O(n^d)$, but we're writing cn^d…

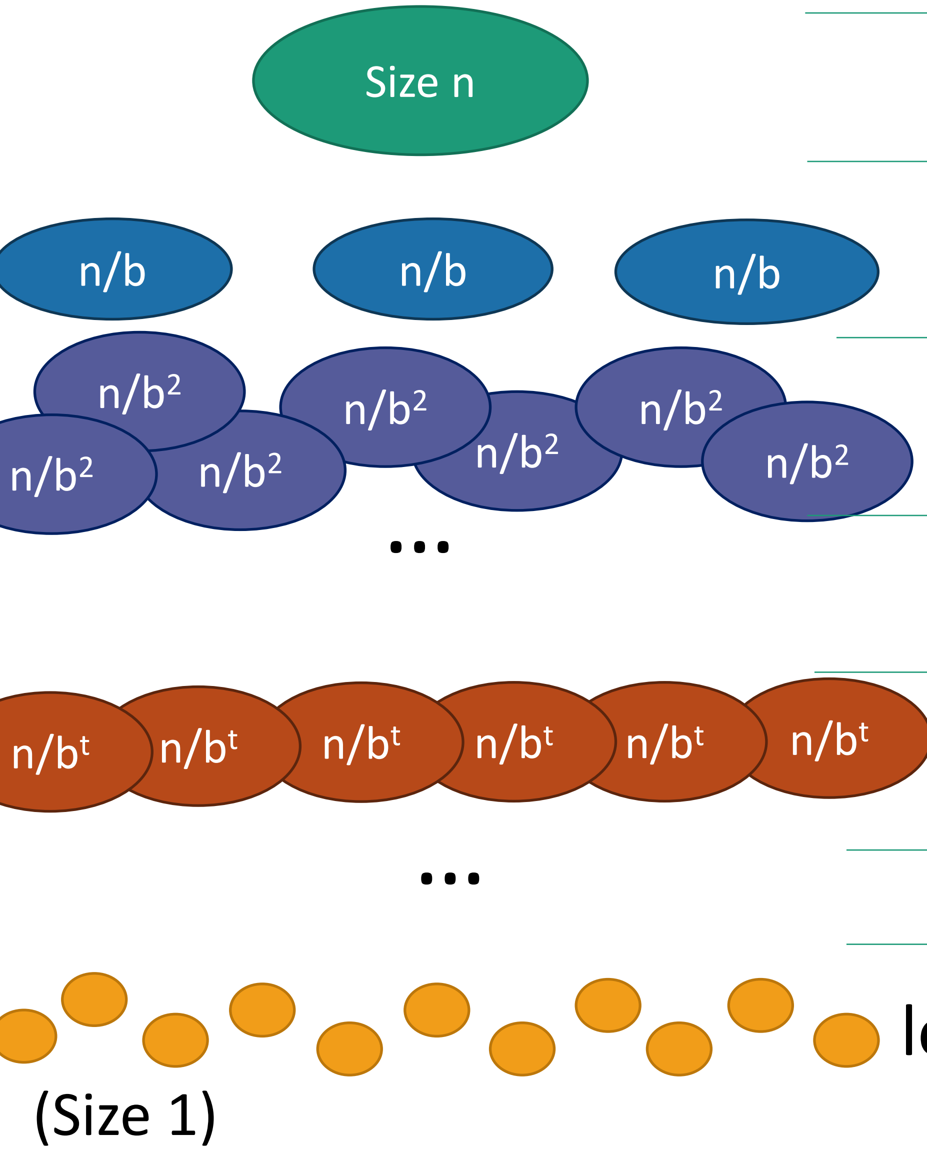
That's true … the hypothesis should be that $T(n) = a \cdot T\left(\frac{n}{b}\right) + O\left(n^d\right)$. For simplicity, today we are essentially assuming that $n_0 = 1$ in the definition of big-Oh. It's a good exercise to verify why that assumption is without loss of generality.

Plucky the Pedantic Penguin

Siggi the Studious Stork

Recursion tree

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$$



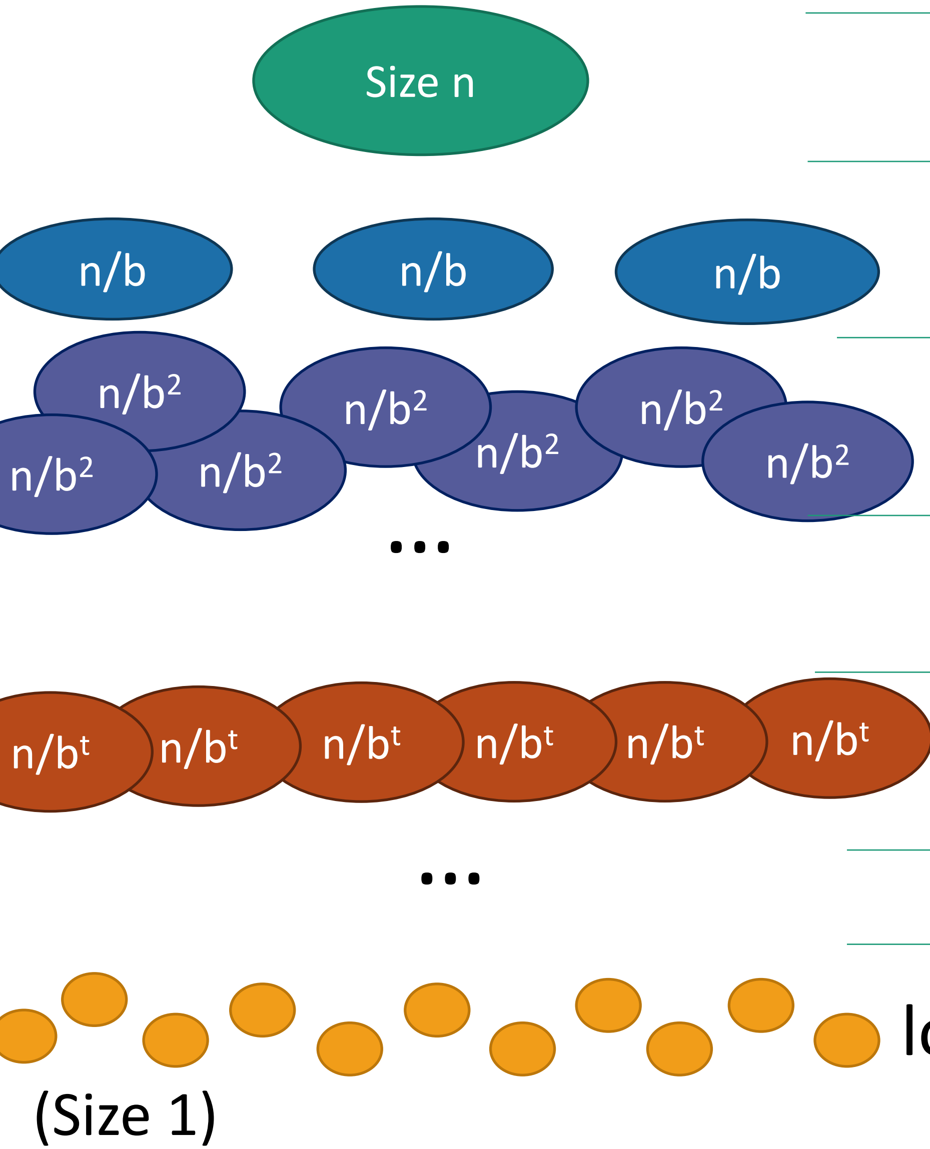
Level	# problems	Size of each problem	Amount of work at this level
0	1	n	
1	a	n/b	
2	a^2	n/b^2	
t	a^t	n/b^t	
…			
$\log_b(n)$	$a^{\log_b(n)}$	1	

Recursion tree

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$$

Help me fill this in!
2 minutes: think
1 minute: share (wait)

How much work at each level?
How much work total?

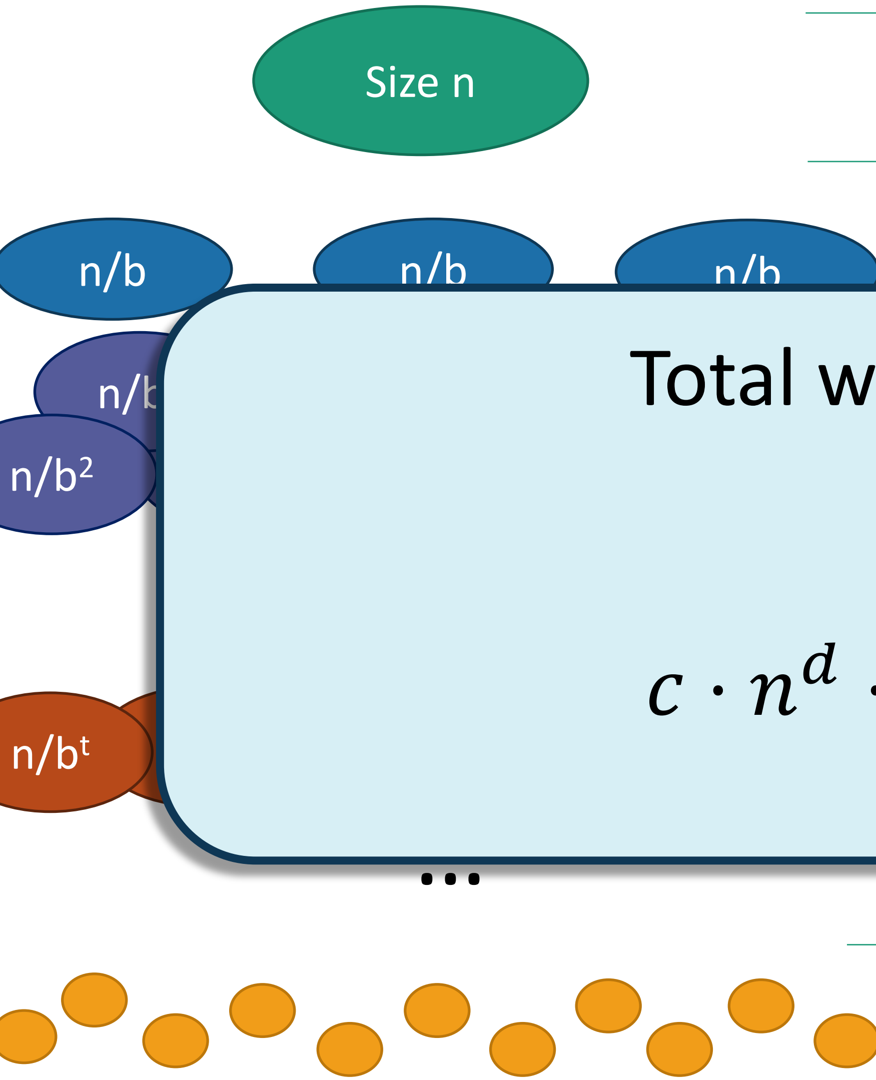


Level	# problems	Size of each problem	Amount of work at this level
0	1	n	$c \cdot n^d$
1	a	n/b	$ac\left(\frac{n}{b}\right)^d$
2	a^2	n/b^2	$a^2c\left(\frac{n}{b^2}\right)^d$
t	a^t	n/b^t	$a^tc\left(\frac{n}{b^t}\right)^d$
…			
$\log_b(n)$	$a^{\log_b(n)}$	1	$a^{\log_b(n)}c$

(Let's pretend that the base case is T(1) = c for convenience).

Recursion tree

$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$



Level	# problems	Size of each problem	Amount of work at this level
0	1	n	$c \cdot n^d$
1	a	n/b	$ac\left(\frac{n}{b}\right)^d$
…			
$\log_b(n)$	$a^{\log_b(n)}$	1	$a^{\log_b(n)}c$

(Let's pretend that the base case isT(1) = c for convenience).

Total work is at most:

$$c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^t$$

Now let's check all the cases

$$T(n) = \begin{cases} O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

Case 1: $a = b^d$

$$T(n) = \begin{cases} O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

- $T(n) = c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^t$ Equal to 1!

$= c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} 1$

$= c \cdot n^d \cdot (\log_b(n) + 1)$

$= c \cdot n^d \cdot \left(\frac{\log(n)}{\log(b)} + 1\right)$

$= \Theta\left(n^d \log(n)\right)$

$$T(n) = \begin{cases} O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

Case 2: $a < b^d$

- $T(n) = c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^t$

Less than 1!

Aside: Geometric sums

- What is $\sum_{t=0}^{N} x^t$?
- You may remember that $\sum_{t=0}^{N} x^t = \frac{x^{N+1}-1}{x-1}$ for $x \neq 1$.
- Morally:

$$x^0 + x^1 + x^2 + x^3 + \cdots + x^N$$

If $0 < x < 1$, this term dominates.

$$1 \leq \frac{x^{N+1}-1}{x-1} \leq \frac{1}{1-x}$$

(Aka, $\Theta(1)$ if x is constant and N is growing).

(If $x = 1$, all terms the same)

If $x > 1$, this term dominates.

$$x^N \leq \frac{x^{N+1}-1}{x-1} \leq x^N \cdot \left(\frac{x}{x-1}\right)$$

(Aka, $\Theta(x^N)$ if x is constant and N is growing).

Case 2: $a < b^d$

$$T(n) = \begin{cases} O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

- $T(n) = c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^t$ Less than 1!

$= c \cdot n^d \cdot [\text{some constant}]$

$= \Theta\left(n^d\right)$

Case 3: $a > b^d$

$$T(n) = \begin{cases} O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

- $T(n) = c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^t$

$$= \Theta\left(n^d \left(\frac{a}{b^d}\right)^{\log_b(n)}\right)$$

$$= \Theta\left(n^{\log_b(a)}\right)$$

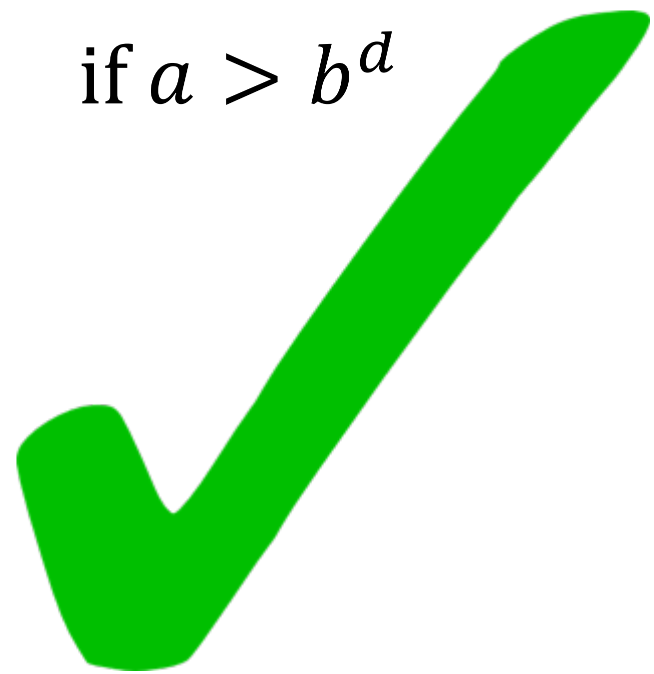
Larger than 1!

Convince yourself that this step is legit!

We'll do this step on the board!

Now let's check all the cases

$$T(n) = \begin{cases} \mathrm{O}\left(n^d \log(n)\right) & \text{if } a = b^d \\ \mathrm{O}\left(n^d\right) & \text{if } a < b^d \\ \mathrm{O}\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$



Understanding the Master Theorem

- Let $a \geq 1, b > 1,$ and d be constants.
- Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O\left(n^d\right)$. Then

$$T(n) = \begin{cases} O\left(n^d \log(n)\right) & \text{if } a = b^d \\ O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^{\log_b(a)}\right) & \text{if } a > b^d \end{cases}$$

- What do these three cases mean?

The eternal struggle

Branching causes the number of problems to explode!
The most work is at the bottom of the tree!

The problems lower in the tree are smaller!
The most work is at the top of the tree!

Consider our three warm-ups

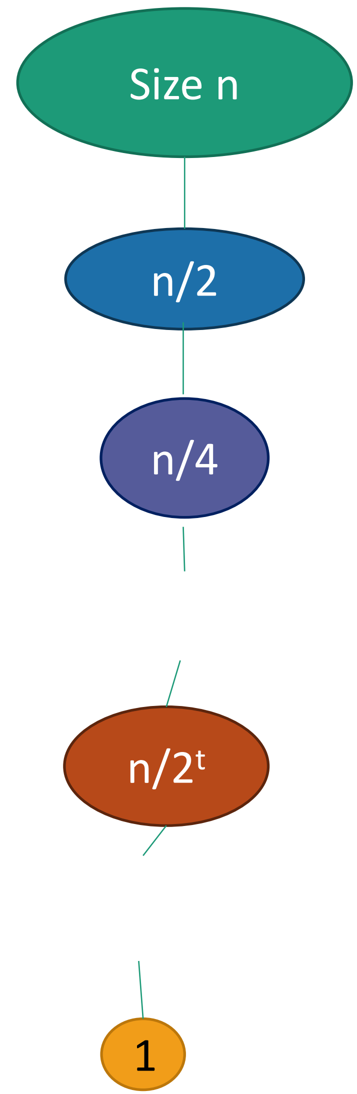
1. $T(n) = T\left(\frac{n}{2}\right) + n$
2. $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n$
3. $T(n) = 4 \cdot T\left(\frac{n}{2}\right) + n$

First example: tall and skinny tree

1. $T(n) = T\left(\frac{n}{2}\right) + n, \qquad \left(a < b^d\right)$

- The amount of work done at the top (the biggest problem) swamps the amount of work done anywhere else.

- T(n) = O(work at top) = O(n)

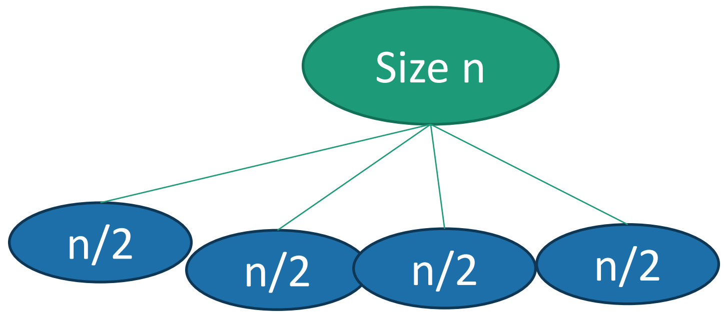


WINNER

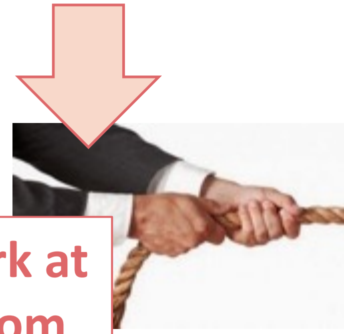
Most work at the top of the tree!

Third example: bushy tree

3. $T(n) = 4 \cdot T\left(\frac{n}{2}\right) + n, \qquad (a > b^d)$



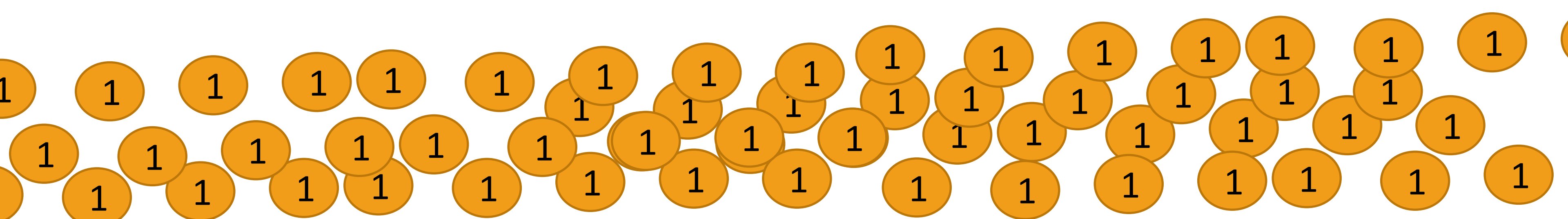
WINNER



Most work at the bottom of the tree!

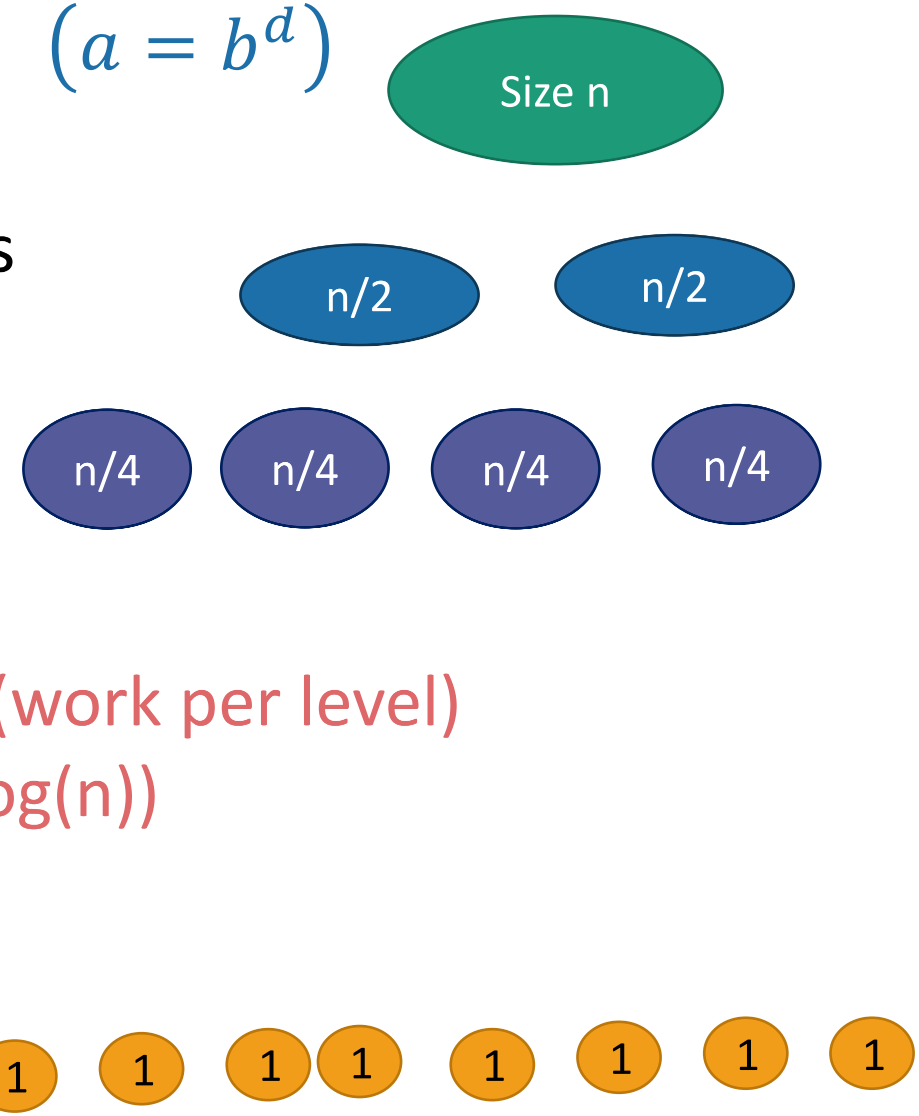
- There are a HUGE number of leaves, and the total work is dominated by the time to do work at these leaves.

- T(n) = O(work at bottom) = O($4^{\text{depth of tree}}$) = O(n^2)



Second example: just right

$$2.\ \ T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n, \qquad \left(a = b^d\right)$$



- The branching **just** balances out the amount of work.
- The same amount of work is done at every level.

- T(n) = (number of levels) * (work per level)
-   = log(n) * O(n) = O(n log(n))

What have we learned?

- The “Master Method” makes our lives easier.
- But it’s basically just codifying a calculation we could do from scratch if we wanted to.

The Substitution Method

- Another way to solve recurrence relations.
- More general than the master method.

- Step 1: Generate a guess at the correct answer.
- Step 2: Try to prove that your guess is correct.
- (Step 3: Profit.)

The Substitution Method

first example

- Let's return to:

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n, \text{ with } T(0) = 0, T(1) = 1.$$

- The Master Method says $T(n) = O(n \log(n))$.
- We will prove this via the Substitution Method.

$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n$, with $T(1) = 1$.

Step 1: Guess the answer

- $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n$
- $T(n) = 2 \cdot \left(2 \cdot T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n$
- $T(n) = 4 \cdot T\left(\frac{n}{4}\right) + 2n$
- $T(n) = 4 \cdot \left(2 \cdot T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + 2n$
- $T(n) = 8 \cdot T\left(\frac{n}{8}\right) + 3n$
- …

Expand $T\left(\frac{n}{2}\right)$

Simplify

Expand $T\left(\frac{n}{4}\right)$

Simplify

You can guess the answer however you want: meta-reasoning, a little bird told you, wishful thinking, etc. One useful way is to try to "unroll" the recursion, like we're doing here.

Guessing the pattern: $T(n) = 2^t \cdot T\left(\frac{n}{2^t}\right) + t \cdot n$

Plug in $t = \log(n)$, and get

$$T(n) = n \cdot T(1) + \log(n) \cdot n = n(\log(n) + 1)$$

$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n$, with $T(1) = 1$.

Step 2: Prove the guess is correct.

- Inductive Hypothesis: $T(n) = n(\log(n) + 1)$.
- Base Case (n=1): $T(1) = 1 = 1 \cdot (\log(1) + 1)$
- Inductive Step:
  - Assume Inductive Hyp. for $1 \leq n < k$:
    - Suppose that $T(n) = n(\log(n) + 1)$ for all $1 \leq n < k$.
  - Prove Inductive Hyp. for n=k:
    - $T(k) = 2 \cdot T\left(\frac{k}{2}\right) + k$ by definition
    - $T(k) = 2 \cdot \left(\frac{k}{2}\left(\log\left(\frac{k}{2}\right) + 1\right)\right) + k$ by induction.
    - $T(k) = k(\log(k) + 1)$ by simplifying.
    - So Inductive Hyp. holds for n=k.
- Conclusion: For all $n \geq 1, T(n) = n(\log(n) + 1)$

We're being sloppy here about floors and ceilings...what would you need to do to be less sloppy?

Step 3: Profit

- Pretend like you never did Step 1, and just write down:

- **Theorem:** $T(n) = O(n \log(n))$
- **Proof: [Whatever you wrote in Step 2]**

What have we learned?

- The substitution method is a different way of solving recurrence relations.

- Step 1: Guess the answer.
- Step 2: Prove your guess is correct.
- Step 3: Profit.

- We'll get more practice with the substitution method next lecture!

Another example (if time)

(If not time, that's okay; we'll see these ideas in Lecture 4)

- $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 32 \cdot n$
- $T(2) = 2$

- Step 1: Guess: $O(n \log(n))$ (divine inspiration).
- But I don't have such a precise guess about the form for the $O(n \log(n))$ …
  - That is, what's the leading constant?
- Can I still do Step 2?

Aside: What's wrong with this?

- Inductive Hypothesis: $T(n) = O(n \log(n))$
- Base case: $T(2) = 2 = O(1) = O(2 \log(2))$
- Inductive Step:
  - Suppose that $T(n) = O(n \log(n))$ for n < k.
  - Then $T(k) = 2 \cdot T\left(\frac{k}{2}\right) + 32 \cdot k$ by definition
  - So $T(k) = 2 \cdot O\left(\frac{k}{2} \log\left(\frac{k}{2}\right)\right) + 32 \cdot k$ by induction
  - But that's $T(k) = O(k \log(k))$, so the I.H. holds for n=k.
- Conclusion:
  - By induction, $T(n) = O(n \log(n))$ for all n.

This is NOT CORRECT!!!

Plucky the Pedantic Penguin

Figure out what's wrong here!!!

Siggi the Studious Stork

Another example (if time)

(If no time, that's okay; we'll see these ideas in Lecture 4)

- $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 32 \cdot n$
- $T(2) = 2$

- Step 1: Guess: $O(n\log(n))$ (divine inspiration).
- But I don't have such a precise guess about the form for the $O(n\log(n))$ …
  - That is, what's the leading constant?
- Can I still do Step 2?

Step 2: Prove it, working backwards to figure out the constant

- Guess: $T(n) \leq C \cdot n \log(n)$ for some constant C TBD.
- Inductive Hypothesis (for $n \geq 2$) : $T(n) \leq C \cdot n \log(n)$
- Base case: $T(2) = 2 \leq C \cdot 2 \log(2)$ as long as $C \geq 1$
- Inductive Step:

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 32 \cdot n$$
$$T(2) = 2$$

Inductive step

- Assume that the inductive hypothesis holds for n<k.
- $T(k) = 2T\left(\frac{k}{2}\right) + 32k$
- $\qquad \leq 2C\frac{k}{2}\log\left(\frac{k}{2}\right) + 32k$
- $\qquad = k(C \cdot \log(k) + 32 \ - C)$
- $\qquad \leq k(C \cdot \log(k))$ as long as $C \geq 32$.
- Then the inductive hypothesis holds for n=k.

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 32 \cdot n$$
$$T(2) = 2$$

Step 2: Prove it, working backwards to figure out the constant

- Guess: $T(n) \leq C \cdot n \log(n)$ for some constant C TBD.
- Inductive Hypothesis (for $n \geq 2$): $T(n) \leq C \cdot n \log(n)$
- Base case: $T(2) = 2 \leq C \cdot 2 \log(2)$ as long as $C \geq 1$
- Inductive step: Works as long as $C \geq 32$
  - So choose $C = 32$.
- Conclusion: $T(n) \leq 32 \cdot n \log(n)$

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 32 \cdot n$$
$$T(2) = 2$$

Step 3: Profit.

- ***Theorem:*** $T(n) = O(n\log(n))$
- ***Proof:***
  - Inductive Hypothesis: $T(n) \leq 32 \cdot n\log(n)$
  - Base case: $T(2) = 2 \leq 32 \cdot 2\log(2)$ is true.
  - Inductive step:
    - Assume Inductive Hyp. for n<k.
    - $T(k) = 2T\left(\frac{k}{2}\right) + 32k$ By the def. of T(k)
    - $\leq 2 \cdot 32 \cdot \frac{k}{2}\log\left(\frac{k}{2}\right) + 32k$ By induction
    - $= k(32 \cdot \log(k) + 32 - 32)$
    - $= 32 \cdot k\log(k)$
    - This establishes inductive hyp. for n=k.
  - Conclusion: $T(n) \leq 32 \cdot n\log(n)$ for all $n \geq 2$.
    - By the definition of big-Oh, with $n_0 = 2$ and $c = 32$, this implies that $T(n) = O(n\log(n))$

Why two methods?

- Sometimes the Substitution Method works where the Master Method does not.
- More on this next time!

Next Time

- What happens if the sub-problems are different sizes?
- And when might that happen?

BEFORE Next Time

- Pre-lecture 4 exercises!