Lecture 17

Gale-Shapley (Deferred Acceptance) Algorithm

Announcements

- HW8 due on Wednesday.
- This week's lectures (including this one) are NOT on the final exam.
- EthiCS pre-recorded lectures (5 short videos) are fair game for the final exam.
- Final exam: Two pages front-and-back of notes allowed.

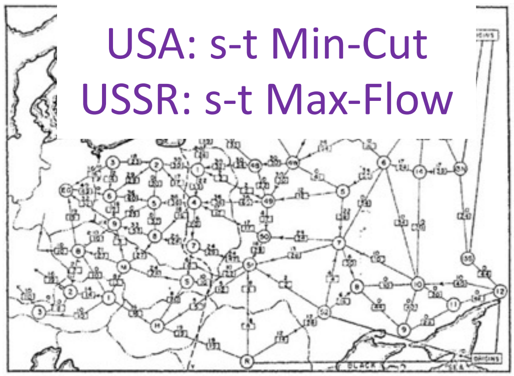
Recap: One way to greedy algorithms

- Greedy algorithms
  - Make a series of choices.
    - Choose this activity, then that one, ..
    - Never backtrack.
  - Show (or hope) that your choice never rules out success.
    - At every step, there exists an optimal solution consistent with the choices we've made so far.
  - At the end of the day:
    - you've built only one solution,
    - never having ruled out success,
    - **so your solution must be correct.**

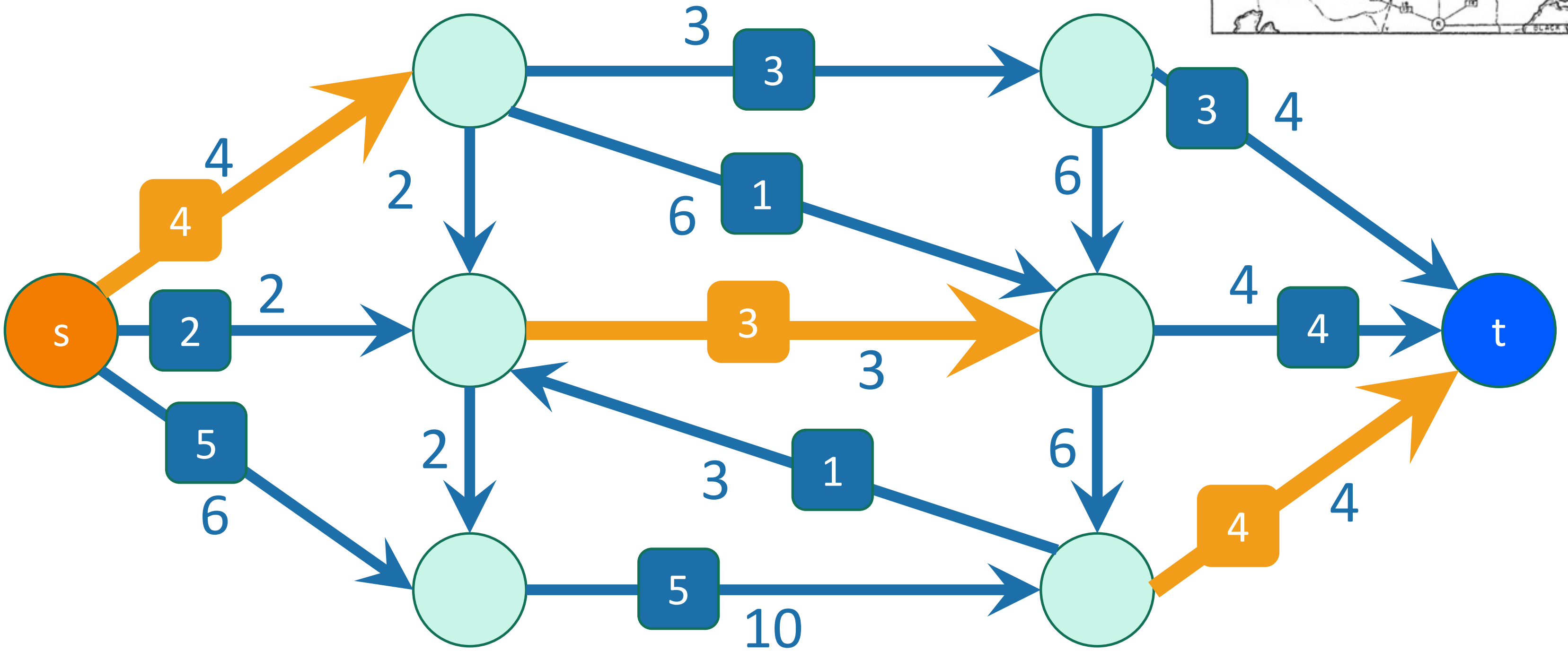
Recap: A different approach to greedy

- Greedy algorithms
  - Make a series of choices.
    - Choose this activity, then that one, ..
    - ~~Never backtrack.~~
  - **Instead:** At each step, free to revert any of the choices we've already made – as long as the solution is improving!

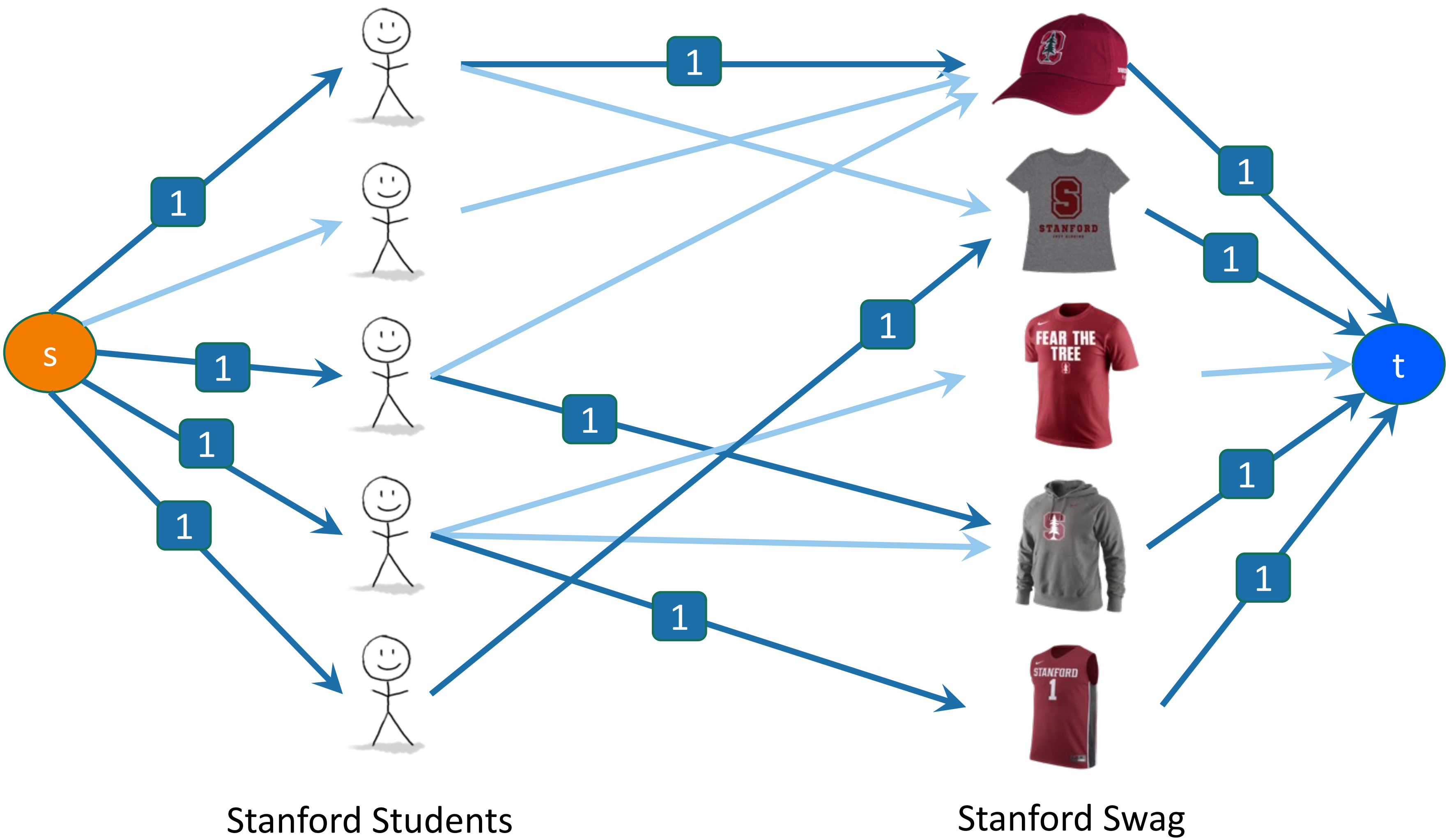
Recap: Ford-Fulkerson algorithm for s-t min-cut / max-flow



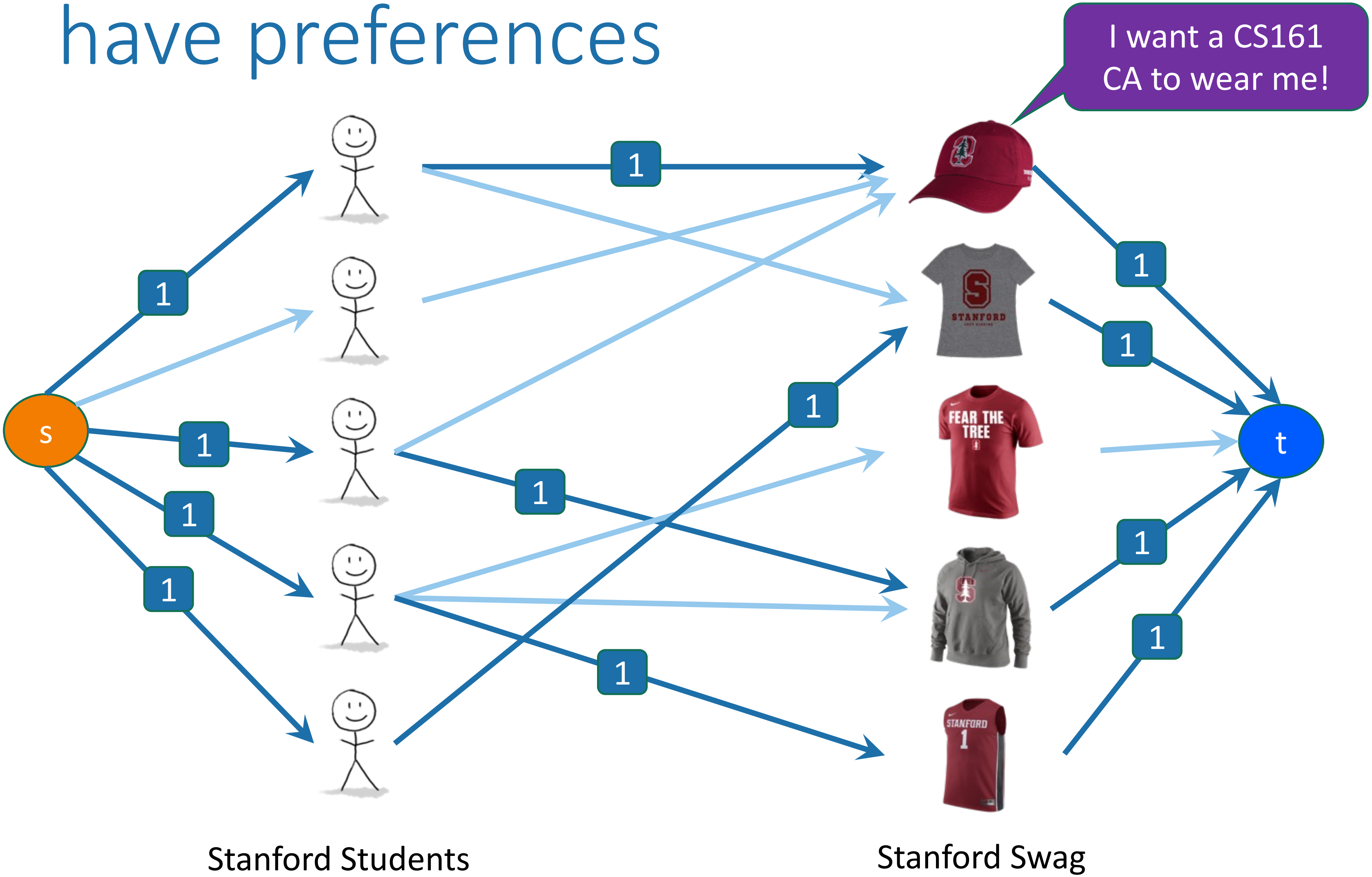
The value of a max flow from s to t
is equal to
the cost of a min s-t cut.



Recap: used s-t max-flow to solve assignment problems



Today: matching when both sides have preferences



Today

- Hospitals/residents problem

- Stable matchings
  - Solve the hospitals/residents problem
  - But can we find them?

- Deferred Acceptance Algorithm
  - Find stable matchings!

- Discussion, applications and non-applications

The hospital residency problem

- After completing medical school, students are finally ready to start their "residency" (similar to job internship):
  - In contrast, I'm told that many of you can get an internship after completing CS161…

- Each applicant has a preference over different residency programs.
- Each program has a preference over the applicants.

How should you match applicants to residencies?

Simplifying assumption today:
Each program has 1 slot

The hospital residency problem

- After completing medical school, students are finally ready to start their "residency" (similar to job internship):
  - In contrast, I'm told that many of you can get an internship after completing CS161…
- Each **doctor** has a preference over different **hospitals**.
- Each **hospital** has a preference over the **doctors**.

How should you match doctors with hospitals?

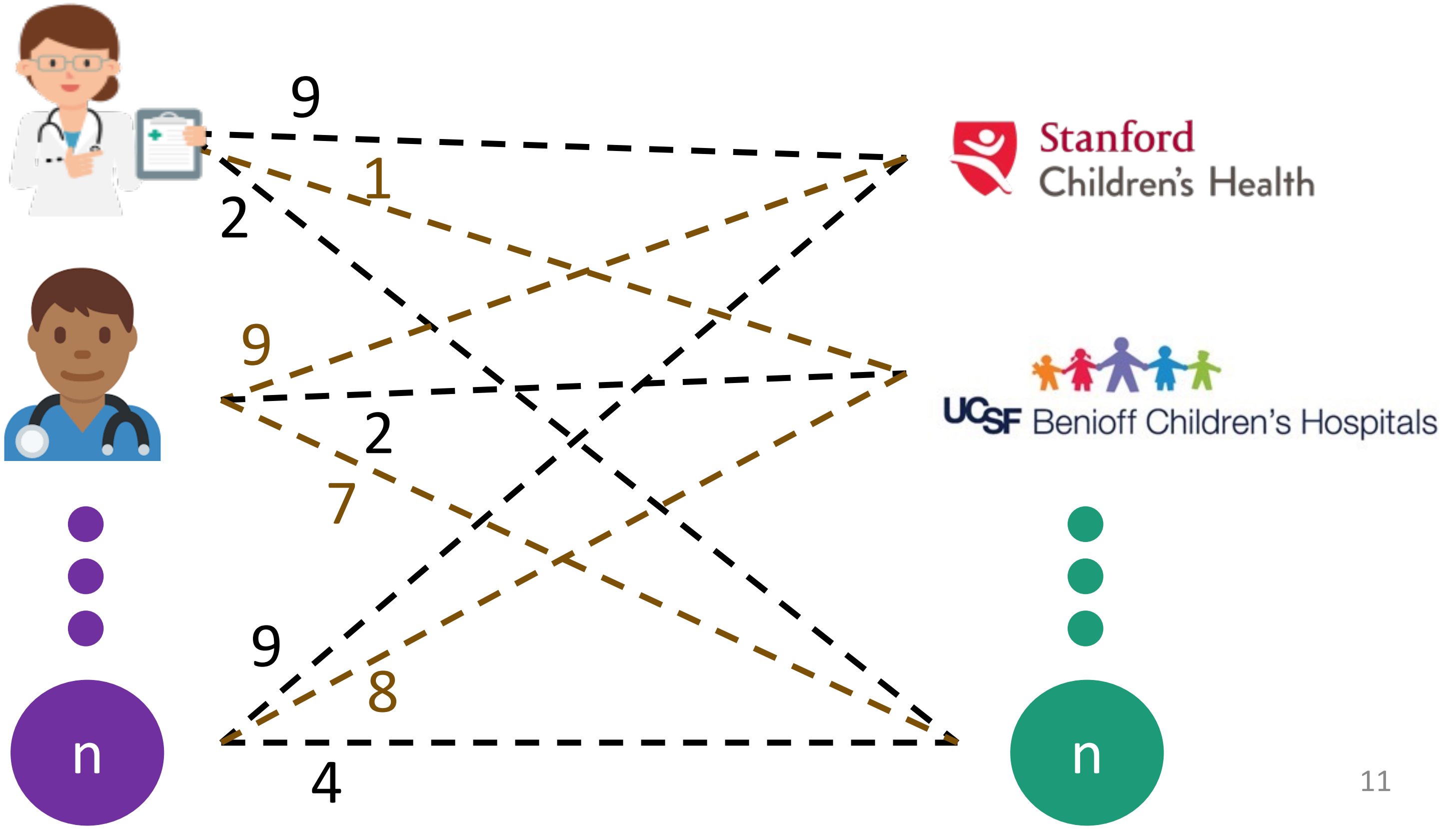
Simplifying assumption today:
Each hospital has 1 slot

This slide just for intuition!

One way to model this problem

- Each doctor has a preference over hospitals
- Each hospital has a preference over the doctor

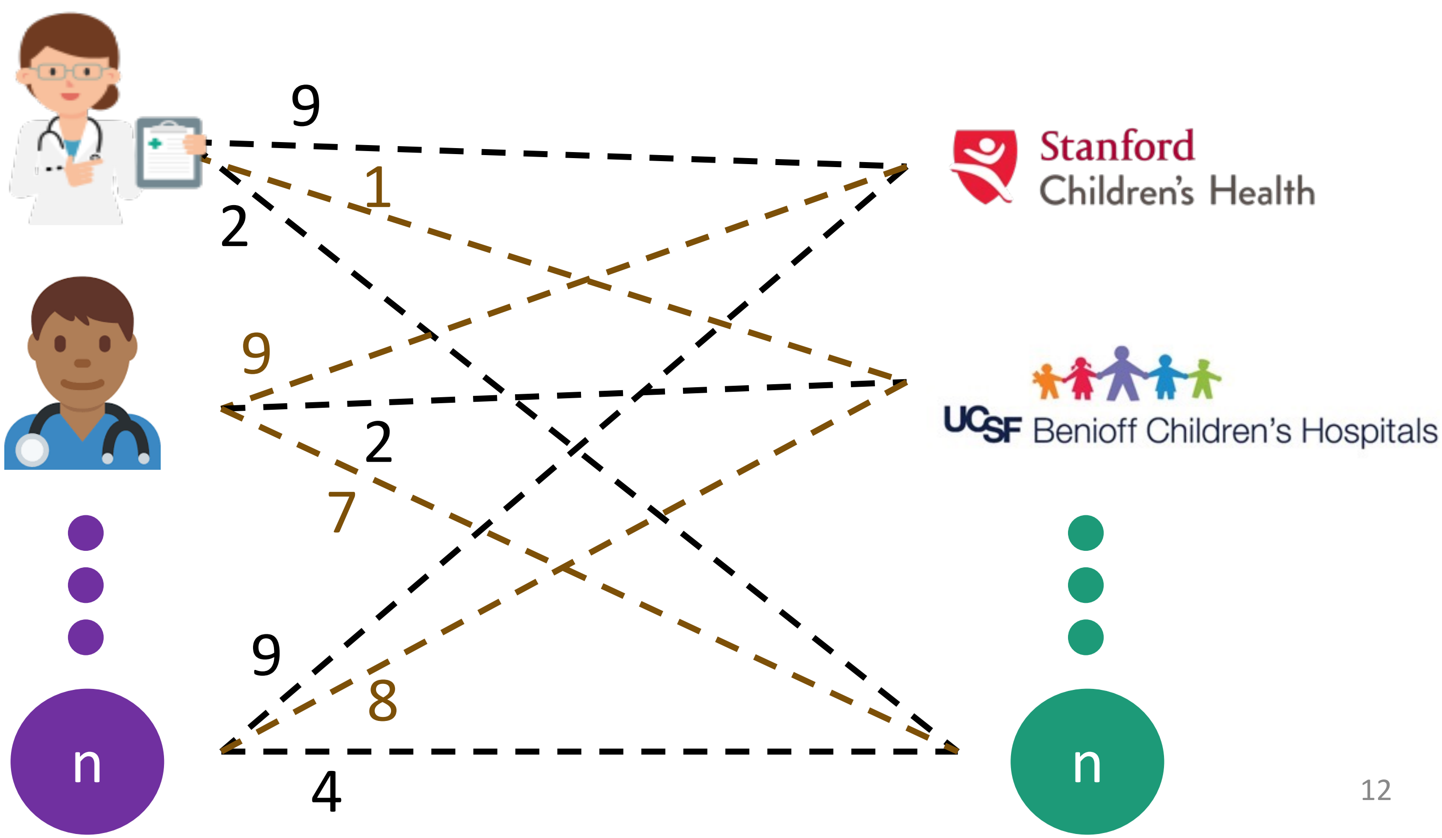
How should you match doctors with hospitals?



This slide just for intuition!

One way to model this problem

- Bipartite graph between doctors and hospitals
- Weights on edges = some function of preferences (highest weight = most preferred)

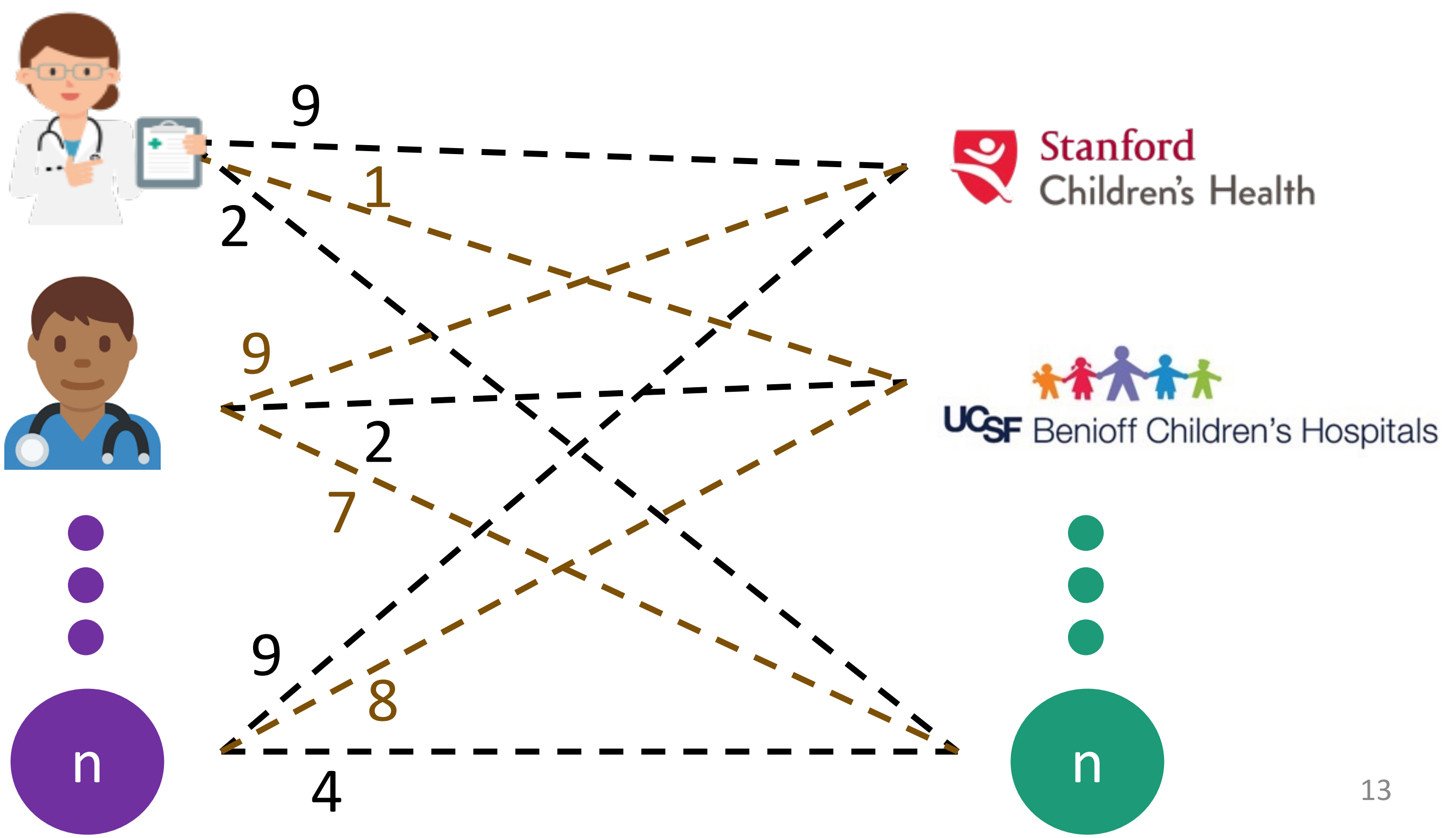


This slide just for intuition:
You don't need to know Hungarian Algorithm!

One way to model this problem

- Bipartite graph between doctors and hospitals
- Weights on edges = some function of preferences

"Hungarian Algorithm" (CS261) finds a max weight matching



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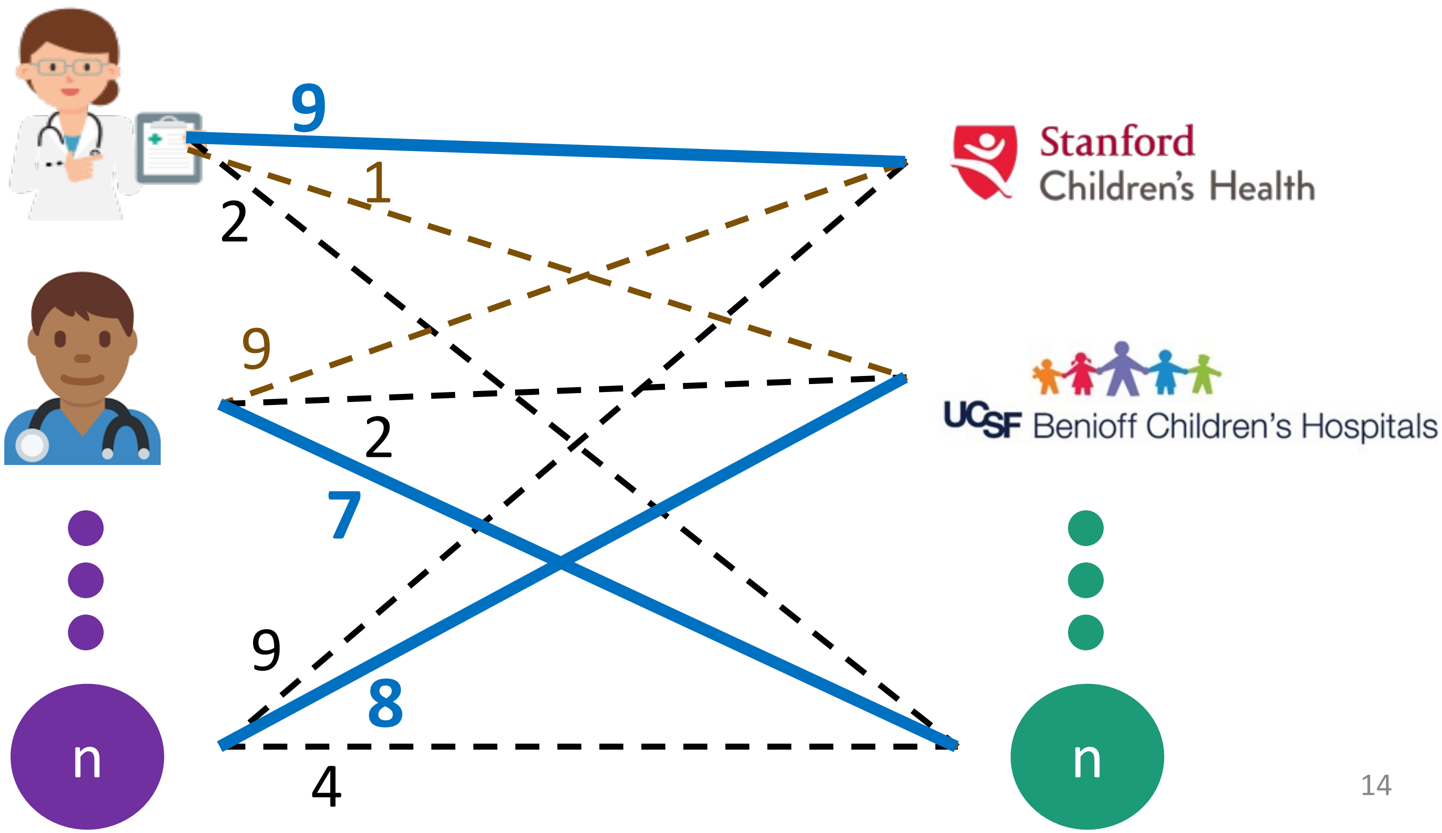


“Each hospital/doctor has a list of preferences”

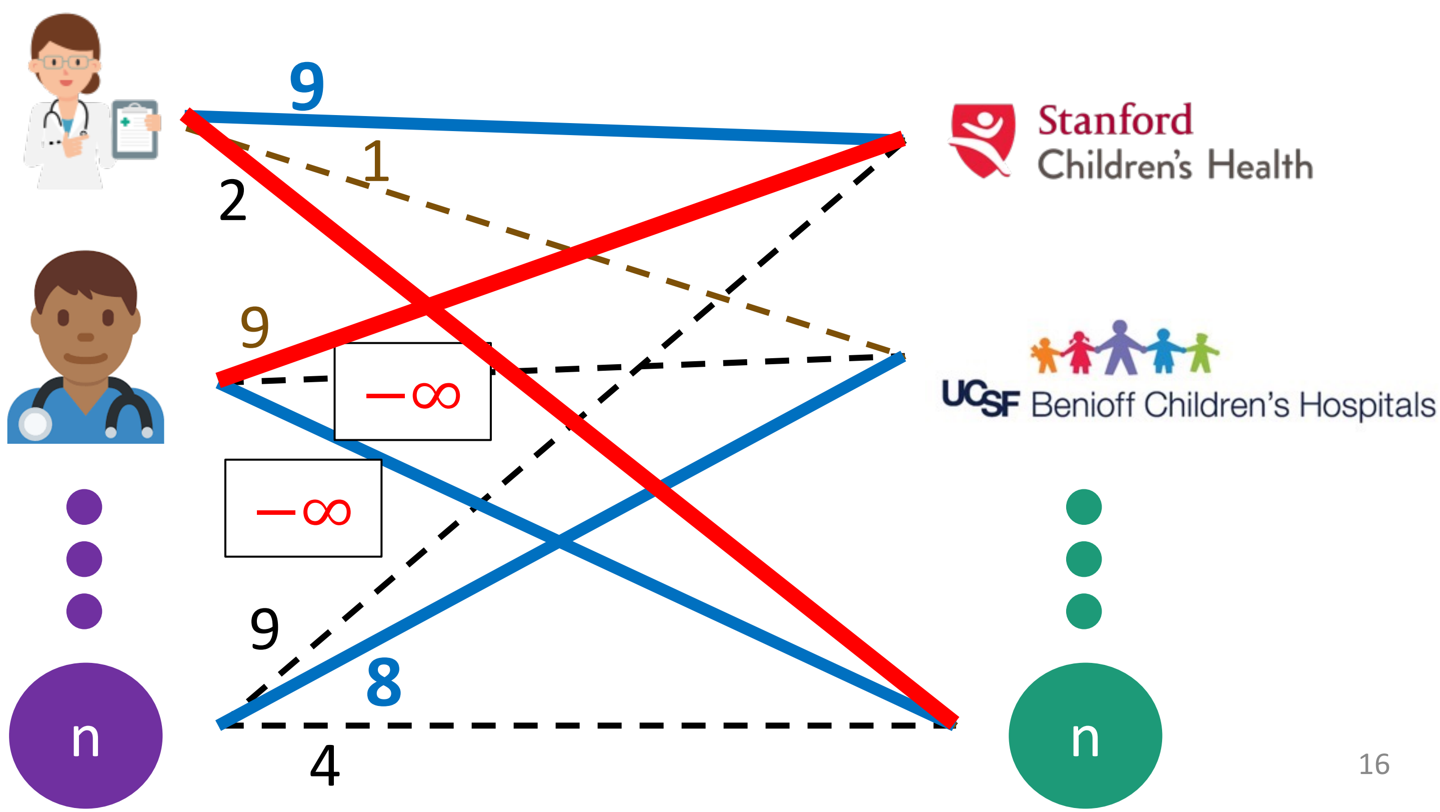
Missing step:

How does the *algorithm* get the preferences?

Where does your input come from?

... and what can go wrong if we don't think about it carefully:

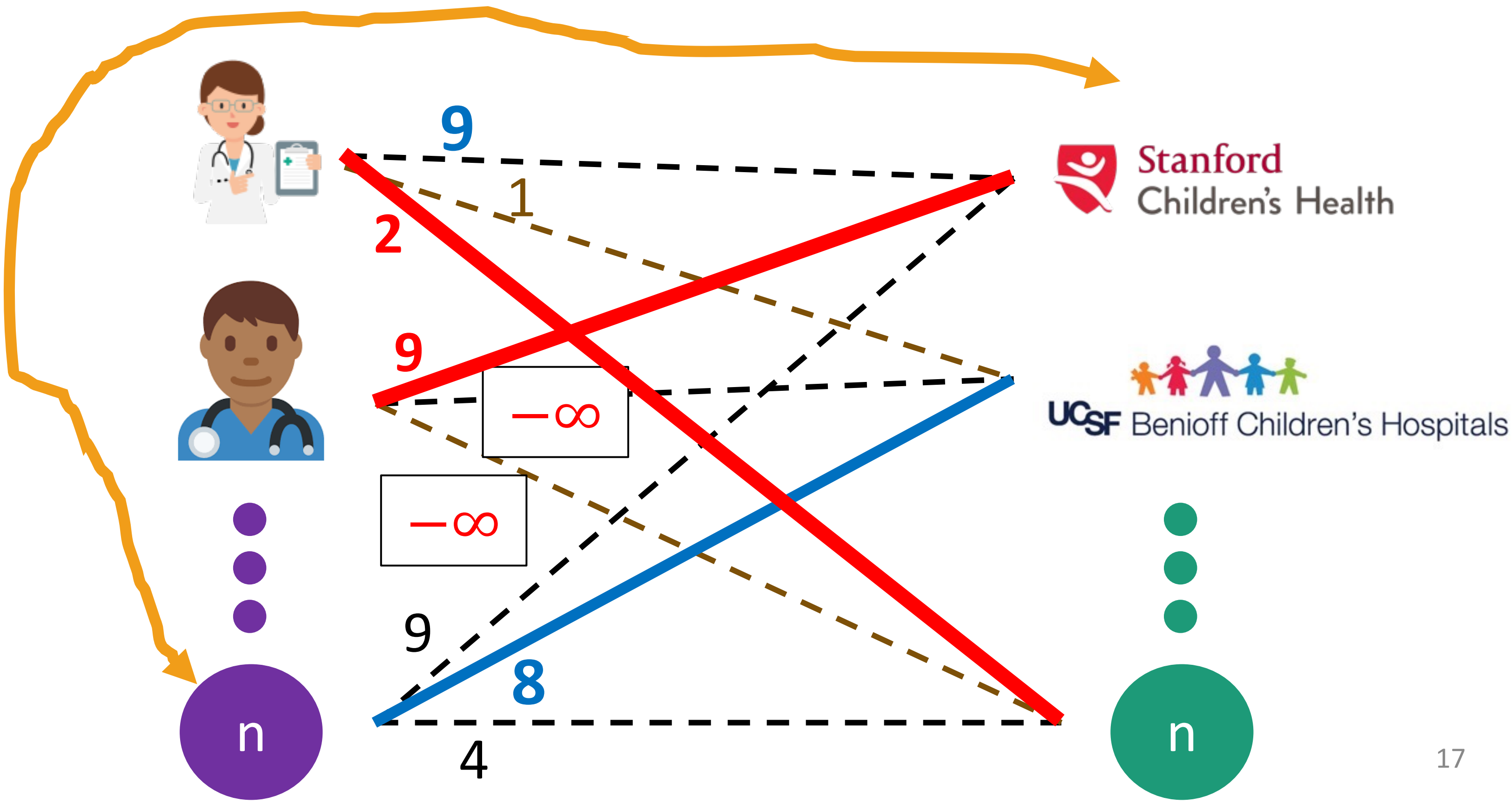
1. Some doctors may misreport their preferences



Where does your input come from?

… and what can go wrong if we don't think about it carefully:

1. Some doctors may misreport their preferences
2. Some doc+hospital may match outside your algorithm



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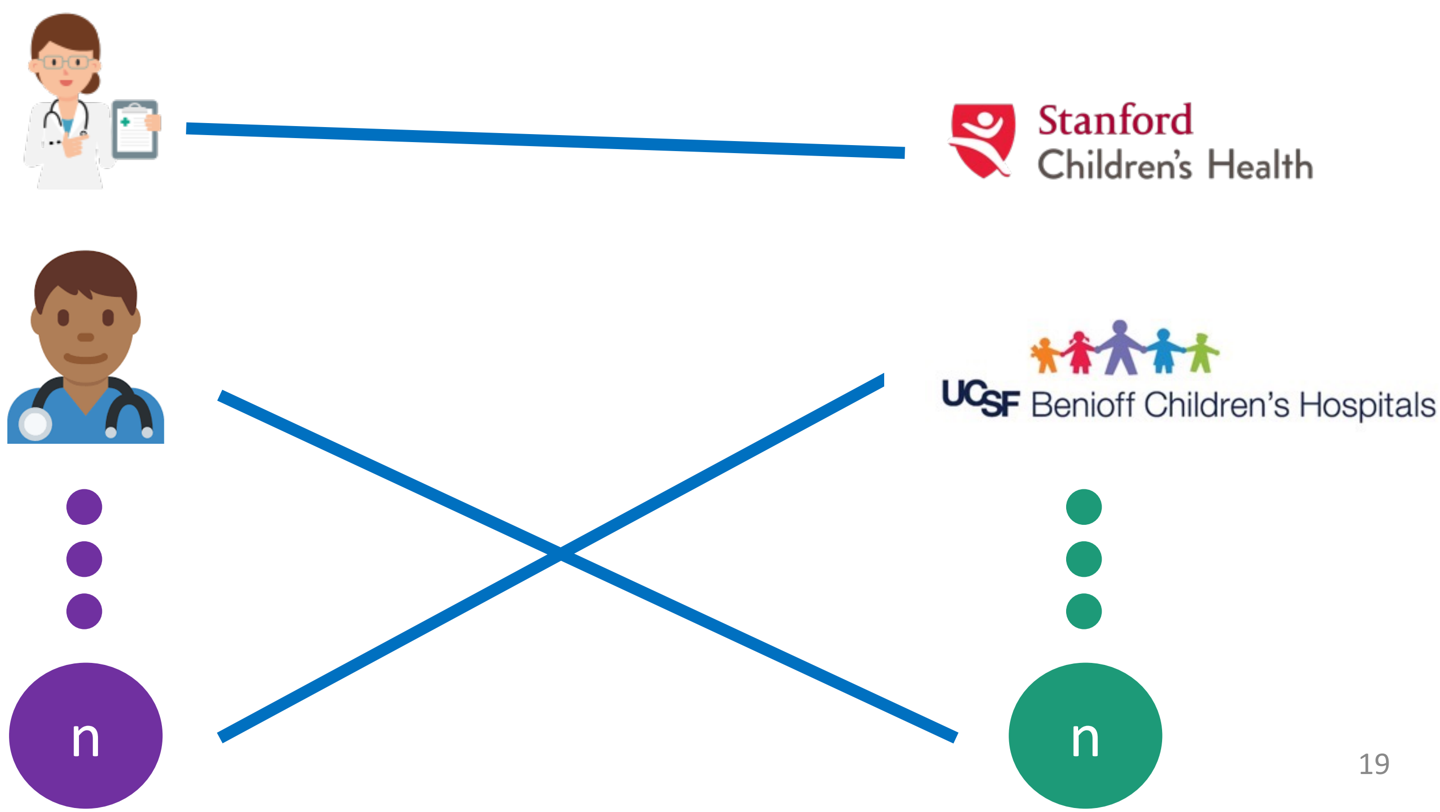
- Discussion, applications and non-applications

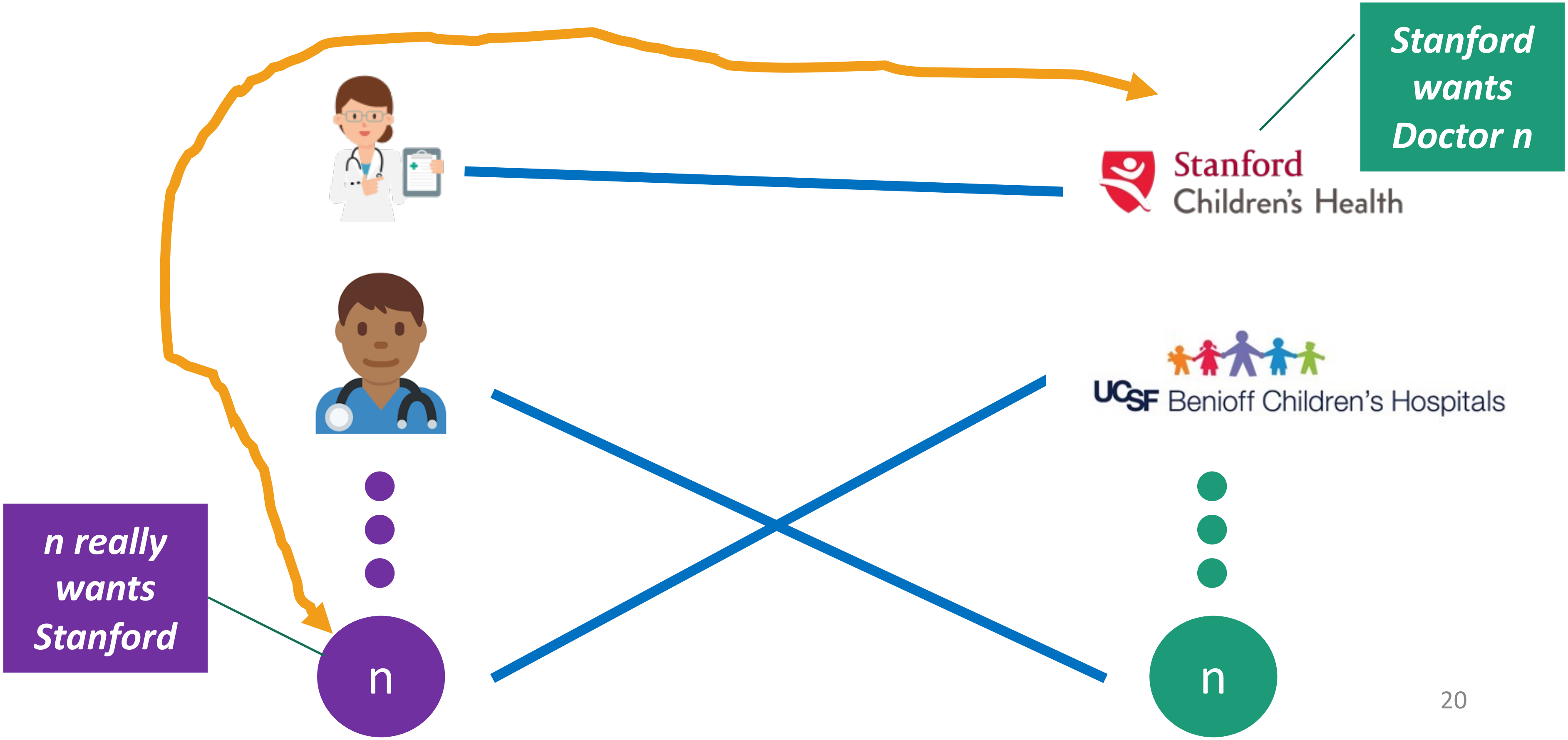
Stable Matching



Stable Matching

Definition (blocking pair):

Given Matching M, (Doctor i, Hospital j) are a ***blocking pair*** if they prefer each other to their assignment in M



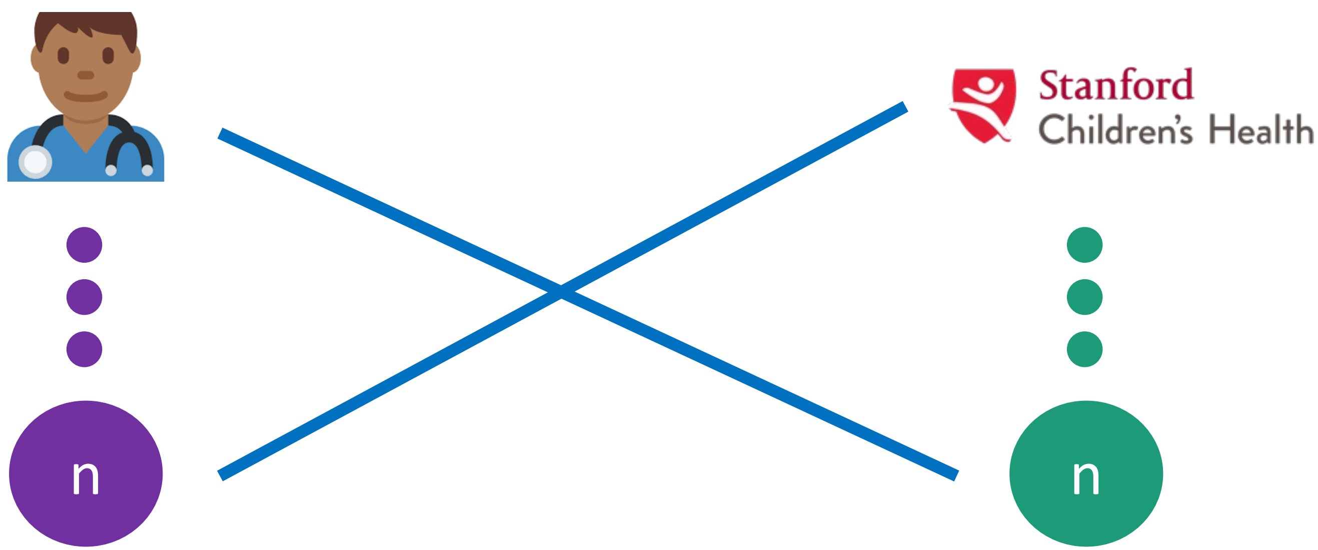
Stable Matching

Definition (blocking pair):

Given Matching M, (Doctor i, Hospital j) are a ***blocking pair*** if they prefer each other to their assignment in M

Definition (stable matching):

M is a ***stable matching*** if there are no blocking pairs.



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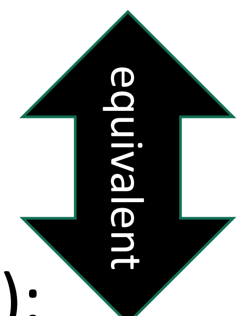


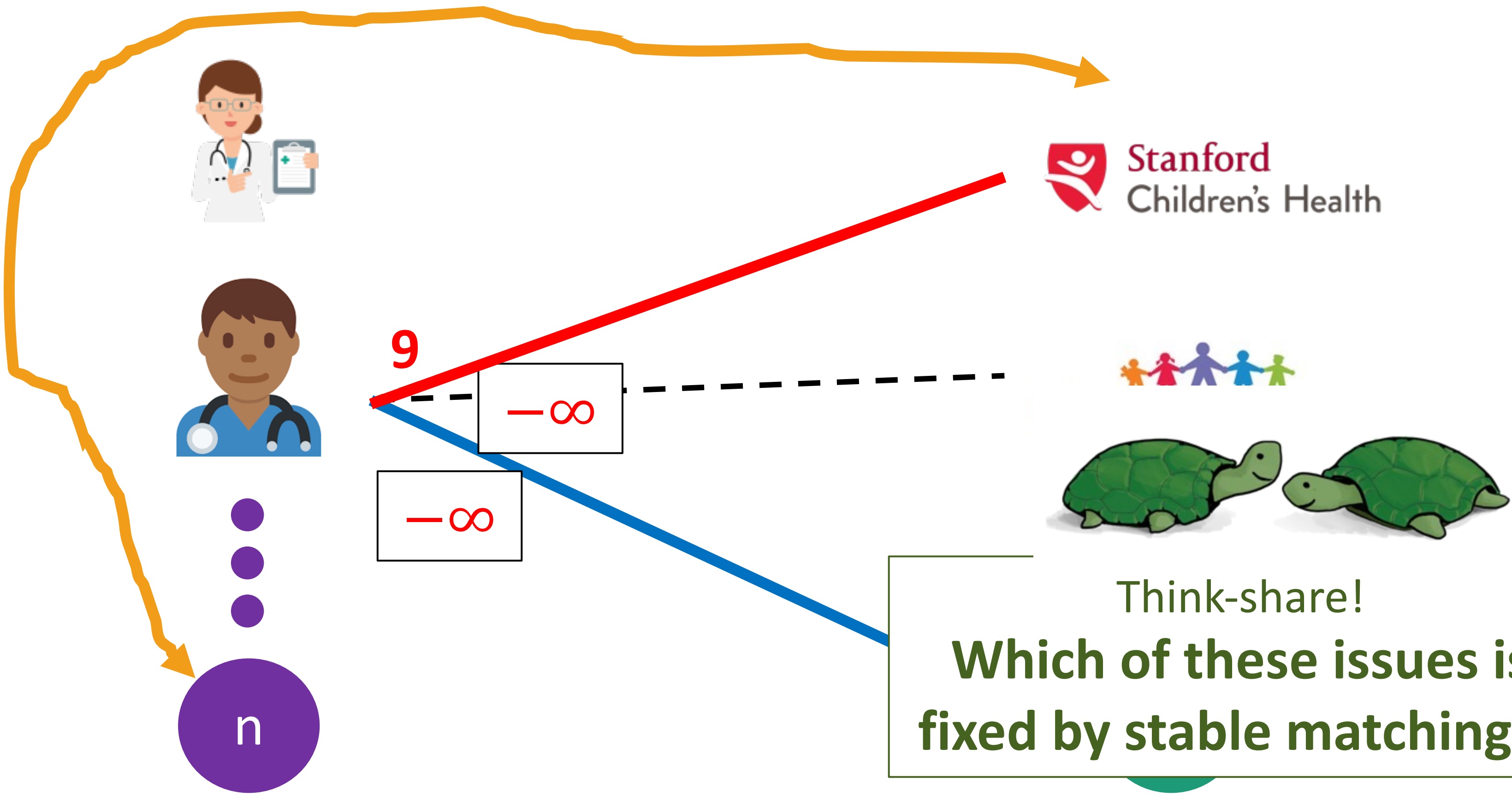
For every unmatched pair (i,j):

- Doctor i prefers Hospital M(i) over Hospital j, or;
- Hospital j prefers Doctor M(j) over Doctor i

*Un*stable Matching and incentives

Problems we identified with unstable matchings:

1. Some doctors may misreport their preferences
2. Some doc+hospital may match outside your algorithm



Think-share!

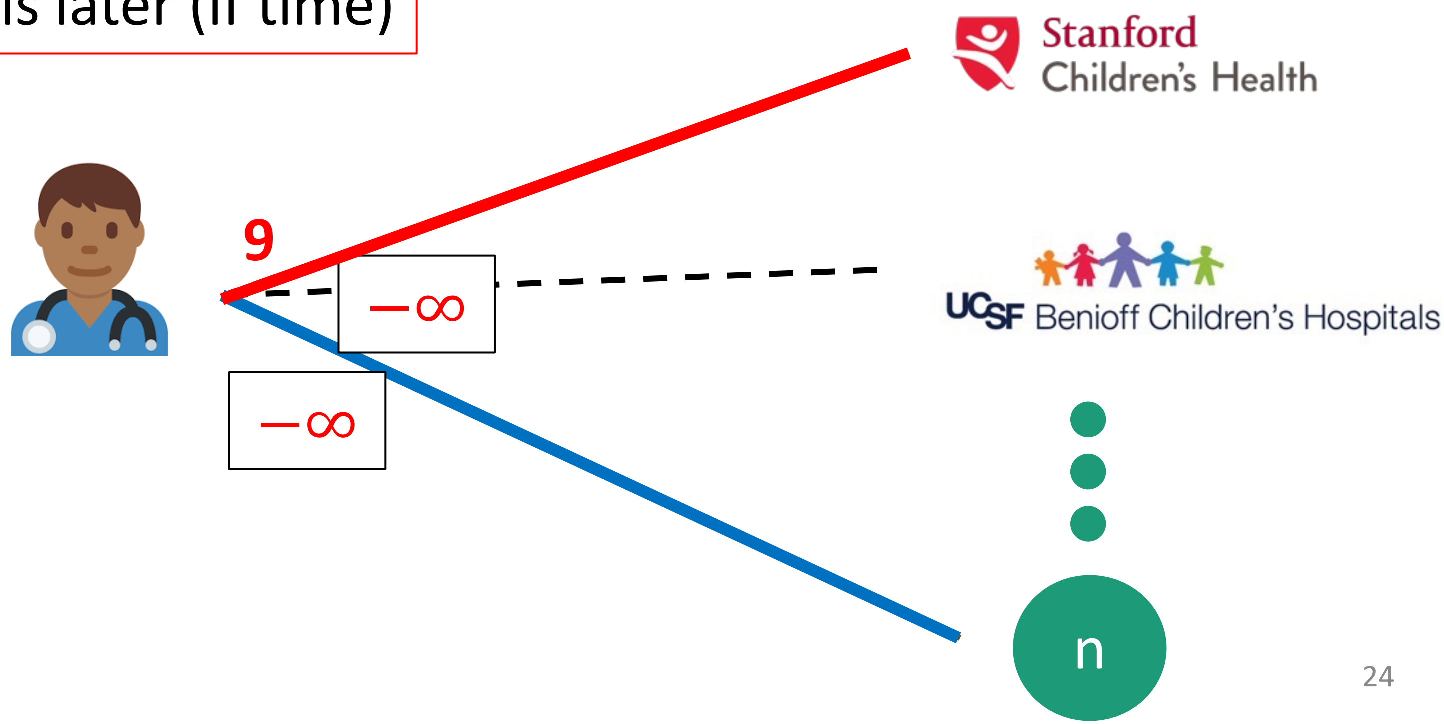
Which of these issues is fixed by stable matchings?

Stable Matching and incentives

With **stable matching**:

1. Will doctors misreport their preferences?

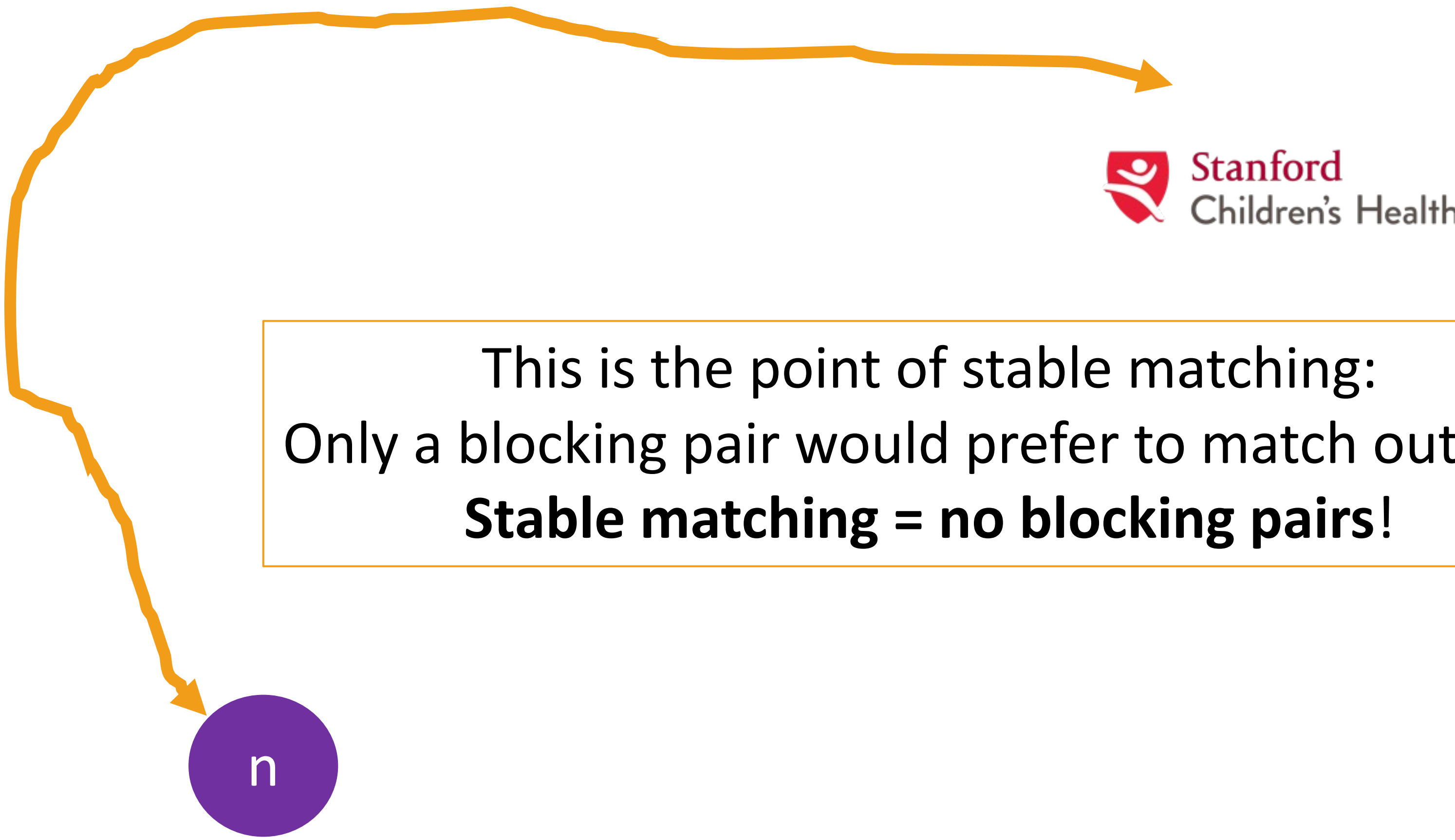
Not obvious!
We'll come back
to this later (if time)



Stable Matching and incentives

With **stable matching**:

- Doctor+hospital ***never*** prefer to match outside algorithm!



This is the point of stable matching:
Only a blocking pair would prefer to match outside.
Stable matching = no blocking pairs!

Stable Matching Problem

How to find stable matchings!
(do they even exist?)

Stable Matching Problem

Alice's preferences	
1^{st}	Stanford
2^{nd}	n
…	…
n^{th}	UCSF

Stanford's preferences	
1^{st}	Alice
2^{nd}	n
…	…
n^{th}	Bob

Stable Matching Problem

Input: each doctor/hospital submits a ranking (permutation) of {1,…,n}

Output: a **stable matching**

Definition (blocking pair):

Given Matching M, (Doctor i, Hospital j) are a ***blocking pair*** if they prefer each other to their assignment in M

Definition (stable matching):

M is a ***stable matching*** if there are no blocking pairs.

Naïve attempt #1

Greedy algorithm:

Step 1- match all the pairs (i,j) such that
j is i's top choice, and i is j's top choice

Step 2- hopefully recurse on the rest somehow…

- Observation: Step 1 never rules out any solution ☺

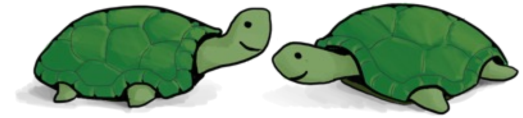
A slightly more ambitious attempt

Greedy attempt #2:

Step 1- try to match every doctor to her favorite hospital

- Break ties by hospital preference

Step 2- hopefully recurse on the rest somehow…



A slightly more an...

Think-pair-share!

Matching (C,y) was a bad idea…

How could we avoid it?

Greedy attempt #2:

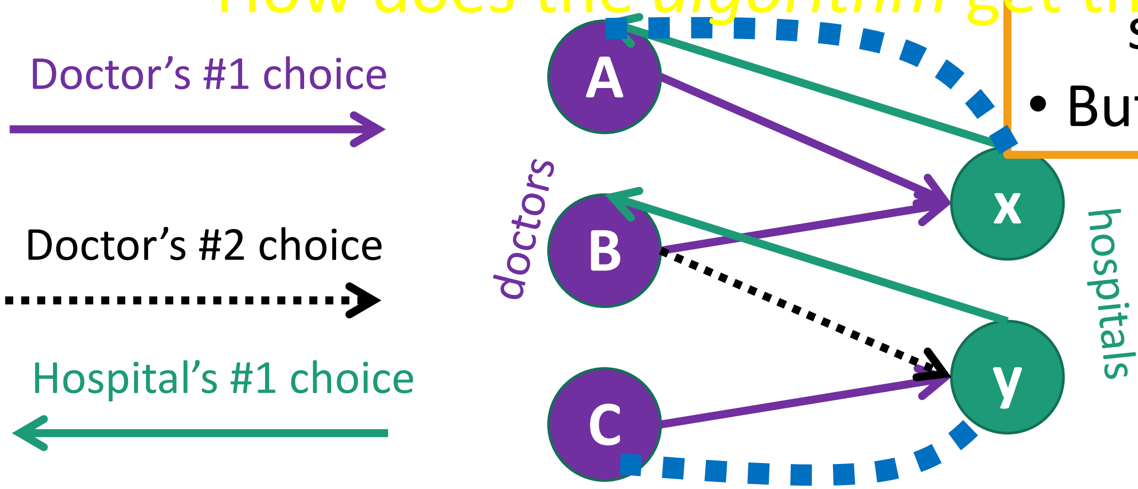
Step 1- try to match every doctor to her favorite hospital

- Break ties by hospital preference

We're already wrong!

How does the *algorithm* get the preferences?

- Step 1: A,B want x, C wants y
  so we **match** (A,x) and (C,y)
- But now (B,y) is **blocking**!



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Questions?

Definition (blocking pair):

Given Matching M, (Doctor i, Hospital j) are a ***blocking pair*** if they prefer each other to their assignment in M

Definition (stable matching):

M is a ***stable matching*** if there are no blocking pairs.

⇕ equivalent

For every unmatched pair (i,j):

- Doctor i prefers Hospital M(i) over Hospital j, or;
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Deferred Acceptance Algorithm

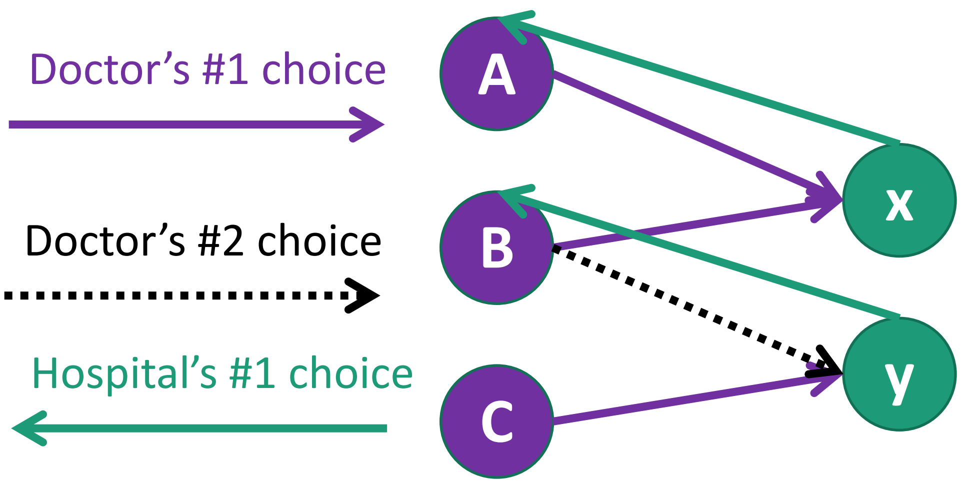
[Gale Shapley ‘62] -> 2012 Nobel Prize* in Econ!

*- Joint w/ Al Roth from Stanford

Deferred Acceptance Algorithm

Main idea: ***try*** to match each doctor to top choice;

if you discover a blocking pair, just switch the matching!

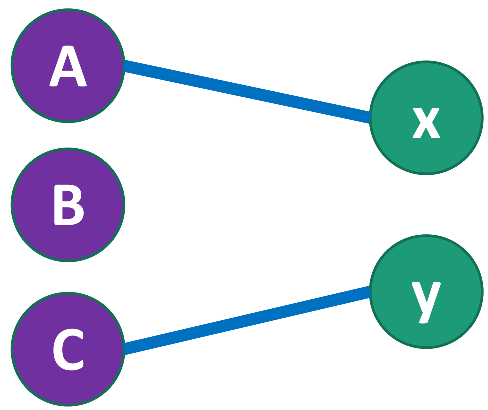


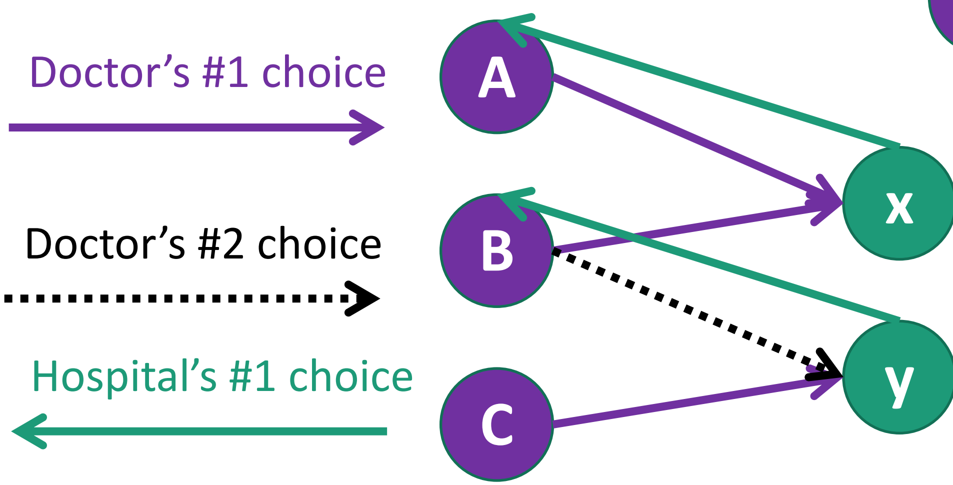
The issue was:
A,B want x, C wants y
we tried to match (A,x) and (C,y)
but then (B,y) was **blocking**!

Deferred Acceptance Algorithm

Main idea: ***try*** to match each doctor to top choice;
if you discover a blocking pair, just switch the matching!

Algorithm iteration 1:
A, B want x; C wants y
So we match (A,x) and (C,y)





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but then (B,y) was **blocking**!

Deferred Acceptance Algorithm

Main idea: ***try*** to match each doctor to top choice;

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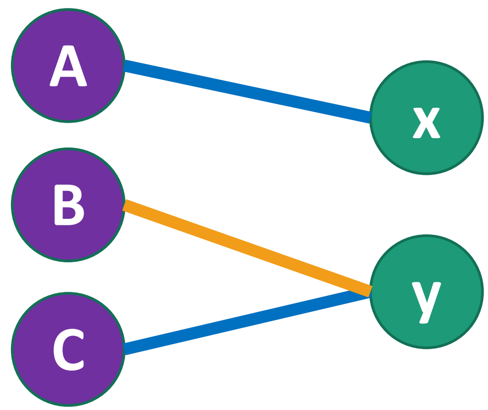
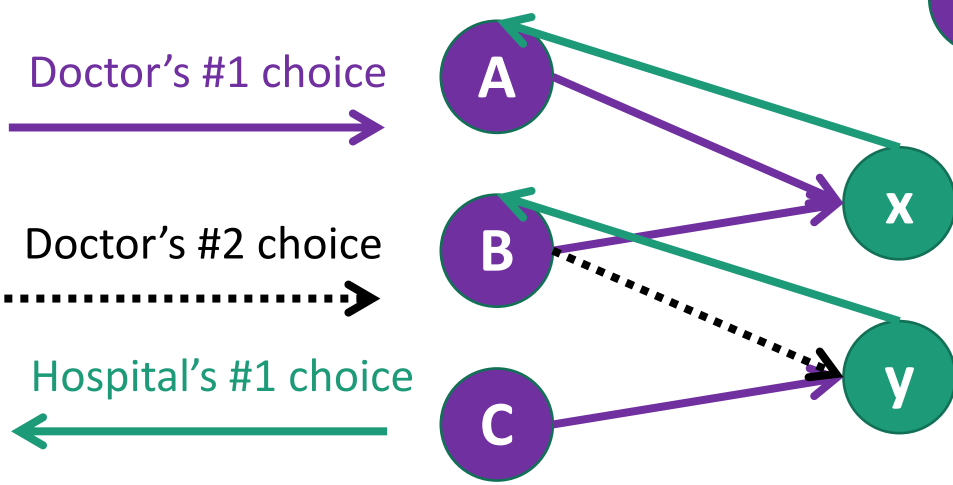
Algorithm iteration 2(a):

Now notice that (B,y) is **blocking** ☹





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Deferred Acceptance Algorithm

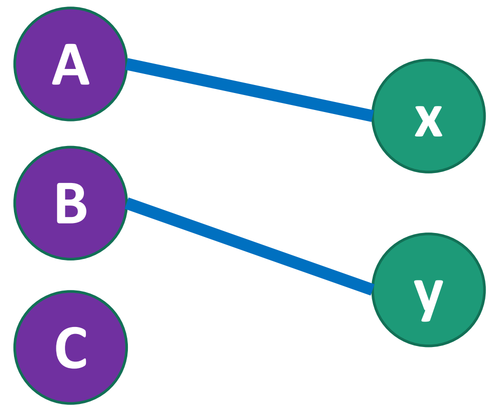
Main idea: ***try*** to match each doctor to top choice;

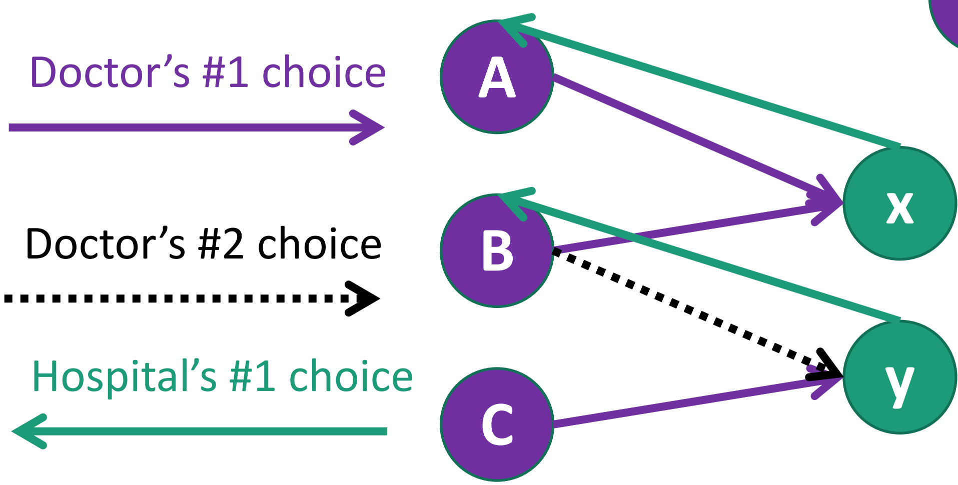
if you discover a blocking pair, just switch the matching!

Algorithm iteration 2(b):

Add (B,y) to the matching ☺

(and remove (C,y))





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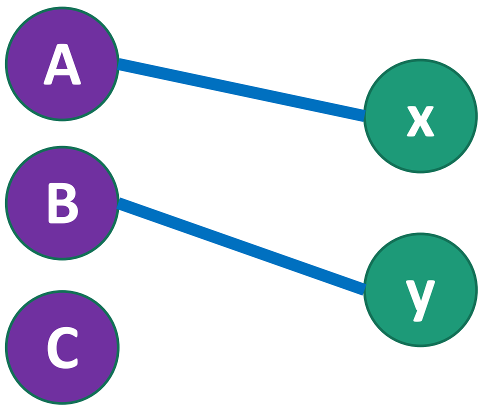


Don't worry

Just switch around

until no blocking pairs!

Lucky the Lackadaisical Lemur

The issue was:

A,B want x, C wants y

we tried to match (A,x) and (C,y)

but then (B,y) was **blocking**!

Deferred Acceptance Algorithm

Main idea: ***try*** to match each doctor to top choice;

if you discover a blocking pair, just switch the matching!

Almost-pseudo-code:

While there is an unmatched doctor **i**:

Try to match **i** to next-favorite hospital on her list;

If this hospital doesn't have a doctor yet:

Both Doctor **i** and hospital are happy with this new match ☺

Else-if this hospital prefers its current match **i'** over **i**:

Doctor **i** remains unmatched

Else-if this hospital prefers **i** over **i'**:

Unmatch **i'**; Match (**i**, hospital)

Example run-through

DA Example Run 1

Alice

X, Y, Z

B, A, C

X

Bob

Y, X, Z

A, B, C

Y

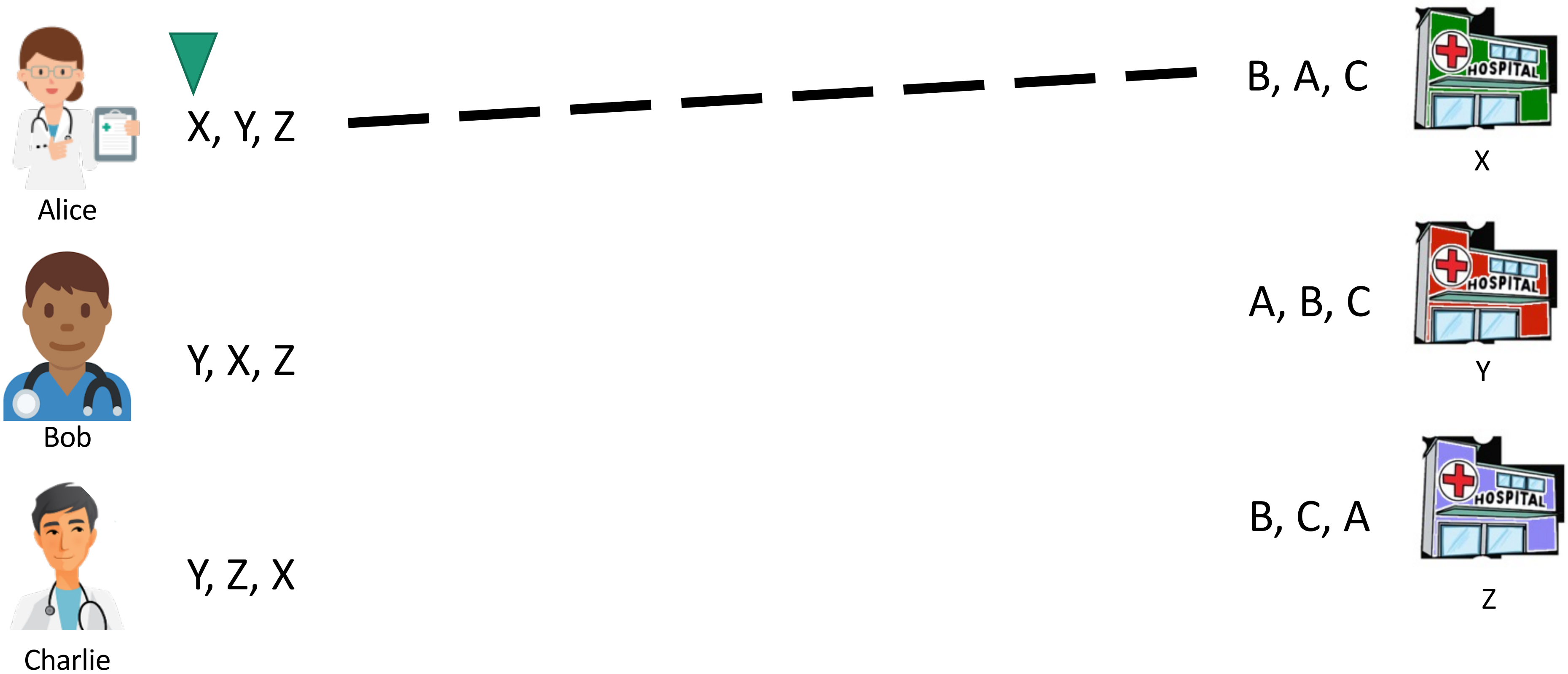
Charlie

Y, Z, X

B, C, A

Z

DA Example Run 1



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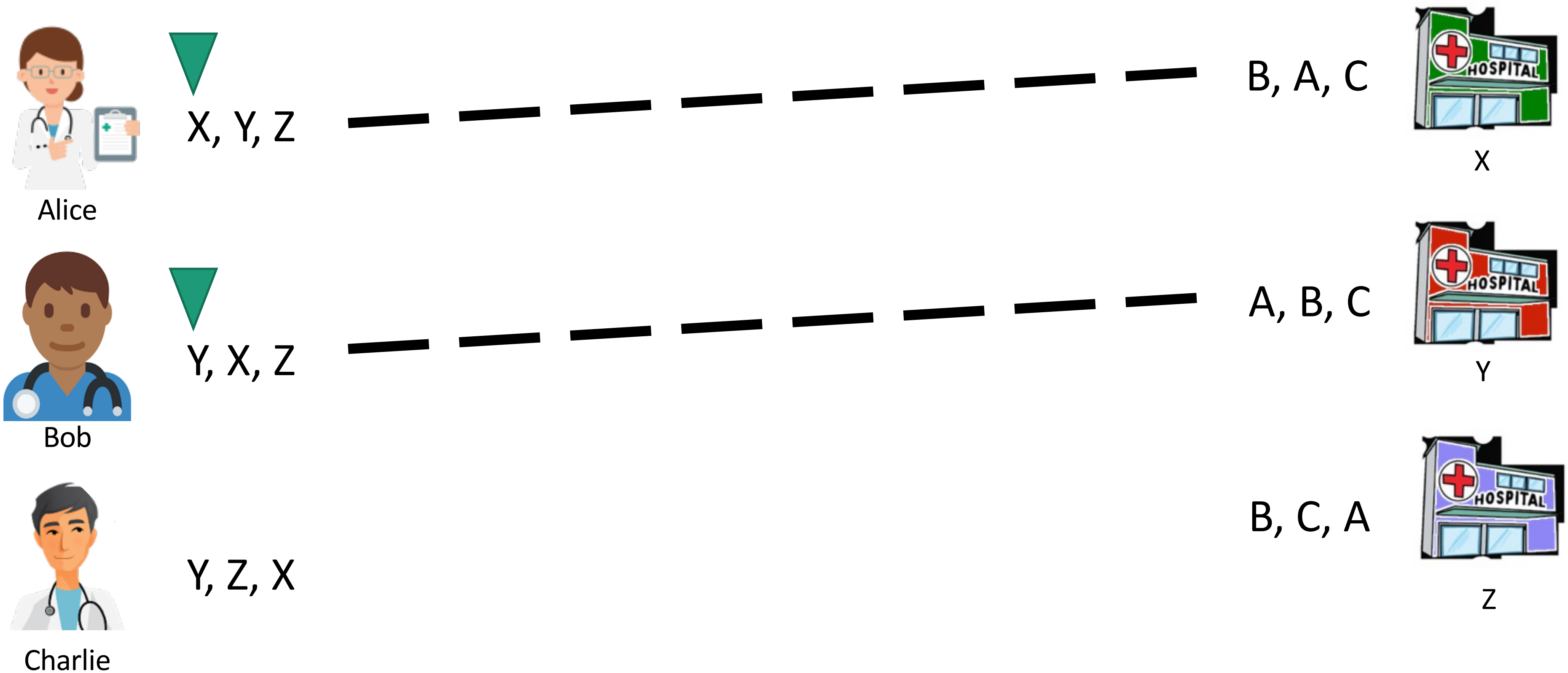

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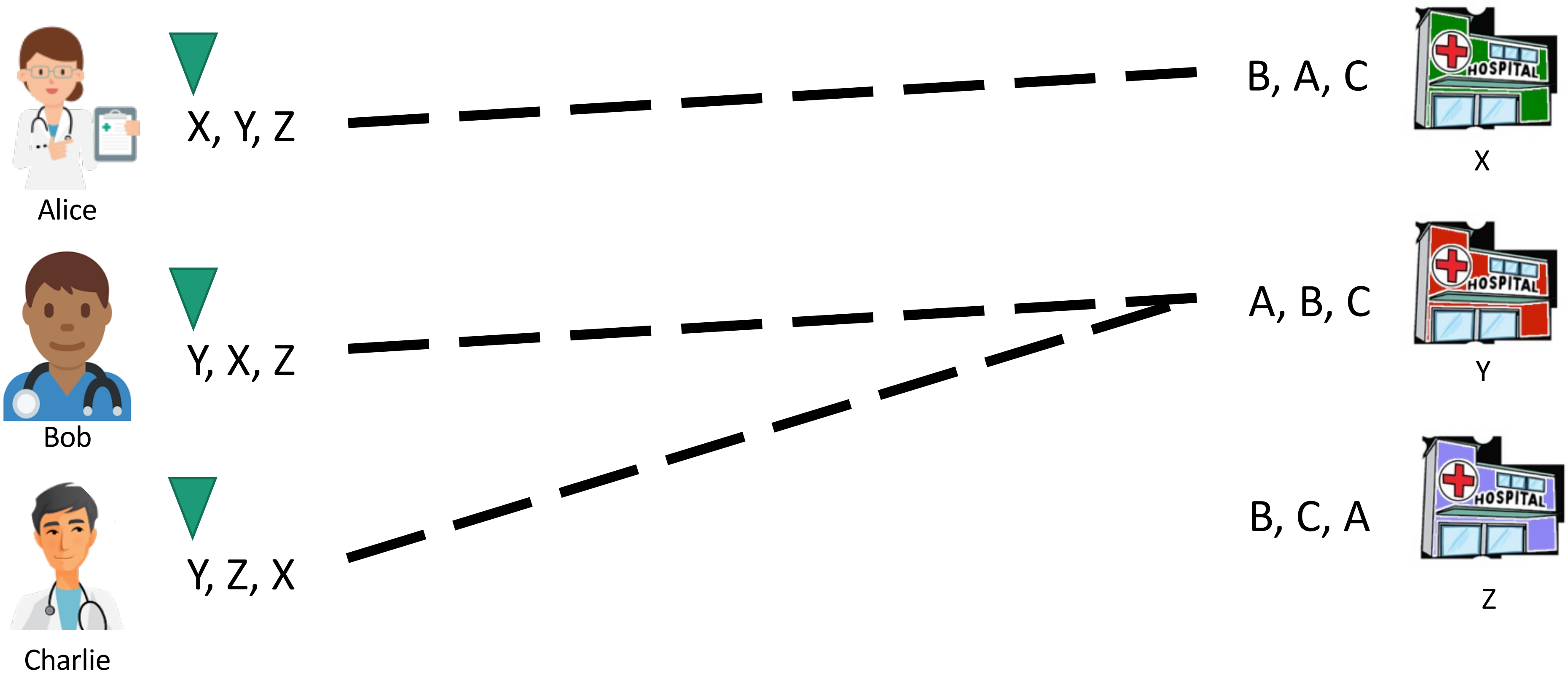

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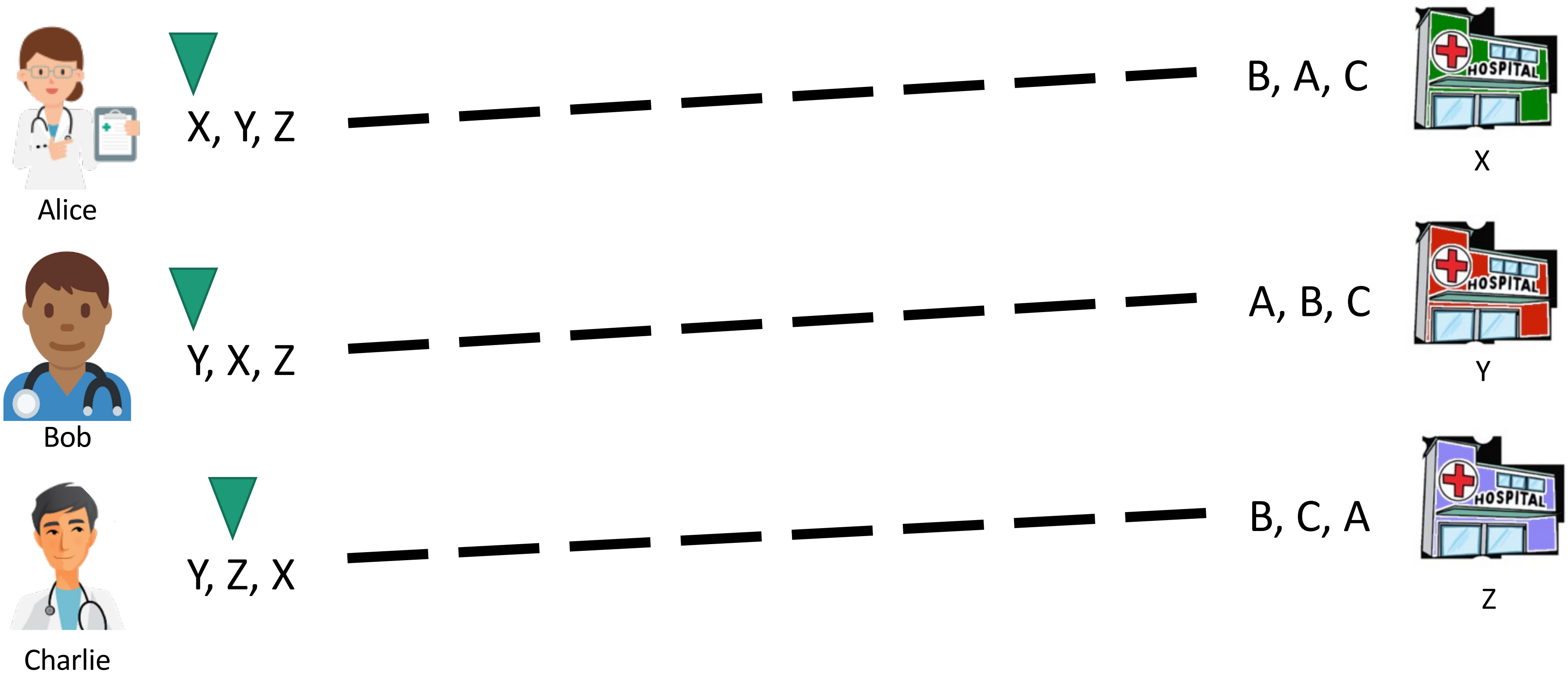

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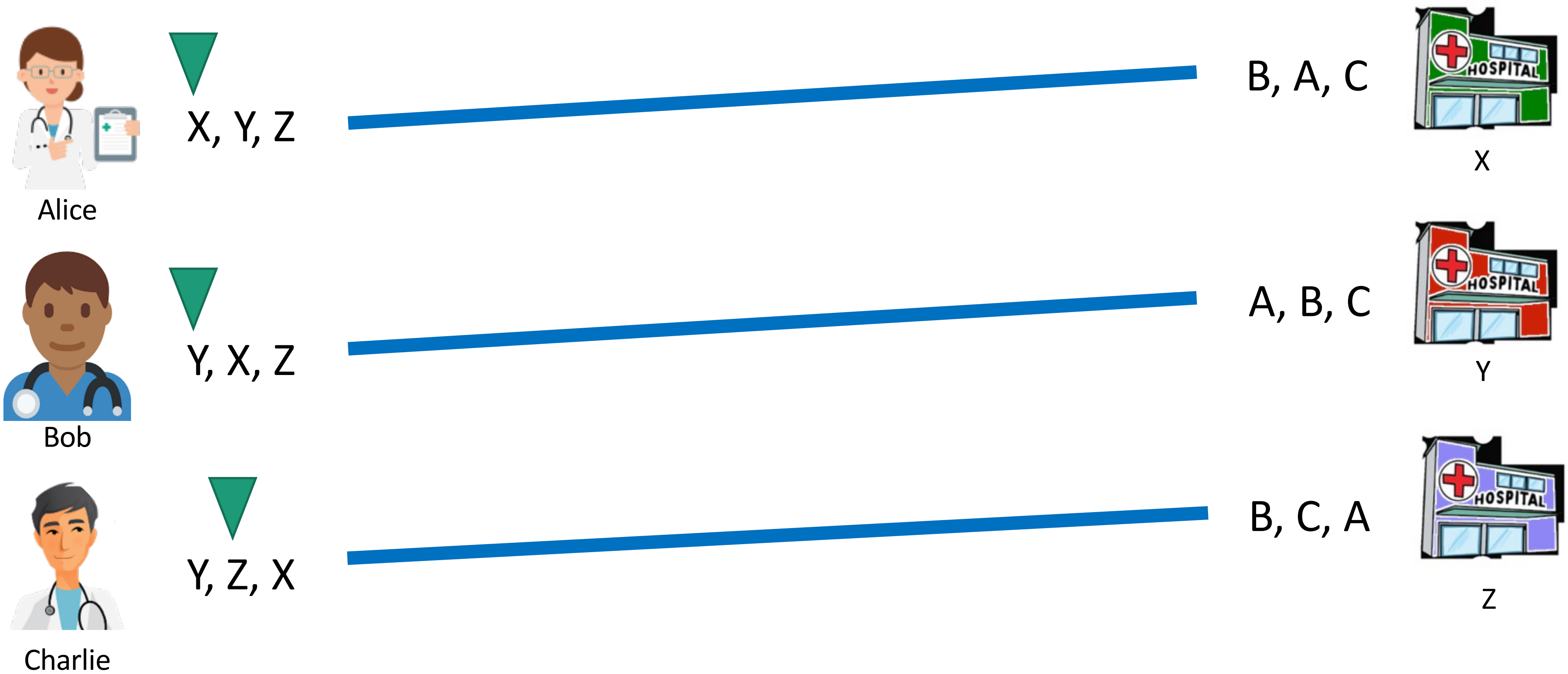


Another example

DA Example Run 2

Alice

X, Y, Z

Bob

Y, X, Z

Charlie

Y, Z, X

B, A, C

X

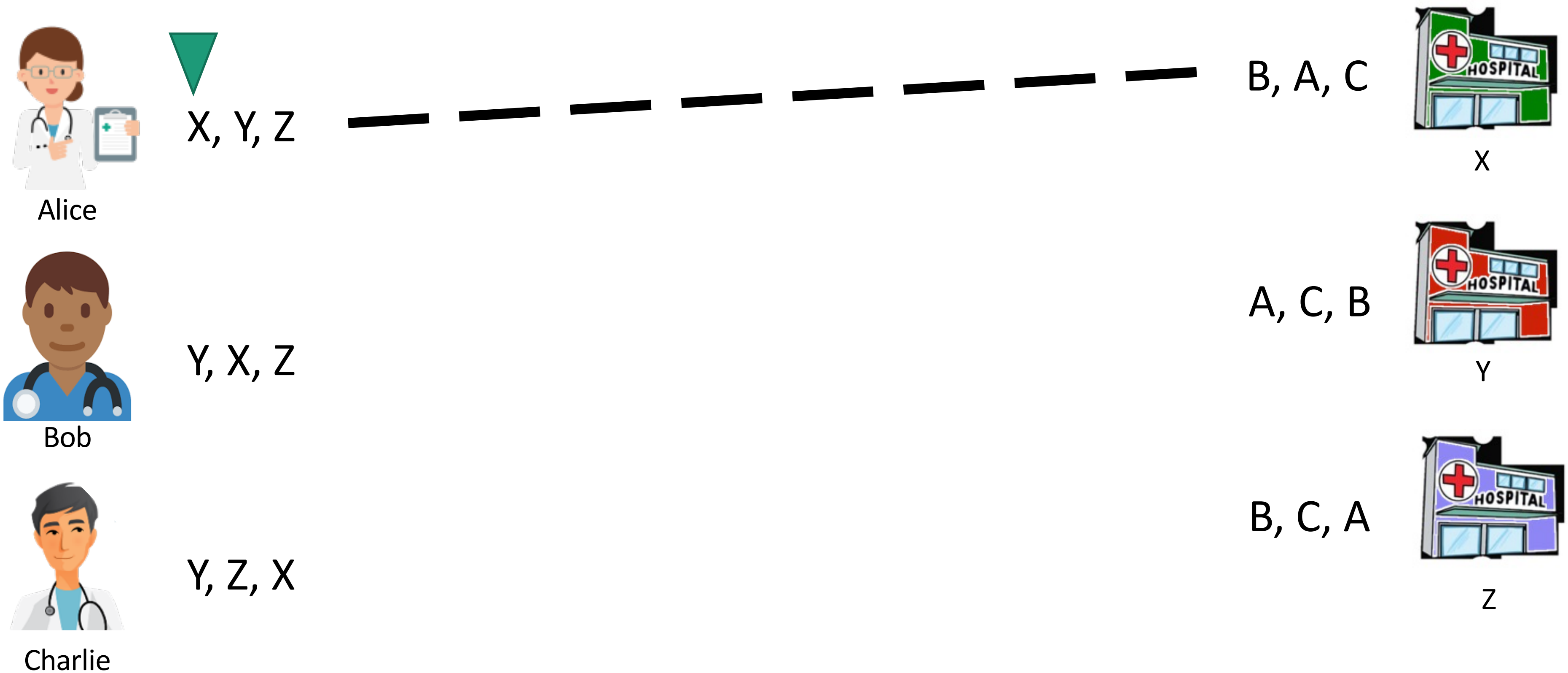
A, C, B

Y

B, C, A

Z

DA Example Run 2



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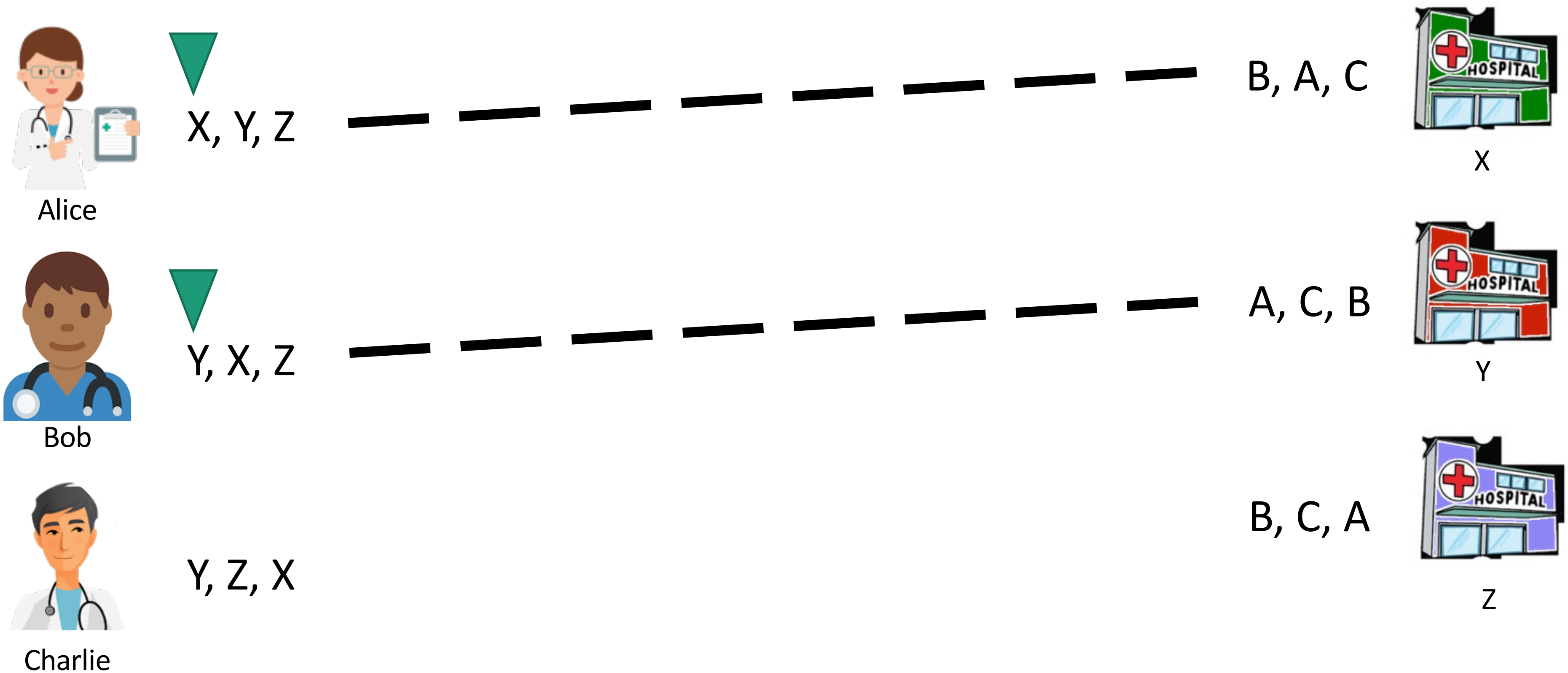

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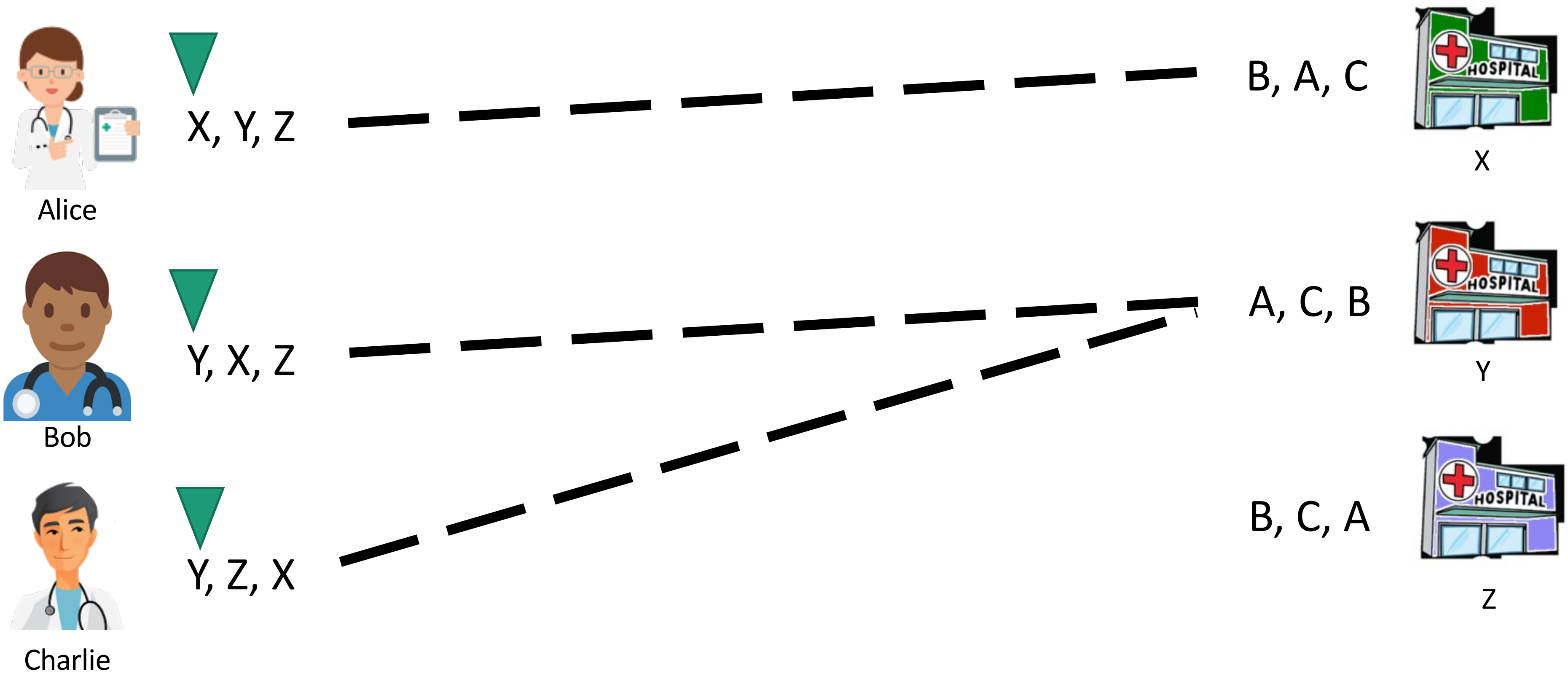

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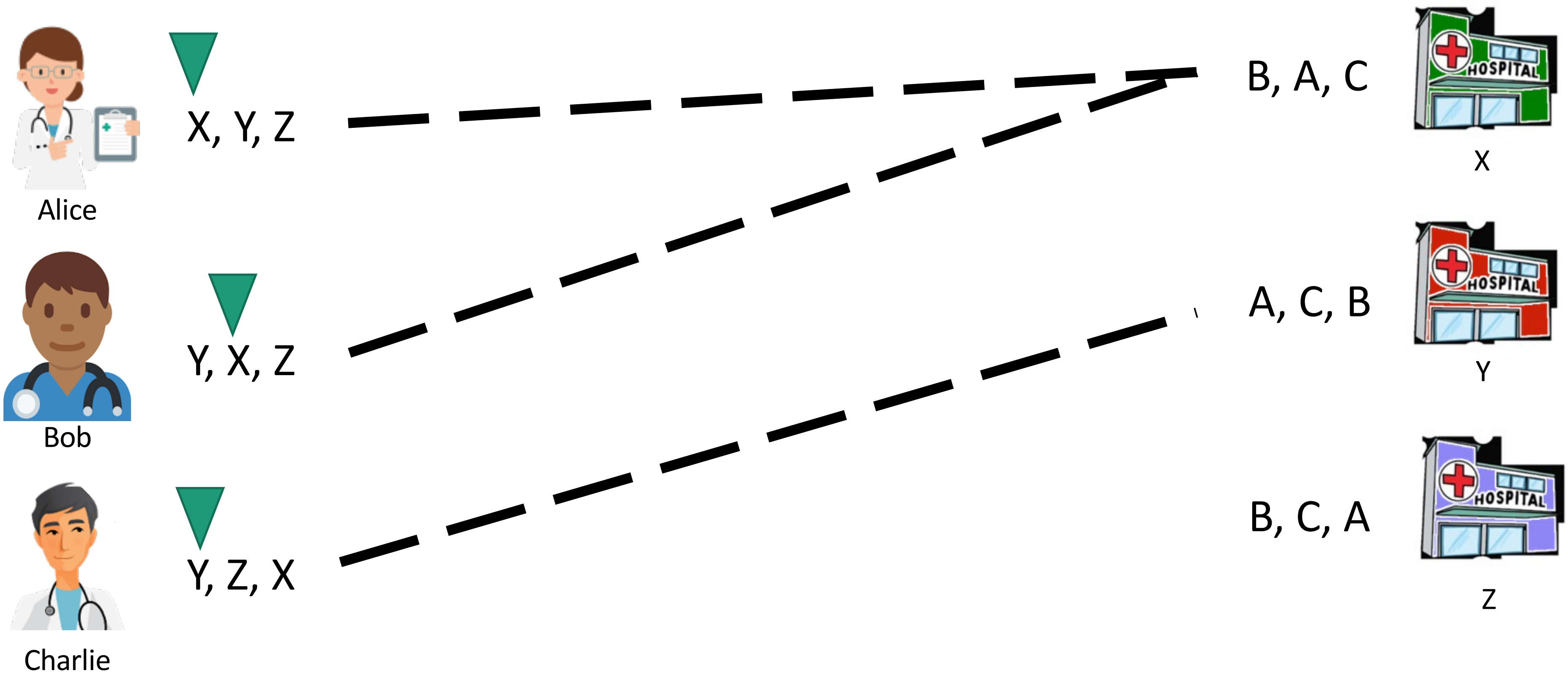

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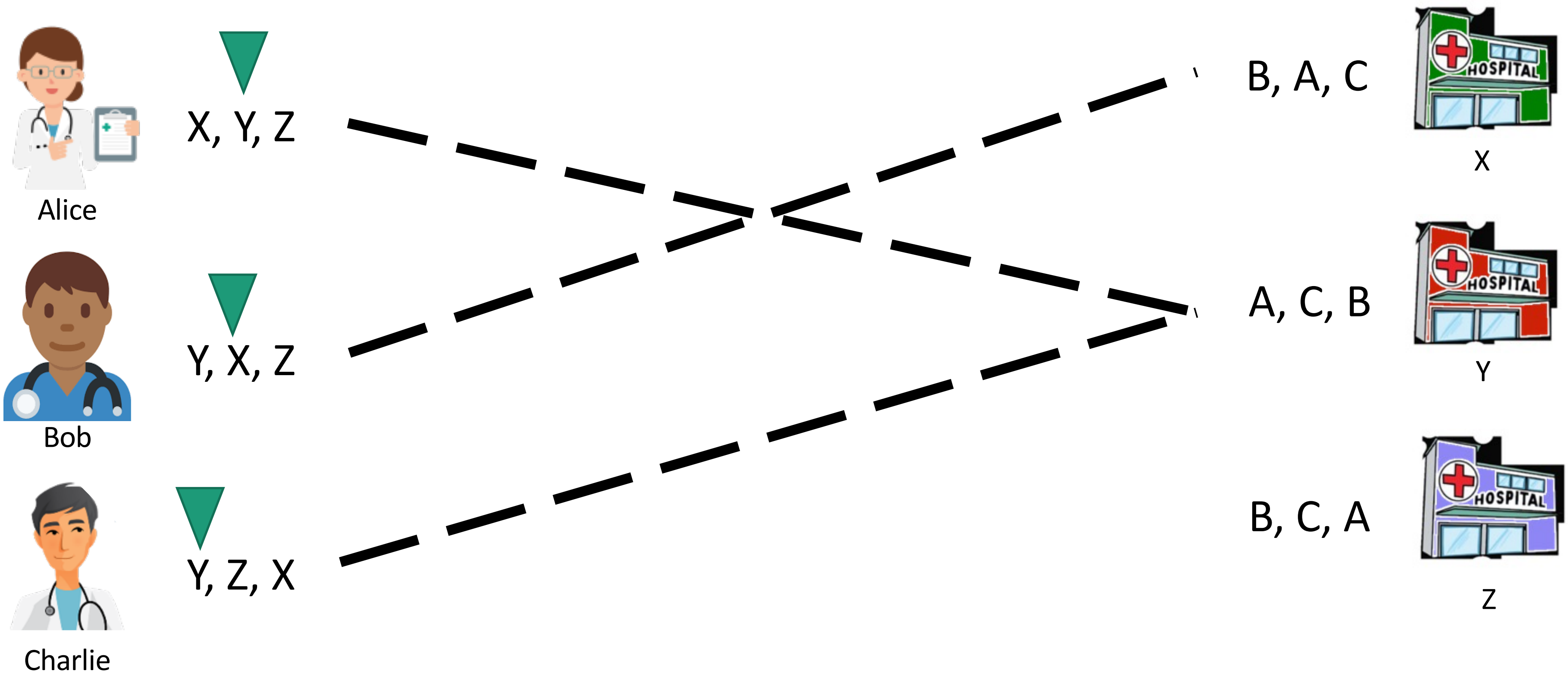

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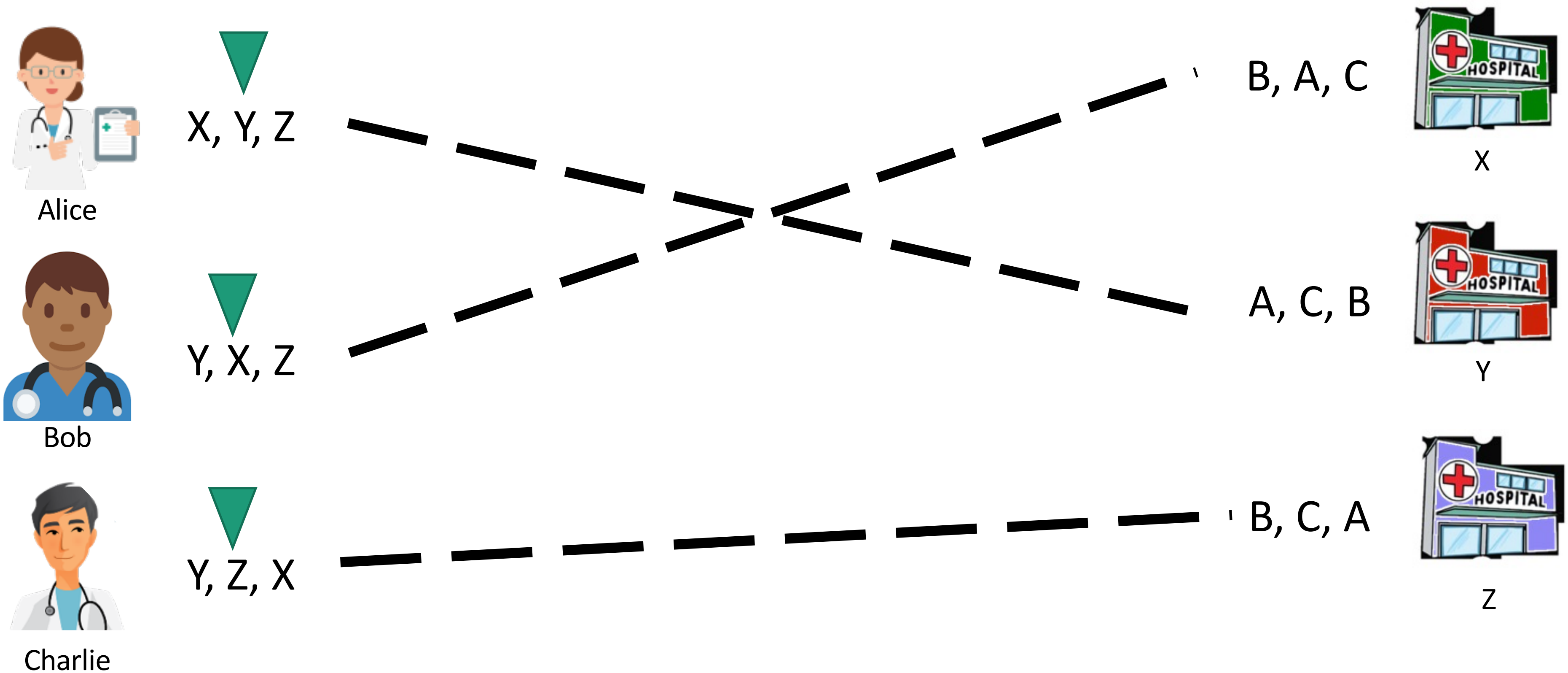

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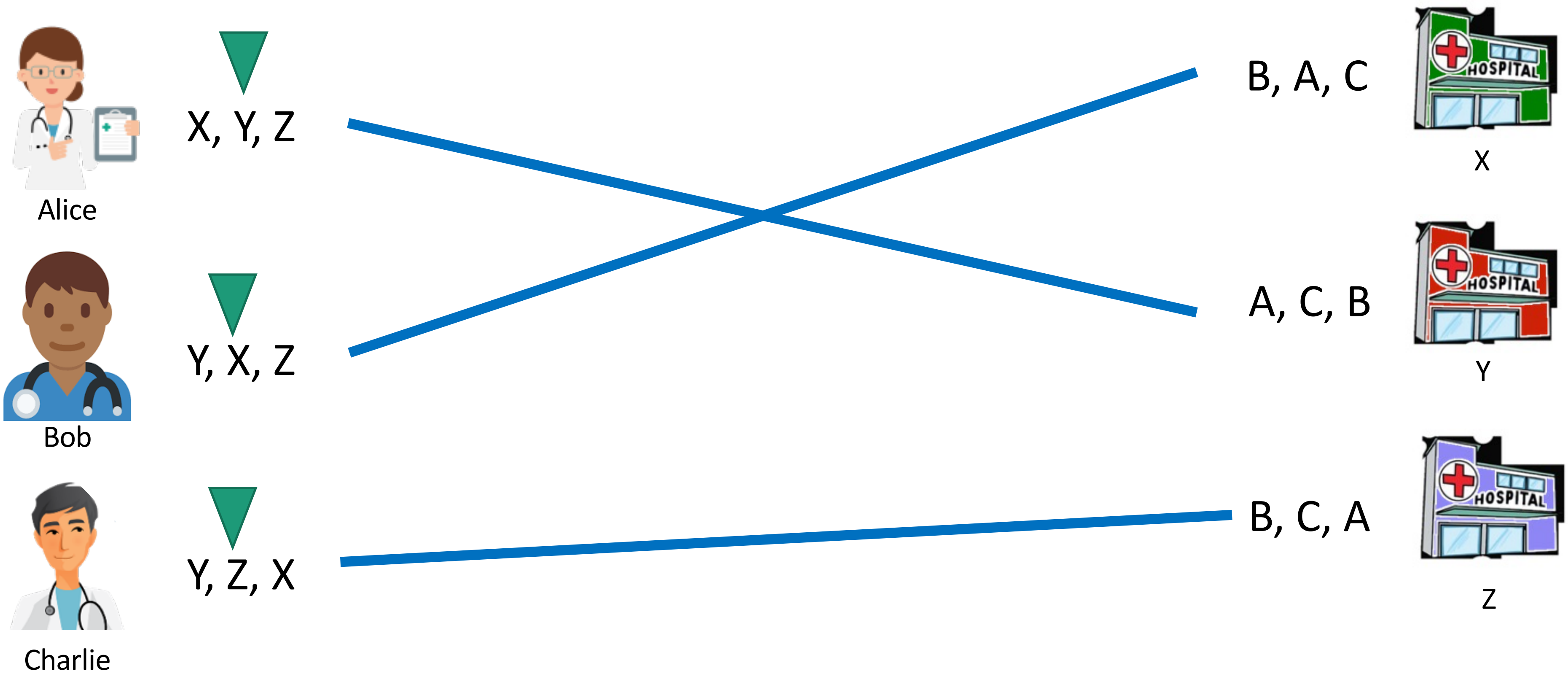


Deferred Acceptance Algorithm

Deferred-Acceptance(Doctors,Hospitals):

// initialize

freeDoctors ← Doctors

for all d in Doctors:

    d.current ← 0

for all h in Hospitals:

    h.doctor ← NIL

// main loop

while (exists d in freeDoctors)

    h ← d.ranking[d.current++]   // h is d's next favorite

    if (h is free)

        h.doctor ← d

        remove d from freeDoctors

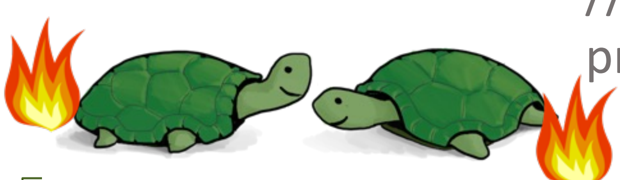
    else-if (h.rank[d] < h.rank[h.doctor])   // h prefers d to previous match

        add h.doctor to freeDoctors

        h.doctor ← d

        remove d from freeDoctors

return (h,h.doctor) for all h in Hospitals



Think-share!

Running time?

Deferred Acceptance Algorithm

Running time:

Each iteration of while loop = O(1)

Each iteration: We +1 d.current for some doctor

We always have: d.current $\leq n$ for every doctor

(There are n doctors…)

Therefore, total run-time = $O(n^2)$

```
        //  main loop
while (exists d in freeDoctors)
    h ← d.ranking[d.current++]   // h is d's next favorite

    if (h is free)
        h.doctor ← d
        remove d from freeDoctors

    else-if (h.rank[d] < h.rank[h.doctor])
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        h.doctor ← d
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return (h,h.doctor) for all h in Hospitals
```

DA algorithm

- Does it work?
  - Yes!

- Is it fast?
  - $O(n^2)$ - this is linear in the input size!

At worst exhaust through every doctor's preference list

Deferred Acceptance works!

Theorem: Given n doctors and n hospitals,
DA algorithm outputs a complete stable matching.

Corollary: A stable matching exists.

(This is not obvious!)

Proof of Theorem

Theorem: Given n doctors and n hospitals,
DA algorithm outputs a complete stable matching.

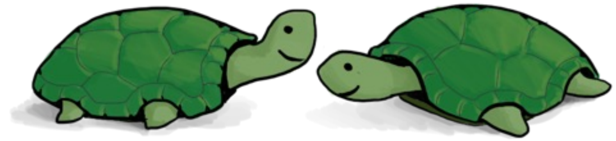
Proof: Follows from Claims 1+3 below…

Claim 1: At every iteration, current match is stable w.r.t. non-free doctors and hospitals.

Claim 2: Once a hospital is matched, it remains matched (possibly to a different doctor) until end of algorithm.

Claim 3: At the end of algorithm, every doctor/hospital is matched.

Proof of claims



Think-share:
Prove these!

Claim 1: At every iteration, current match is stable w.r.t. non-free doctors and hospitals.

Proof by contradiction: Suppose (d,h) blocking pair.
→ d is currently matched to worse hospital than h.
→ d already tried to match to h.
→ h either refused d or left d later. Why?
→ h must be matched to better doctor than d – contradiction!

Claim 2: Once a hospital is matched, it remains matched (possibly to a different doctor) until end of algorithm.

"**Proof**": obvious from algorithm

Claim 3: At the end of algorithm, every doctor/hospital is matched.

Proof by contradiction: Suppose (d,h) still free.
End of algorithm → d already tried to match to h.
→ after that step, h wasn't free → by Claim 2, contradiction!

Deferred Acceptance works!

Theorem: Given n doctors and n hospitals,
DA algorithm outputs a complete stable matching.

Corollary: A stable matching exists.

Claim 1: At every iteration, current match is stable w.r.t. non-free doctors and hospitals.

Claim 2: Once a hospital is matched, it remains matched (possibly to a different doctor) until end of algorithm.

Claim 3: At the end of algorithm, every doctor/hospital is matched.

What have we learned?

Blocking Pair: A doctor and hospital that prefer each other over their respective matches.

Stable Matching: A matching without blocking pairs!

Deferred Acceptance Algorithm

"Tentatively match each free doctor to best interested hospital. Allow the hospital to leave match when a better doctor arrives."

Runs in time $O(n^2)$ = linear in input size ☺

Today

- Hospitals/residents problem

- Stable matchings
  - Solve the hospitals/residents problem
  - But can we find them?

- Deferred Acceptance Algorithm
  - Find stable matchings!

- Discussion, applications and non-applications

The optimal stable matching?

DA algorithm found *a* stable matching…

- Is it *optimal*?
- What does optimality mean?

Theorem: The matching returned by DA is **doctor-optimal**, i.e. every doctor is matched to favorite hospital possible in any stable matching.

Corollary: Order of popping from freeDoctors does not change the output.

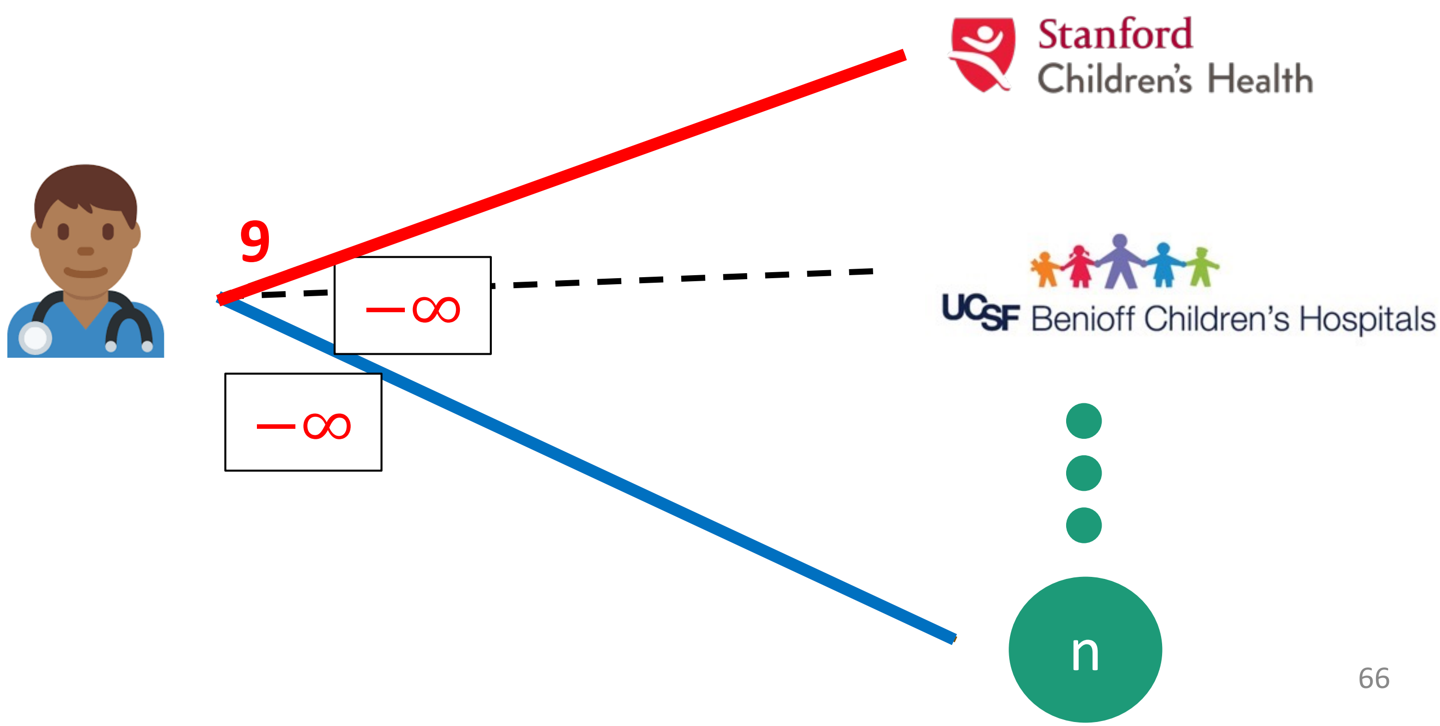
Theorem: Doctors cannot gain from misreporting their preferences.

Prove this theorem!

Stable Matching and Incentives

- Doctor 2 may tell you he only wants to go to Stanford, but…

Corollary: This won't help him
if we find Stable Matching with DA!

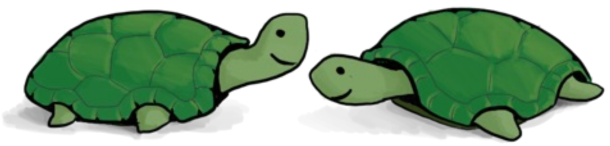


Prove this theorem too!

The optimal stable matching?

Theorem: The matching returned by DA is **hospital-*worst***, i.e. every hospital is matched to *least*-favorite doctor possible in any stable matching.

Caution: Hospitals *can* gain from misreporting their preferences.



Think-share:

How would you find a hospital-optimal stable matching?
Should actual matching be doctor- or hospital-optimal?

What have we learned?

Doctor-optimality: The matching returned by DA is **doctor-optimal** (but hospital-*worst*)

Truthful preferences corollary: Doctors cannot gain from misreporting their preferences (but hospitals *can*).

Point:

It's important to **think** about how **our algorithms affect people.**

Theorems can help!

Today

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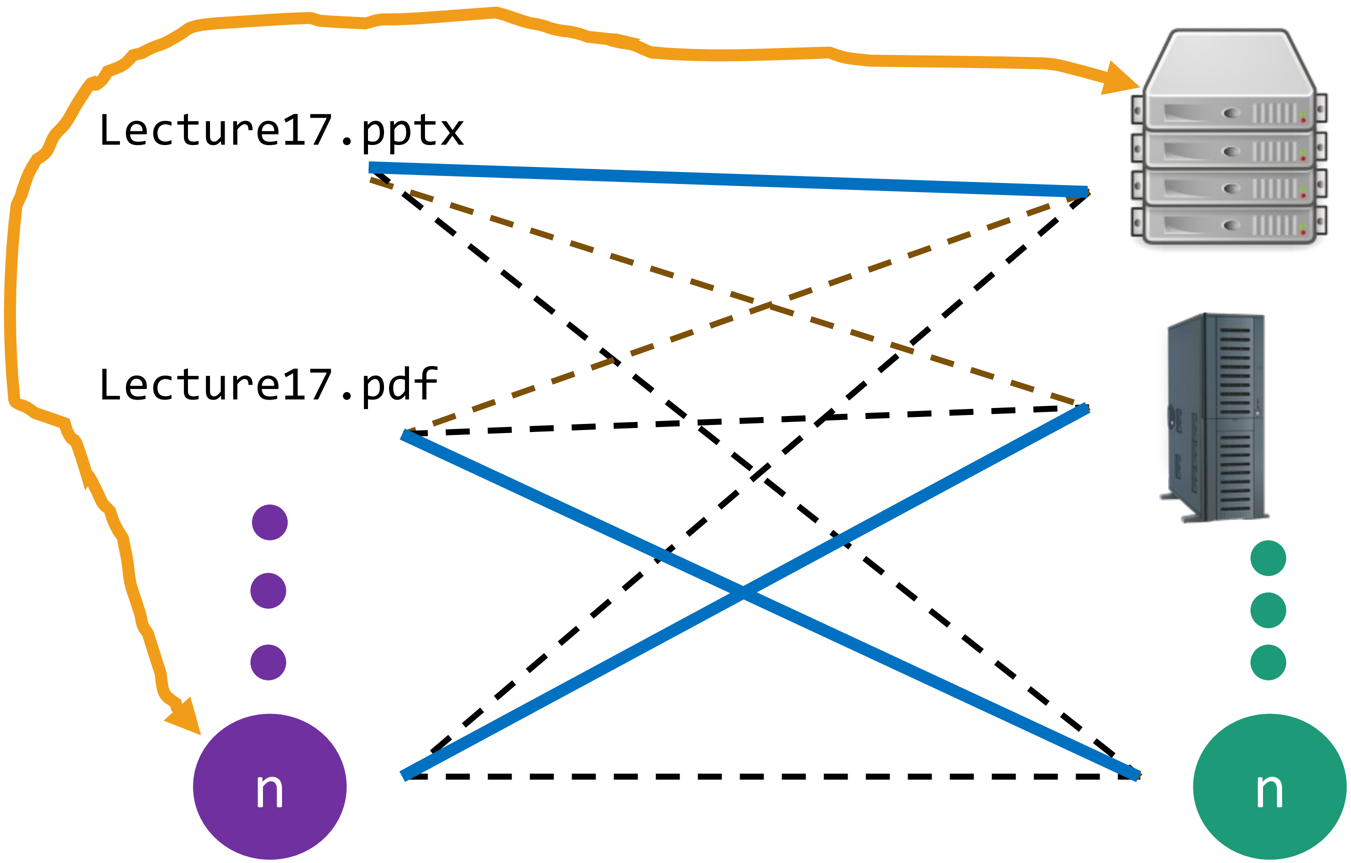
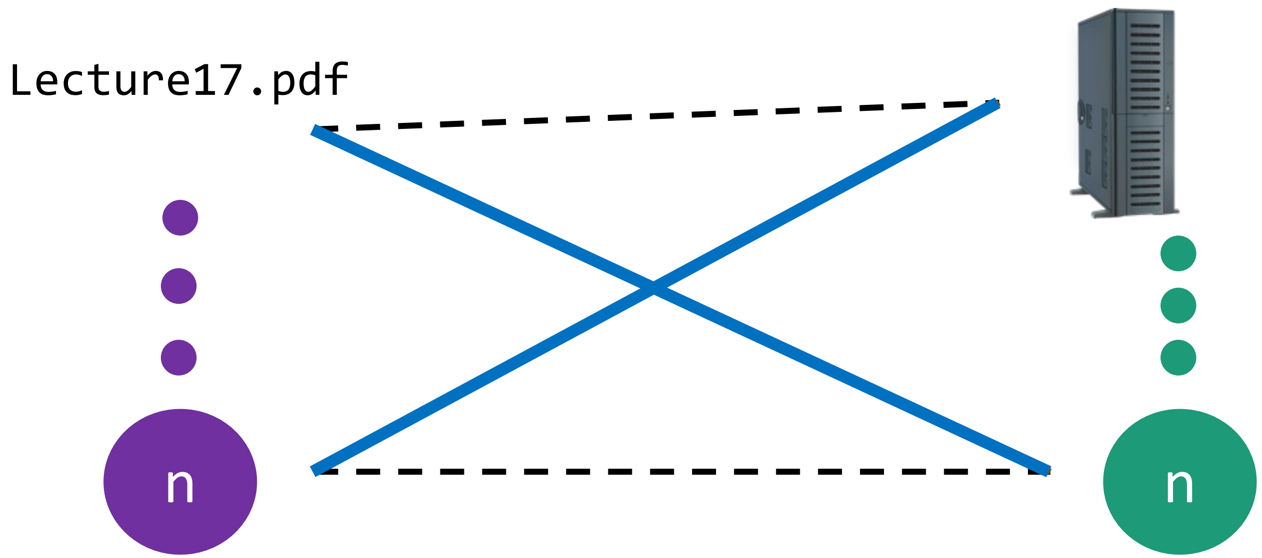
Doctors vs Packets

- Suppose that instead of doctors and hospitals, you want to match packets to servers on the internet.



Doctors vs Packets

- Suppose that instead of doctors and hospitals, you want to match packets to servers on the internet.
- When you *own all the servers*, you don't have to worry about them matching outside your algorithm...
- But it turns out that Deferred Acceptance is just very fast in practice ☺



Doctors vs Packets

- Suppose that instead of doctors and hospitals, you want to match packets to servers on the internet.
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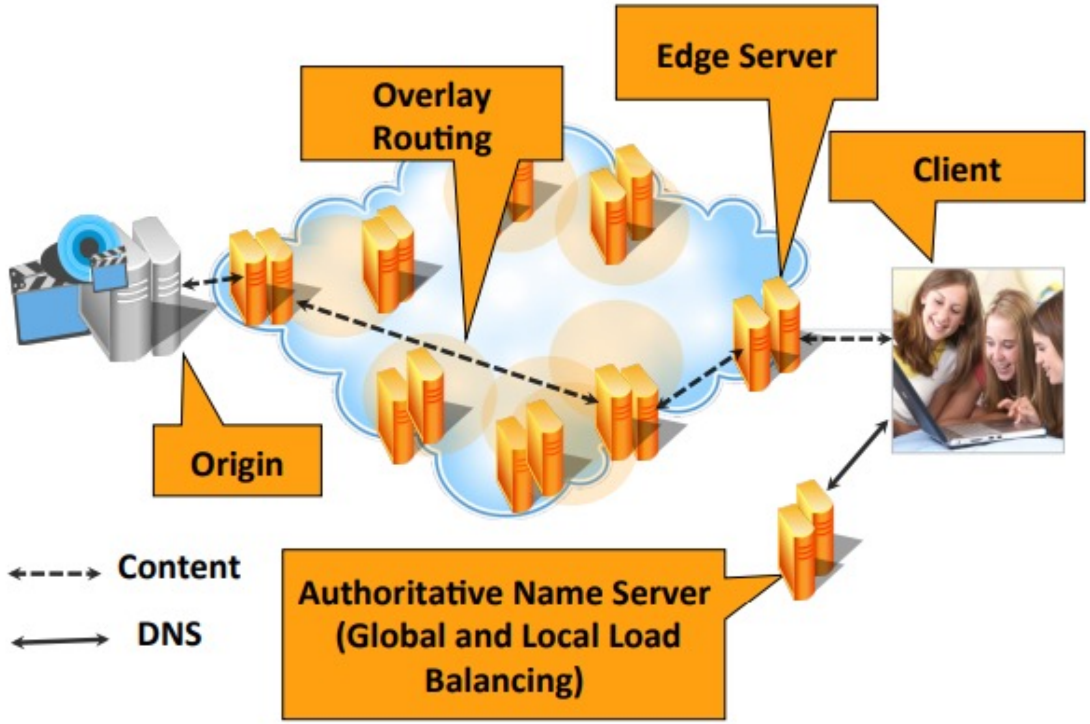
Truncated preference lists
Packets typically get one of top servers
Total running time closer to $O(n)$!

Highly **distributed**:
Every packet looks for its own server!

Doctors vs Packets



See “Algorithmic Nuggets in Content Delivery“ (Maggs & Sitaraman, CCR’15)
for details on how Akamai uses Deferred Acceptance to match packets to servers

Stanford Marriage Pact

Stanford Marriage Pact

- Matches between Stanford students who want to make a *pact*: "If we don't get married by time X, we'll marry each other."

- Historically, Gale-Shapley's original paper talked about *Stable Marriage*
  - men = doctors; women = hospitals.

- Original Marriage Pact used variant of Deferred Acceptance
  - It doesn't any more…

Preference graph is not bipartite!

Marriage Pact doesn't need stability:
It is meant to be a back-up match-
Couples are *encouraged* to find outside matches!

Recap

- Hospitals/residents problem
- Stable matchings
  - Solve the hospitals/residents problem
  - But can we find them?
- Deferred Acceptance Algorithm
  - Find stable matchings!
- Discussion, applications and non-applications

Next time

- Quick and hand-wavey recap of past lectures.
- Algorithms beyond 161 …