# Randomized Algorithms

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## 1 Random Variables and Expectation

Plucky has an $n$-sided die that will generate numbers in $\{1, 2, \ldots, n\}$ uniformly at random. She is bored and she has decided to start and keep rolling her die until she has seen all the numbers in $\{1, 2, \ldots, n\}$ at least once.

We want to calculate how many die rolls it takes in expectation for Plucky to stop.

For each $i \in \{1, 2, \ldots, n\}$, we define a random variable $X_i$: its value is equal to the number of additional die rolls we need to see the $i$-th unique value after having already seen $i-1$ unique values.

What is the type of probability distribution that the random variable $X_i$ follows?

- ○ Binomial
- ○ Bernoulli
- ○ Poisson
- ● Geometric

Correct

Assume Plucky has started rolling her die and she has seen $i-1$ unique values so far. What is the probability of seeing a new number that she has not seen before in her next die roll?

- ○ $\frac{1}{n}$
- ○ $\frac{i}{n}$
- ● $\frac{n-i+1}{n}$
- ○ $\frac{1}{i-1}$

Correct

What is $\mathbb{E}[X_i]$?

- ○ $n$
- ○ $\frac{n}{i}$
- ● $\frac{n}{n-i+1}$
- ○ $i-1$

Correct

What is the expected total number of die rolls, until Plucky sees all the n values at least once?

- ○ $\Theta(n)$
- ● $\Theta(n \log n)$
- ○ $\Theta(n^2)$
- ○ $\Omega(n^2 \log n)$

Correct

## 2 Randomized Algorithms

Can we use the random pivot selection idea in QuickSort for the selection problem?

Assume we modify the $k$-select algorithm that we saw in previous lectures; instead of picking the pivot cleverly, we just pick a uniformly random element as the pivot each time. We call the resulting algorithm QuickSelect.

What is the worst case runtime of QuickSelect?

- ○ $\Theta(n)$
- ○ $\Theta(n \log n)$
- ● $\Theta(n^2)$
- ○ $\Omega(n^2 \log n)$

Correct

What is the probability that our random pivot partitions the array into two parts, each of size at most $\frac{3n}{4}$?

- ○ $\frac{3}{4} \pm O(1/n)$
- ● $\frac{1}{2} \pm O(1/n)$
- ○ $\frac{1}{3} \pm O(1/n)$
- ○ $\frac{1}{4} \pm O(1/n)$

Correct

Assume we group QuickSelect's recursive calls into multiple phases. Phase $i$ is when the size of the array is in the interval

$$\left((3/4)^{(i+1)}n, (3/4)^{i}n\right].$$

Note that we start at phase 0 with an array of size $n$.

For each phase we define a random variable $X_i$, whose value is the number of recursive calls in that phase. Using the answers to previous questions, calculate an upper bound for $\mathbb{E}[X_i]$. Which of the following is the (asymptotically) smallest upper bound on $\mathbb{E}[X_i]$?

- ○ $3 + O(1/n)$
- ● $2 + O(1/n)$
- ○ $n$
- ○ $O(\log n)$

Correct

What is the expected (average case) runtime of QuickSelect?

- ● $\Theta(n)$
- ○ $\Theta(n \log n)$
- ○ $\Theta(n^2)$
- ○ $\Omega(n^2 \log n)$

Correct