Prereq Review #2: Induction

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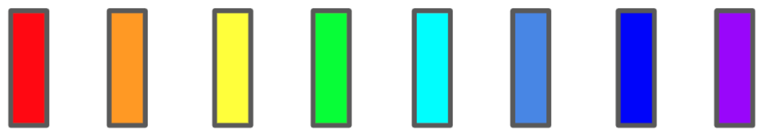
Induction is a tremendously powerful proof technique that we'll use again and again in CS161. But it's easy to get lost in the formalism and boilerplate, so let's start with the big picture. My favorite metaphor for induction is knocking over a line of dominoes:



For all the dominoes to fall, it is necessary and sufficient to have **both** of these things:

- The first domino has to be knocked over.
- Knocking over any domino ensures that the next domino in line will be knocked over.

Without that second bullet point, even knocking over the first domino isn't enough, because at some point there will be a pair of dominoes that are placed too far apart for one to knock over the next. RIP.

On the other hand, without that first bullet point, the dominoes might be set up perfectly, but with nothing to knock over the first domino, it will never fall. Also RIP.

Now imagine that line of dominoes going on arbitrarily long – perhaps forever – and replacing each domino with a mathematical claim, and now we have induction! Specifically, suppose we want to prove a claim of the form

Claim: $P(n)$ is true, for all integers $n \geq n_0$.

where P is a proposition based on n, and n_0 is a fixed integer. To prove the claim, we show it is true for every value of $n \geq n_0$, i.e., we knock over the first and every other domino.

One thing that can make induction hard to appreciate at first is that instructors often choose an example where the claim is obviously true – something like "any number ending in 1 is odd". Then, after a boring proof, one is left wondering why induction was needed in the first place. So we'll choose a claim that isn't so obvious. Let's make our proposition $P(n)$ the following:

If we flip n fair coins, the total number of heads is equally likely to be even or odd.

and in this case, let's say $n_0 = 1$. Now this claim might feel like it *could* be true, especially if we do the math for some small values of n, but I think most of us would be hard pressed to be *sure* it was true.[1]

Then it suffices to

- show that the claim holds for $n = 1$ (knock over the first domino)
- show that – for any $k \geq 1$ – if the claim holds for $n = k$, then it also holds for $n = k+1$ (i.e., show that knocking over any domino knocks over the next domino)

We can organize our argument as follows.

- **Claim:** For all $n \geq 1$, if we flip n fair coins, the total number of heads is equally likely to be even or odd.
- **Proof of claim:** We will proceed via induction:
  - **Base case:** For $n = 1$, there is only one coin. Since it is fair, it is equally likely to come up heads (in which case the total number of heads is 1, which is odd) or tails (in which case the total number of heads is 0, which is even). Therefore the claim holds for $n = 1$.

    Now, to complete the proof, we will show that for any $k \geq 1$, if the claim holds for $n = k$, it also holds for $n = k + 1$.
  - **Inductive hypothesis:** Let k be some arbitrary[2] integer ≥ 1. Suppose the claim holds for such k.
  - **Inductive step:** Suppose we flip $k + 1$ fair coins. Consider the results from the first k of those; by the inductive hypothesis, the total number of heads among those coins is equally likely to be odd or even. Now consider the $k + 1$-th coin.
    * Half of the time, it comes up heads, which changes the total number of heads from odd to even or even to odd. But since even and odd numbers of heads were equally likely before the $k + 1$-th flip, swapping those equal quantities does nothing, and they are still equally likely after that flip.
    * The other half of the time, that coin comes up tails, which leaves the total number of heads the same. Again, since even and odd numbers of heads were equally likely before the $k+1$-th flip, they're still equally likely after that flip.

    Therefore the claim holds for $n = k + 1$ as well. This completes the proof – we see that the claim holds for all $n \geq 1$.

[1] An equivalent formulation of this turns out to be: if you color the entries of a row of Pascal's triangle alternatingly blue and red, do the blue numbers add up to the same total as the red numbers? I think it's far from obvious that, say, in 1 4 6 4 1 , the blue and red numbers both sum to 8.

[2] This usage of "arbitrary" may be unfamiliar. It doesn't mean that we are proving the claim for only one particular k. Rather, we are demonstrating an argument that will hold for **any** k.

Let's unpack some details of that proof that were probably not obvious:

- **The base case is usually relatively simple.** If you find yourself doing significant work to show that the claim holds for a base case – in particular, if it seems to duplicate the argument you make in the inductive step – it probably means you should have chosen a smaller value of n. (If it is especially hard to show that the base case holds, it may be a sign that induction is not the best method for the problem.)

- **The base case determines the range of the inductive hypothesis.** We couldn't have stated the inductive hypothesis for $k \geq 2$, for example, because then there would have been a hole in our dominoes: we would have shown that the claim holds for $n = 1$, and that the claim holding for $n = 2$ implies that it holds for $n = 3$ and so on, but we would not have demonstrated that the claim holding for $n = 1$ implies that it holds for $n = 2$. A hole like this renders a proof invalid.

- **The inductive step has to actually bring in some new insight.** If you find yourself writing an inductive step argument that seems trivial, or feels like just rearranging previous parts of the proof setup, it's possible that you have accidentally written a circular argument.

  The thing the inductive step is trying to prove is: if the claim holds for $n = k$, then it also holds for $n = k + 1$. That is, it has to move from one domino to the next. If you find yourself arguing about $n = k$ using the claim for $n = k$, something is wrong.

- **The use of k and $k + 1$ was one cosmetic choice.** You may prefer to assume that the claim holds up to $n = k - 1$ and then show that it holds for $n = k$. The argument is equivalent. Just make sure that your indexing is consistent – you don't want to accidentally introduce a gap between the base case and inductive hypothesis.

- **Wait, did we really need induction?** Would it have sufficed to informally argue "flipping the next coin is equally likely to change or not change the parity[3] of the total number of heads"? Well, we'd still need to show that those two parities were equally likely to begin with, and by then we would have pretty much written an implicit induction proof anyway.

In CS161, your proofs do not need to be quite as formal as what I have written above, but I think the structure helps.

Here are a couple more things to be aware of:

- **You can use either "strong" or "weak" induction.** In the example above, we used weak induction: we assumed that the claim held for just the previous domino, so to speak. In strong induction, we assume that the claim holds for *all* previous dominoes; in the above example, we would have said "suppose the claim holds for all $n \leq k$."

[3]Parity = whether an integer is even or odd.

Strong induction is more powerful, in the sense that it might make it easier to set up the proof. But there is no shame in using it, and it is not necessarily more elegant or preferable to use weak induction. In fact, the two turn out to be mathematically equivalent. Just use whichever you prefer for the proof at hand; you don't need to explicitly declare which type you're using as long as it's clear.

- **You might need more than one base case.** For example, an induction proof about the Fibonacci sequence – in which each value depends on the *two* previous values – might require two base cases to work gracefully.

- **Induction is not the only game in town.** For example, another very common and powerful proof technique is *proof by contradiction* – to show that something is false, we step into the world in which it is true, and show that that world is internally inconsistent / math breaks somehow. It is also possible to prove some statements just by citing the right list of facts. If a proof involves showing that something is true for all natural numbers, though, induction is surely worth considering.