CS 161 (Stanford, Winter 2024) Homework 3

Style guide and expectations: Please see the "Homework" part of the "Resources" section on the webpage for guidance on what we are looking for in homework solutions. We will grade according to these standards. You should cite all sources you used outside of the course material.
What we expect: Make sure to look at the "**We are expecting**" blocks below each problem to see what we will be grading for in each problem!

Exercises. The following questions are exercises. We suggest you do these on your own. As with any homework question, though, you may ask the course staff for help.

1 Finding the median of sorted arrays

As shown in lecture, there is an $O(n)$ algorithm to find the median of any array. However, there are ways to improve this runtime if we know something about the sortedness of the input arrays.

1.1 Two unsorted arrays (2 pt.)

Suppose you are given two arrays of length n. Describe an algorithm that returns the median of the combined array and explain why it has the best possible runtime.

[We are expecting: A runtime and a brief description of your algorithm.**]**

1.2 Two sorted arrays (8 pt.)

Suppose you are given two arrays of length n, but now that the arrays are sorted. Can you devise a faster algorithm to find the median?

[We are expecting: A short English description, Pseudocode, runtime analysis. You may assume that the length of the arrays is a power of 2.**]**

1.3 One somewhat-sorted array (6 pt.)

Suppose you are given a somewhat-sorted array. We will define a somewhat-sorted array to be such that for any two elements in the array, if the array elements have a distance of more than k, then they are sorted relative to each other. There are no such guarantees for elements within k distance of one another. Devise an algorithm to find the median of this array that runs in $O(k)$. For simplicity, you may assume the length of the array is odd.

[We are expecting: A brief description of your algorithm and a justification for correctness.**]**

2 Randomized Algorithms (12 pt.)

In this exercise, we'll explore different types of randomized algorithms. We say that a randomized algorithm is a **Las Vegas algorithm** if it is always correct (that is, it returns the right answer with probability 1), but the running time is a random variable. We say that a randomized algorithm is a **Monte Carlo algorithm** if there is some probability that it is incorrect. For example, QuickSort (with a random pivot) is a Las Vegas algorithm, since it always returns a sorted array, but it might be slow if we get very unlucky.

To get more insight on randomized algorithms, we will revisit the problem of finding an experienced dogwalker from a group where the majority are experienced. You may assume WalkAndRate runs in $O(1)$, and that isExperienced is a correct algorithm.

Algorithm	Monte Carlo or Las Vegas?	Expected running time	Worst-case running time	Probability of returning an experienced dogwalker
Algorithm 1				
Algorithm 2				
Algorithm 3				

[We are expecting: Your filled in-table, and a short justification for each entry of the table. You may use asymptotic notation for the running times; for the probability of returning a majority element, give the tightest bound that you can given the information provided. Fill in the table below, and justify your answers.**]**

Algorithm 1: findExperiencedWalker1

```
Input: A population P of n dogwalkers
while true do
    Choose a random p ∈ P;
    if isExperienced(P, p) then
        return p;
```

Algorithm 2: findExperiencedWalker2

```
Input: A population P of n dogwalkers
for 100 iterations do
    Choose a random p ∈ P;
    if isExperienced(P, p) then
        return p;
return P[0];
```

Algorithm 3: findExperiencedWalker3

```
Input: A population P of n dogwalkers
Put the elements in P in a random order.;
/* Assume it takes time Θ(n) to put the n elements in a random order
   */
for p ∈ P do
    if isExperienced(P, p) then
        return p;
```

Algorithm 4: isExperienced

```
Input: A population P of n dogwalkers and a dogwalker p ∈ P
Output: True if p is an experienced dogwalker
count ← 0;
for q ∈ P do
    if p ≠ q then
        WalkAndRate(p, q);
        if q rates p as experienced then
            count++;
if count ≥ (n − 1)/2 then
    return True;
else
    return False;
```

Problems. The following questions are problems. You may talk with your fellow CS 161-ers about the problems. However:

- Try the problems on your own *before* collaborating.
- Write up your answers yourself, in your own words. You should never share your typed-up solutions with your collaborators.
- If you collaborated, list the names of the students you collaborated with at the beginning of each problem.

3 Sorting with low adaptivity

In practice, when the steps of a given algorithm do not depend on one another (i.e. the inputs to each step are pre-determined and do not depend on the outputs of other steps of the algorithm), we can often execute the steps simultaneously using multiple machines (or multiple cores/processors of the same machine) and thereby gain speed. This approach of using multiple machines to execute different independent parts (steps) of the algorithm is called *parallelization*. Outside this question, we will not spend much time discussing it in this class, but we will still give an example to motivate the following question.

Multiplying a vector by a scalar is an algorithm where the steps do not depend on each other, hence it is *parallelizable*. More precisely, each multiplication of the given scalar with an element of the given vector is naturally independent of the other elements of the vector. So, different computers (or processors) can be used to perform these multiplications in parallel, without talking to one another.

In contrast, consider a contrived algorithm that computes $f(f(x))$ for a given input x and an arbitrary function f, by computing $y = f(x)$ and then using y to compute the final output $f(y)$. This algorithm is not parallelizable; instead, it is *sequential*.

In this question, we will be analyzing *comparison-based* sorting algorithms. A *comparison-based* sorting algorithm is one which cannot access the values of the elements, instead, it can only compare pairs of elements and find out which element is bigger or smaller. Observe that mergesort and quicksort, both are comparison-based sorting algorithms as the algorithms only perform comparisons without accessing the values of the elements.

3.1 Non-Adaptive Sort (10 pt.)

We say that a comparison-based sorting algorithm is *non-adaptive* if it commits in advance to the pairs of elements that it will compare. For example, consider a sorting algorithm which commits in advance to compare all possible pairs of elements. And then it uses the comparisons to find to find k^{th} smallest element for each k, thereby sorting the input array.

Prove that any non-adaptive sorting algorithm requires $\Omega(n^2)$ comparisons.

[We are expecting: A succinct but rigorous proof. It's OK to prove the lower bound for deterministic algorithms, although you may want to challenge yourself and prove it for randomized algorithms (with expected number of queries).**]**

3.2 Adaptivity of MergeSort (10 pt.)

We say that a comparison-based sorting algorithm has *adaptivity* t if it proceeds in $t+1$ stages, where the pairs to be compared in the i-th stage only depend on the outcome of comparisons in stages $1, \ldots, i-1$ (but not on other comparisons in the i-th stage). Observe that a non-adaptive algorithm has 0 adaptivity.

Let's consider an example of an adaptive algorithm with adaptivity 1. Consider the problem of sorting *exactly* three given numbers a, b, c. Let's say that the algorithm commits to comparing a, b and b, c in stage 1. For a given input, if the outputs of the comparisons indicate that $a < b$ and $b < c$, then the algorithm outputs $a < b < c$ without performing any further comparisons. However, if the comparisons indicate that $a < b$ and $c < b$, the algorithm performs an additional comparison a, c in stage 2. Since the decision to compare the specific pair a, c (and to make no further comparisons in the earlier case) in stage 2 was determined by the comparisons made in stage 1, this algorithm has an adaptivity of 1. Based on this final comparison made in stage 2, the algorithm returns the sorted output.

What is the adaptivity of the MergeSort algorithm?

[We are expecting: Adaptivity in terms of big-Θ notation, and a proof supporting the answer**]**

3.3 Adaptivity of QuickSort (15 pt.)

What is the expected adaptivity of QuickSort?

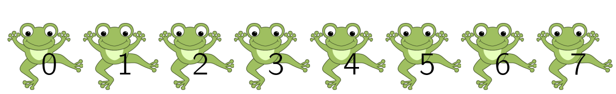
[We are expecting: Adaptivity in terms of big-Θ notation, and a proof supporting the answer**]**

Hint: There are many ways of solving this problem. One that seems useful is to consider "comparable pairs" i.e. pairs of elements that might still be compared by the algorithm, and analyze how the expected total number of comparable pairs changes over iterations of the algorithm.

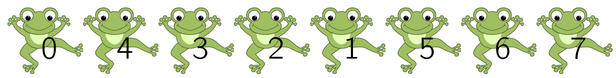
4 Flipping frogs

You have encountered a troupe of n frogs who are each labeled with a number $0, \ldots, n-1$, where n is a power of 2.

The n frogs dance in a line, and they only know one type of dance move, called `Flip(i,j)`: for any i and j so that $0 \leq i < j \leq n$, `Flip(i,j)` flips the order of the frogs standing in positions $i, i+1, \ldots, j-1$. For example, if the frogs started out like this:



then after executing `Flip(1,5)`, the frogs would look like this:



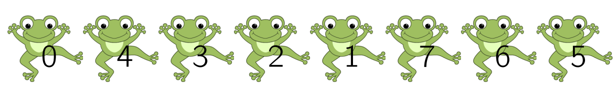
Executing this dance move is pretty complicated: it takes the frogs $O(|i - j|)$ seconds to implement `Flip(i,j)`.

In this problem, you will design and analyze an algorithm to sort the n frogs by their labels, which uses `Flip`, the only move the frogs know. However, you don't know the original order of the frogs until you see them, and so you may also need to spend some extra time in between the `Flip` operations computing which indices i and j you want to call `Flip` on.

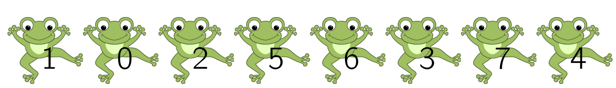
Below, you will consider algorithms which take as input an array that contains the frog's initial positions (for example, if the frogs were originally organized as the picture above (after the flip), then the input array would be $A = [0, 4, 3, 2, 1, 5, 6, 7]$). In addition to normal computation, which can be used to compute indices i and j, your algorithm is allowed to call `Flip(i,j)`, which will cause the frogs to implement the `Flip(i,j)` dance. The algorithm will wait until the frogs are done executing that dance move to continue its computations. Thus, the total running time of the algorithm includes *both* the time spent computing, and the time that the frogs spend dancing.

4.1 Partitioning the frogs (15 pt.)

First, you will design an algorithm that tells the frogs how to perform a dance called `Partition(i)`. For this dance, the frogs will partition themselves around the frog in position i, so that all of the frogs whose labels are smaller than that frog's label will end up to the left, and all the frogs whose labels are larger will end up to the right. For example, if the situation were this:



then after executing `Partition(3)`, the frogs partition themselves around the frog in position 3, whose label is 2. The resulting frogs formation could look like this:



Notice that it doesn't matter what order the smaller frogs and larger frogs are in.

Give an algorithm that instructs the frogs to implement `Partition` that runs in time $O(n \log n)$.

Hint: Try divide-and-conquer. After one call of `Flip(i, k)` *(think of a value for k that works well)*, can you identify two frogs - one for each half $(0, 1, \ldots, k - 1$ and $k + 1, \ldots, n - 1)$ - such that recursively partitioning the two halves about the identified frogs, gives a favourable arrangement of frogs. And executing Flip once more completes the dance. Also, be careful with off-by-one errors in your indexing.

[We are expecting: Pseudocode **AND** a clear English expanation of what your algorithm is doing. In addition, we expect an informal justification of the running time. You may appeal to the Master Theorem if it is relevant.**]**

4.2 Sorting the frogs (5 pt.)

You are excited about the `Partition` algorithm from part (a), because it allows you to tell the frogs how to perform a dance called `Sort()`, which puts the frogs in sorted order. The algorithm is as follows:

```
def Sort(A):
    //A is an array of length n, with the positions of the n frogs
    //Assume that n is a power of 2.

    //base case:
    if n == 1:
        return

    //get the index of the median, using the Select algorithm
    //from lecture 4
    i = Select(A,n/2)

    //tell the frogs to partition themselves around the i'th frog:
    Partition(i)

    //recurse on both the left and right halves of the frog array.
    Sort(A[:n/2])
    Sort(A[n/2:])
```

That is, this algorithm first partitions the frogs around the median, which puts the smaller-labeled frogs on the left and the larger-labeled frogs on the right. Then it recursively sorts the smaller frogs and the larger frogs, resulting in a sorted list.

Let $T(n)$ be the running time of `Sort(A)` on an array A of length n. Write down a recurrence relation that describes $T(n)$.

[We are expecting: A recurrence relation of the form $T(n) = a \cdot T(n/b) + O(\text{something...})$, and a short explanation.**]**

4.3 Runtime (10 pt.)

Explain, without appealing to the master theorem, why the running time of `Sort(A)` is $O(n \log^2 n)$ on an array of length n. (Note: "$\log^2 n$" means $(\log n)^2$).

If it helps, you may ignore the big-Oh notation in your answer in part (b). That is, if your answer in part (b) was $T(n) = aT(n/b) + O(\mathit{something})$, then you may drop the big-Oh and assume that your recurrence relation is of the form $T(n) = aT(n/b) + something$. You may assume that $T(1) \leq 1$, $T(2) \leq 2$. Recall that we are assuming that n is a power of 2.

Hint: Either the tree method or the substitution method is a reasonable approach here.

[We are expecting: An explanation. You do not need to give a formal proof, but your explanation should be convincing to the grader. You should **NOT** appeal to the master theorem, although it's fine to use the "tree method" that we used to prove the master theorem.**]**