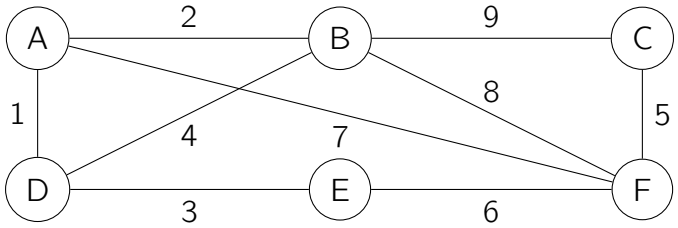
CS 161 (Stanford, Winter 2024) Homework 8

Style guide and expectations: Please see the "Homework" part of the "Resources" section on the webpage for guidance on what we look for in homework solutions. We will grade according to these standards. You should cite all sources you used outside of the course material.
What we expect: Make sure to look at the "**We are expecting**" blocks below each problem to see what we will be grading for in each problem!

Exercises. The following questions are exercises. We suggest you do these on your own. As with any homework question, though, you may ask the course staff for help.

1 Spanning Tree Algorithms

Consider the graph G below.



1.1 Prim (2 pt.)

In what order does Prim's algorithm add edges to an MST when started from vertex C?

[We are expecting: An ordered list of edges.**]**

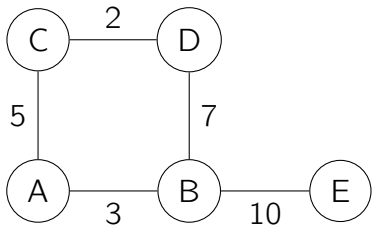
1.2 Kruskal (2 pt.)

In what order does Kruskal's algorithm add edges to an MST?

[We are expecting: An ordered list of edges.**]**

2 Min Cuts and Max Flows

Consider the following graph:



2.1 Min Cut (2 pt.)

Recall that the weight of a cut is the sum of the weights of the edges which cross the cut. (If there is a cut between vertex sets X and Y, the weight of the cut is the sum of the weight of the edges which have one endpoint in X and the other in Y).

How can we partition all of the vertices of the graph above into two non-empty sets X and Y so that the cut between the two is minimized? (This is known as a global min-cut).

[We are expecting: Two sets of vertices X and Y, along with the weight of the cut between them. No explanation is required.**]**

2.2 Max Flow (2 pt.)

What is the maximum flow from C to E? From D to E?

[We are expecting: Just the maximum flows. No justification is required.**]**

Problems. The following questions are problems. You may talk with your fellow CS 161-ers about the problems. However:

- Try the problems on your own *before* collaborating.
- Write up your answers yourself, in your own words. You should never share your typed-up solutions with your collaborators.
- If you collaborated, list the names of the students you collaborated with at the beginning of each problem.

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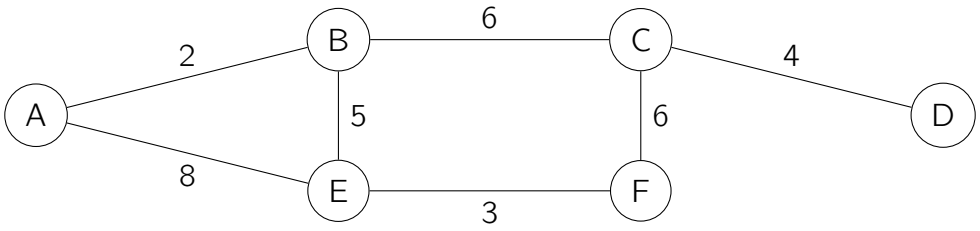
3 Plucky's Subway Adventure

Plucky is planning a weekend visit to her extensive family network. She realizes that she needs to visit every single subway station to visit everyone from her family. Upon reading a subway map, she identifies each station as a vertex in a network, with subway lines linking these points as edges, forming an undirected graph $G = (V, E)$.

The local subway operates on an unusual fare system. The cost of traveling between two stations is indicated by a weight on each edge of the graph.

Plucky plans to buy a special student pass priced at x dollars that allows her to travel for **unlimited trips** between any two stations that take no more than x dollars to travel. In other

words, she can travel through any path P unlimited times, as long as $\max\{w_e \mid e \in P\} \leq x$.



For example, in the graph above, for Plucky to visit every station, she needs to buy a ticket for \$6, which permits unrestricted travel on all routes except for the A, E edge. Note that this example is just for demonstration purposes, and the subsequent questions apply to any undirected graphs G.

Plucky wants to get the cheapest ticket that allows her to reach all the stations. She realizes that she can do this by identifying a spanning tree T of G that minimizes the largest edge weight in that tree

$$x = \max_{e \in T} w_e,$$

Let us call this spanning tree a minimum-maximum tree since it minimizes the largest edge in the tree.

3.1 MST (6 pt.)

Prove that a minimum spanning tree in G is always a minimum-maximum tree. We will provide two hints that suggest two *separate* ways to prove this statement. *DO NOT* try to use both hints within the same proof.

Hint 1: Suppose toward a contradiction that T is an MST but not a minimum-maximum tree, and say that T' is a minimum-maximum tree. Try to come up with a cheaper MST than T (and hence a contradiction).

Hint 2: Use (without proof) the fact that any MST can be created by Kruskal's algorithm.

[We are expecting: A rigorous proof.**]**

3.2 The other way around (3 pt.)

Show that the converse to part 1 is not true. That is, a minimum-maximum tree is not necessarily always a minimum spanning tree.

[We are expecting: A counter-example, with an explanation of why it is a counter-example. **]**

4 Max-Flow

Let $G = (V, E)$ be a flow network with source $s \in V$, sink $t \in V$, and edge capacities for each edge $e \in E$. All edge capacities are positive *integers*. We can represent a flow by a 1-indexed array F, where $F[i]$ is the flow through edge $E[i]$ for $1 \leq i \leq |E|$.

4.1 Flow verification (5 pt.)

Given G and F, design an $O(|V|+|E|)$-time algorithm to determine if the flow F is a maximum flow in G.

[We are expecting: An English description of your algorithm, an informal explanation of why it works, and a runtime analysis.**]**

[Hint: remember to check that F is a valid flow.]

4.2 Flow update I (5 pt.)

Suppose that the capacity of a single edge $e = (u, v) \in E$ is increased by 1. Given G, its maximum flow F before the update, and e, design an $O(|V| + |E|)$-time algorithm to update F so that it is still the maximum flow of G after the update to e.

[We are expecting: An English description of your algorithm, an informal explanation of why it works, and a runtime analysis.**]**

4.3 Flow update II (5 pt.)

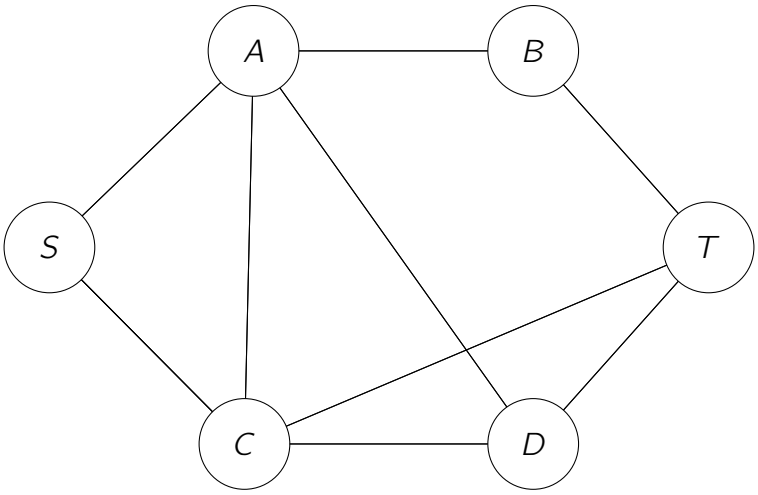
Suppose that the capacity of a single edge $e = (u, v) \in E$ is decreased by 1. Given G, its maximum flow F before the update, and e, design an $O(|V| + |E|)$-time algorithm to update F so that it is still the maximum flow of G after the update to e.

[We are expecting: An English description of your algorithm, an informal explanation of why it works, and a runtime analysis.**]**

5 Truculent Terrapins

Toby the Terrapin has two children who need to get to tap dancing class, but they often quarrel, and so have trouble traveling together. The possible routes that these two terrapins can take to class are represented via an undirected, unweighted graph G. Both of Toby's children start at node s, and they need to finish at node t without having used any of the same edges on their path.

An example graph: (Note that this example is just for demonstration purposes, and the subsequent questions apply to any undirected, unweighted graph G.)



The paths $[S, A, B, T]$ and $[S, C, D, T]$ are valid for both 5.1 and 5.2.

The paths $[S, A, C, T]$ and $[S, C, D, T]$ are valid for 5.1 but not 5.2.

The paths $[S, A, D, T]$ and $[S, C, D, T]$ are valid for neither 5.1 nor 5.2.

5.1 Find Two Paths, No Overlapping Edges (6 pt.)

Help Toby design an algorithm to find two paths without having used any of the same edges. Your algorithm should modify the graph and call Ford-Fulkerson as a subroutine. Your algorithm should either return two lists of vertices visited by the two separate paths, or -1 if no two paths which meet the requirements exist.

[We are expecting: How you will modify the graph, how you will use Ford-Fulkerson, and a justification as to why your algorithm always finds two paths with no overlapping edges if they exist.**]**

5.2 Finding Two Paths, No Overlapping Edges or Nodes (6 pt.)

Toby finds that his children are still quarreling, because even if they don't use the same edges, if they ever end up at the same node they will get into an argument.

Help Toby design a new algorithm that finds two paths from s to t for his children that do not share any edges or nodes (except for s and t). Your algorithm should first modify the graph so that each node can only be passed through once, and then call Ford-Fulkerson as a subroutine. Your algorithm should either return two lists of vertices visited by the two separate paths, or -1 if no two paths which meet the requirements exist.

[We are expecting: How you will modify the graph, how you will use Ford-Fulkerson, and a justification as to why your algorithm always finds two paths with no overlapping edges or nodes if they exist.**]**

5.3 Extending the Algorithm (3 pt.)

Lucky the Lemur has n children, and he wants to put his children on n separate paths to tap dancing as well. (He wants to use Toby's more strict requirement that none of his children ever visit the same node). Extend the algorithm you wrote in part 5.2 to find n separate paths from s to t such that no two of them visit the same edge or node (except for s and t). Again, the algorithm should either return n lists of vertices visited by each separate path, or -1 if it is impossible to create n such paths.

[We are expecting: How you will modify your previous approach, and a justification as to why your algorithm always finds a path for each of Lucky's children if one exists.**]**

6 Smartphone application

Suppose you are choosing between two algorithms for a commercial smartphone application. Algorithm 1 would run very quickly for the 60% of your users who have the latest iPhone, but wouldn't work at all on cheaper or older phones. Algorithm 2 would run on any phone, but at half the speed that Algorithm 1 would run on the latest iPhone.

6.1 Utilitarianism (3 pt.)

How would a utilitarian choose between these two algorithms? Why haven't you been given enough information to make the utilitarian's calculation? Why might it be difficult to collect the missing information?

[We are expecting: (1) a 1 sentence description, in general terms, of how utilitarianism calculates value for different options, (2) 1-2 sentences explaining what missing information you would need to make the utilitarian calculation, and (3) 1-2 sentences explaining why it might be difficult to collect or measure the missing information**]**

6.2 Prioritarianism (3 pt.)

Why might prioritarianism favor Algorithm 2, even if utilitarianism turns out to favor Algorithm 1?

[We are expecting: 2-3 sentences that explain how prioritarianism differs from utilitarianism, and how that difference might matter in this case**]**

6.3 Deontology (2 pt.)

Suppose your company has publicly committed to supporting older hardware. Why might a deontologist say that you should choose Algorithm 2, even if utilitarianism turns out to favor Algorithm 1?

[We are expecting: 1-2 sentences that explain why this new fact about the case might make a difference to a deontologist**]**