Induction

Review Section 1/12

Background on Induction

- Type of mathematical proof
- Typically used to establish a given statement for all natural numbers (e.g. integers > 0)
- Proof is a sequence of deductive steps
  - Show the statement is true for the first number.
  - Show that if the statement is true for any one number, this implies the statement is true for the next number.
  - If so, we can infer that the statement is true for all numbers.

Let P be some predicate.

The principle of mathematical induction states that if

If it starts true…

$P(0)$ **is true**

and

…and it stays true…

$\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))$

then

$\forall n \in \mathbb{N}.\ P(n)$

…then it's always true.

Intuition of Induction

Thinking of climbing a ladder:

1. Show you can get to the first rung (base case)

2. Show you can get between rungs (inductive step)

3. Now you can climb forever!

Components of Inductive Proof

Inductive proof is composed of 3 major parts :

- **Base Case** : One or more particular cases that represent the most basic case. (e.g. n=1 to prove a statement in the range of positive integer)
- **Induction Hypothesis** : Assumption that we would like to be based on. (e.g. Let's assume that P(k) holds)
- **Inductive Step** : Prove the next step based on the induction hypothesis. (i.e. Show that Induction hypothesis P(k) implies P(k+1))

Weak Induction vs Strong Induction:

- In weak induction, we only assume that particular statement holds at k-th step,
- In strong induction, we assume that the particular statement holds at all the steps from the base case to k-th step

Example: Integer Summation

Claim:

$$\textit{Let } S(n) = \sum_{i=1}^{n} i. \textit{ Then } S(n) = \frac{n(n+1)}{2}.$$

Base Case:

We show the statement is true for $n = 1$. As $S(1) = 1 = \frac{1(2)}{2}$, the statement holds.

Induction Hypothesis:

We assume $S(n) = \frac{n(n+1)}{2}$.

Example: Integer Summation

Inductive Step:

We show $S(n+1) = \frac{(n+1)(n+2)}{2}$. Note that $S(n+1) = S(n) + n + 1$. Hence

$$\begin{aligned} S(n+1) &= S(n) + n + 1 \\ &= \frac{n(n+1)}{2} + n + 1 \\ &= (n+1)\left(\frac{n}{2} + 1\right) \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

Example: Sum of Powers of 2

Question: What is the sum of the first n powers of 2?

$$2^0 + 2^1 + 2^2 + ... + 2^n$$

Some observations:

$k = 1: 2^0 = 1$

$k = 2: 2^0 + 2^1 = 1 + 2 = 3$

$k = 3: 2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7$

$k = 4: 2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15$

For general n, the sum is 2^n - 1

Example: Sum of Powers of 2

Base Case: n=1. Clearly, the sum of the first one power is 2^0=1. At the same time, 2^1-1=1.

Inductive Hypothesis: Assume the sum of the first *k* powers of 2 is 2^k-1

Inductive Step: The sum of the first k+1 powers of 2 is

$$\underbrace{2^0 + 2^1 + 2^2 + \ldots + 2^{(k-1)}}_{\text{sum of the first k powers of 2}} + 2^k$$

by inductive hypothesis

$$= 2^k - 1 \quad + 2^k$$

$$= 2(2^k) - 1 = 2^{k+1} - 1$$

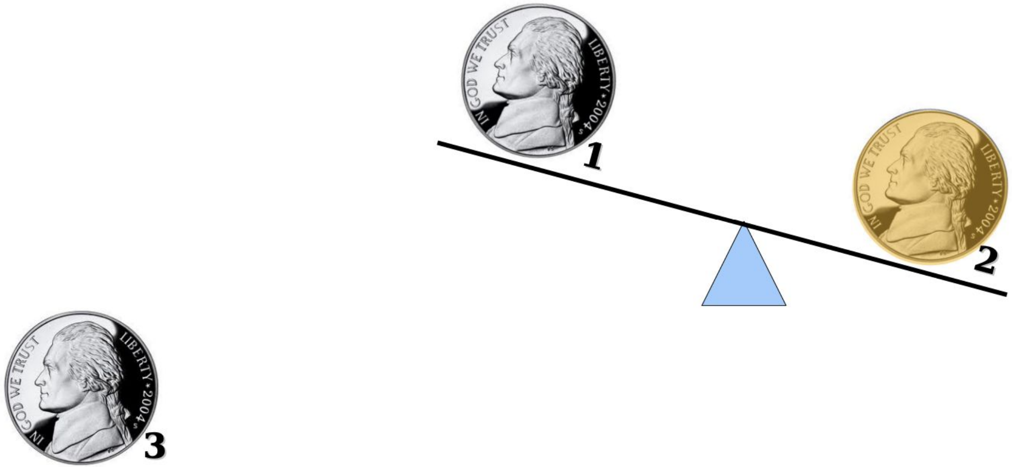
Example: Finding the Counterfeit Coin

Problem Statement:

- You are given a set of three seemingly identical coins, two of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins
- You are given a balance. Using only one weighing on the balance, find the counterfeit coin.

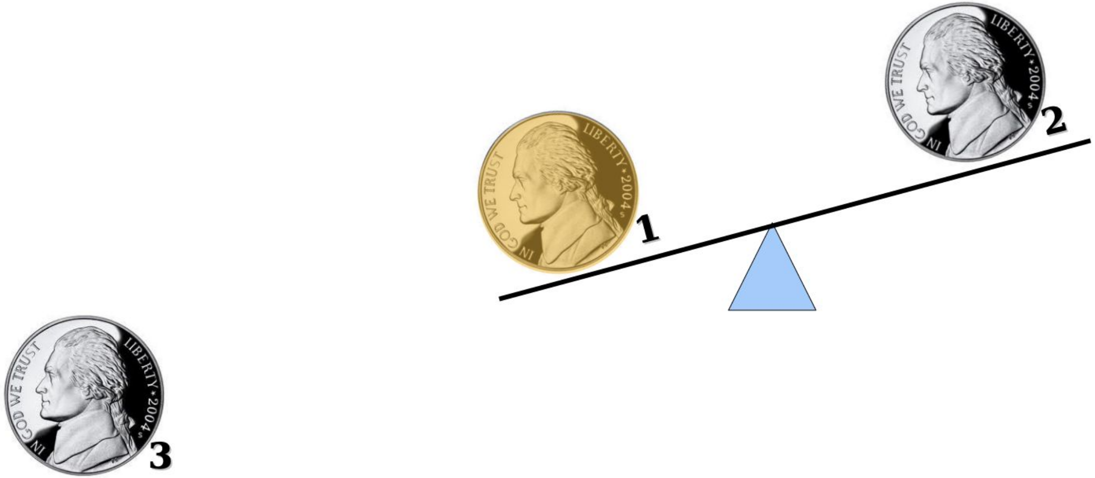
How to do it

Case 1:



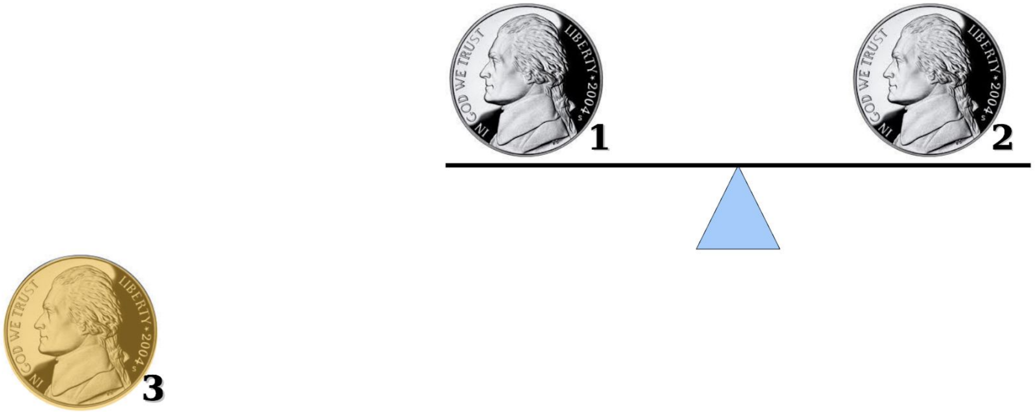
How to do it

Case 2:



How to do it

Case 3:



Example: Finding the Counterfeit Coin

A Harder Problem:

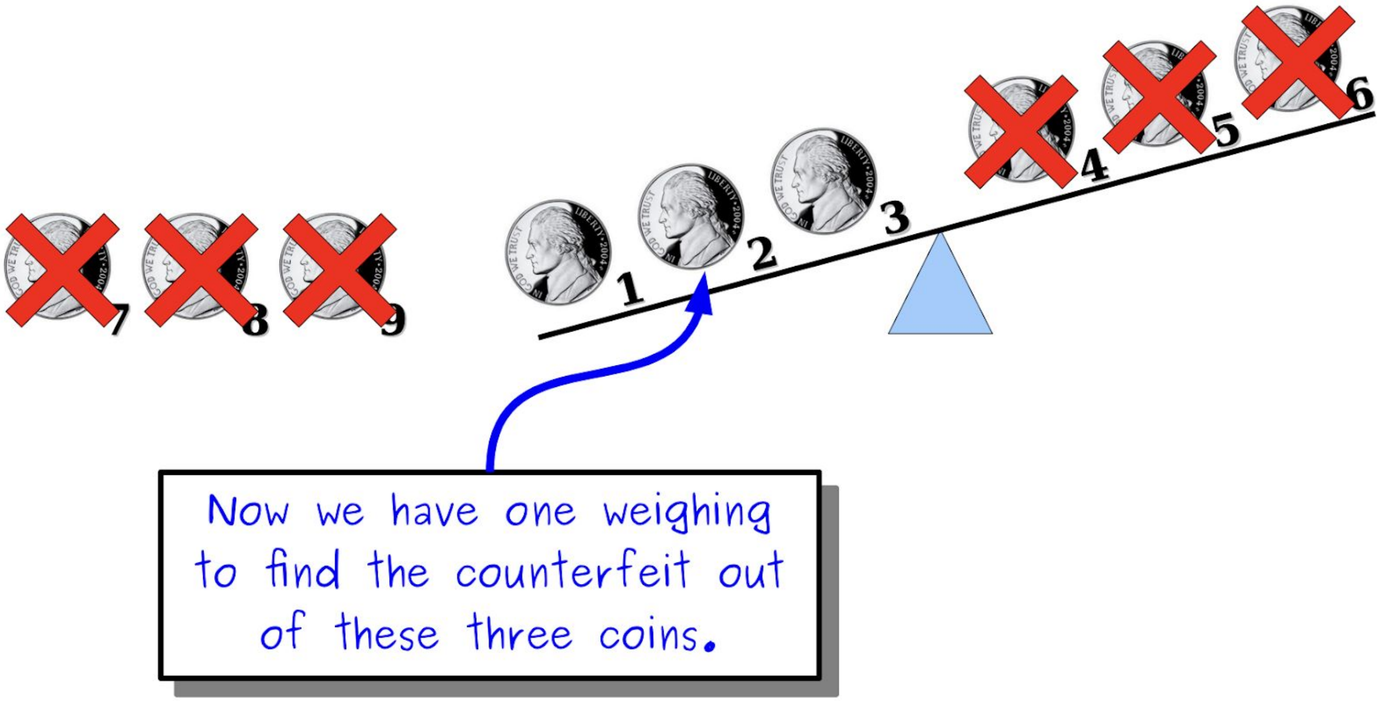
- You are given a set of nine seemingly identical coins, eight of which are real and one of which is counterfeit.

- The counterfeit coin weighs more than the rest of the coins

- You are given a balance. Using only two weighing on the balance, find the counterfeit coin.

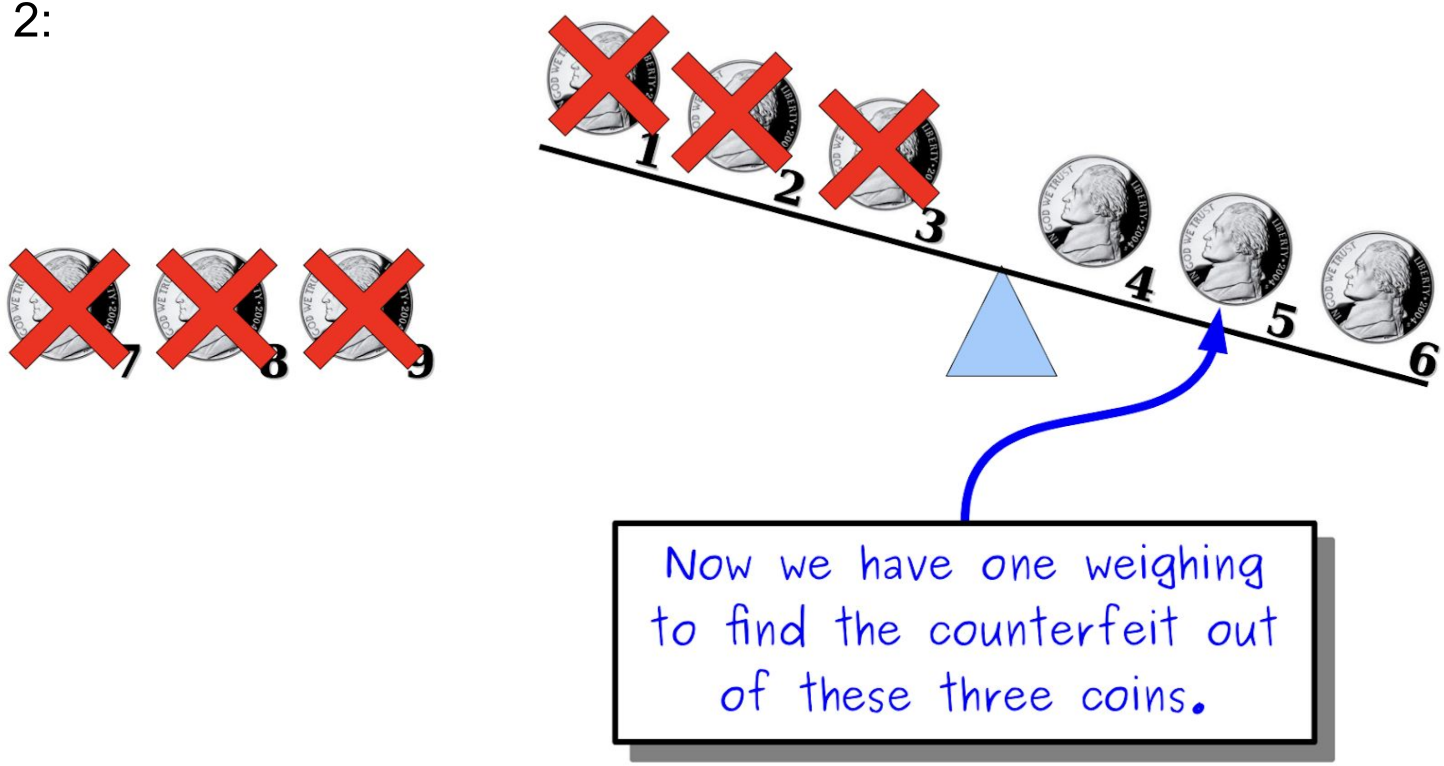
How to do it

Case 1:



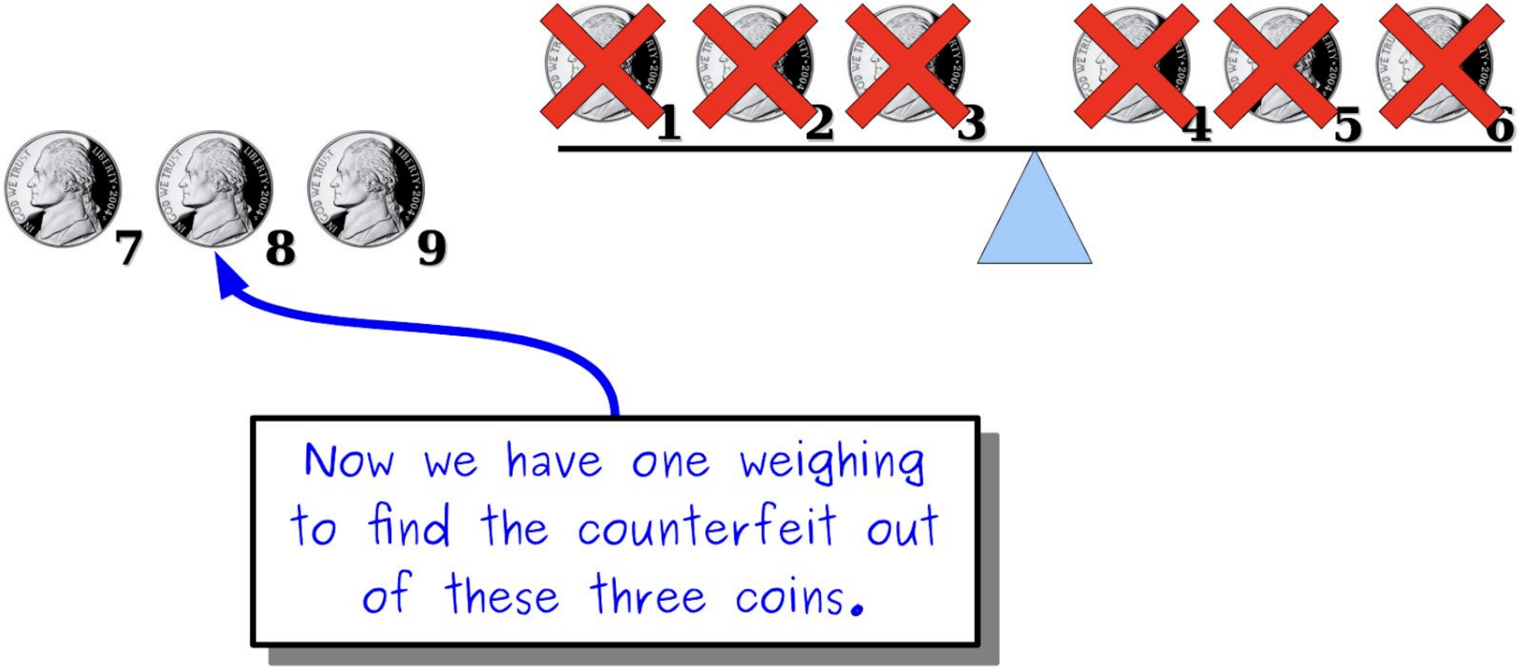
How to do it

Case 2:



How to do it

Case 3:



Example: Finding the Counterfeit Coin

Theorem: If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

Proof: Let $P(n)$ be the following statement:

> If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

Base Case: When $n=1$, already proven.

Inductive Hypothesis: $P(k)$ is true for some arbitrary integer $k>0$.

Example: Finding the Counterfeit Coin

Inductive Step: For the 3^{k+1} coins,

- Split the coins into three groups of 3^k coins each.

- Weigh two of the groups against one another.
  - If one group is heavier than the other, the coins in that group must contain the heavier coin.
  - Otherwise, the heavier coin must be in the group we didn't put on the scale.
  - Therefore, with one weighing, we can find a group of 3^k coins containing the heavy coin.

- Then use k more weighings to find the heavy coin in that group.

Example of **Strong** Induction

Theorem: Every integer greater than or equal to 2 can be factored into primes.

Proof: Let P(n) be the statement that an integer n greater than or equal to 2 can be factored into primes.

Base Case: For n=2, state is true because 2 is itself a prime.

Induction Hypothesis: Assume that for **all** integers less than or equal to k, the statement holds.

Example of Strong Induction

Inductive Step: Consider the number k+1.

- **Case 1:** k+1 is a prime number.
  - The number is a prime factorization of itself, so the statement P(k+1) holds.

- **Case 2:** k+1 is not a prime number.
  - We know $k+1=p \times q$ for integers $p>=2$ and $q>=2$, where both p and q are less or equal to k.
  - By inductive hypothesis, both p and q can be expressed as prime factorizations.
  - We can get the prime factorization for k+1 by multiplying the prime factorizations of p and q

Summary

Template of Inductive Proof:

- **Base Case:** Prove the most basic case.

- **Induction Hypothesis:** Assume that the statement holds for some k or for all numbers less than or equal to k.

- **Inductive Step:** Prove the statement holds for the next step based on induction hypothesis.