Last Time:
Solving Recurrence Relations

- A recurrence relation expresses $T(n)$ in terms of $T(\text{less than } n)$
- For example, $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 11 \cdot n$
- Two methods of solution:
  1. Master theorem (aka, generalized “tree method”)
  2. Substitution method (aka, guess and check)
The Master Theorem

- Suppose $a \geq 1$, $b > 1$, and $d$ are constants (that don’t depend on $n$).
- Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d)$. Then

$$T(n) = \begin{cases} 
O(n^d \log(n)) & \text{if } a = b^d \\
O(n^d) & \text{if } a < b^d \\
O(n^{\log_b(a)}) & \text{if } a > b^d
\end{cases}$$

Three parameters:
- $a$: number of subproblems
- $b$: factor by which input size shrinks
- $d$: need to do $n^d$ work to create all the subproblems and combine their solutions.
The Substitution Method

• Step 1: Guess what the answer is.
• Step 2: Prove by induction that your guess is correct.
• Step 3: Profit.
The plan for today

1. More practice with the Substitution Method.
2. k-SELECT problem
3. k-SELECT solution
4. Return of the Substitution Method.
A fun recurrence relation

- $T(n) \leq T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + n$ for $n > 10$.
- Base case: $T(n) = 1$ when $1 \leq n \leq 10$

Apply here, the Master Theorem does NOT.
The Substitution Method

• Step 1: Guess what the answer is.
• Step 2: Prove by induction that your guess is correct.
• Step 3: Profit.
Step 1: guess the answer

\[ T(n) \leq T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + n \text{ for } n > 10. \]
Base case: \( T(n) = 1 \) when \( 1 \leq n \leq 10 \)

- Trying to work backwards gets gross fast...
- We can also just try it out.
  - (see Python notebook)
- Let’s guess \( O(n) \) and try to prove it.
Aside: Warning!

- It may be tempting to try to prove this with the inductive hypothesis “$T(n) = O(n)$”
- But that doesn’t make sense!

- Formally, that’s the same as saying:
  - Inductive Hypothesis for $n$:
    - There is some $n_0 > 0$ and some $C > 0$ so that, for all $n \geq n_0$, $T(n) \leq C \cdot n$.

- Instead, we should pick $C$ first…
Step 2: prove our guess is right

\[ T(n) \leq T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + n \text{ for } n > 10. \]

Base case: \( T(n) = 1 \) when \( 1 \leq n \leq 10 \)

- **Inductive Hypothesis:** \( T(n) \leq Cn \)
- **Base case:** \( 1 = T(n) \leq Cn \) for all \( 1 \leq n \leq 10 \)
- **Inductive step:**
  - Let \( k > 10 \). Assume that the IH holds for all \( n \) so that \( 1 \leq n < k \).
  - \( T(k) \leq k + T\left(\frac{k}{5}\right) + T\left(\frac{7k}{10}\right) \)
    \[ \leq k + C \cdot \left(\frac{k}{5}\right) + C \cdot \left(\frac{7k}{10}\right) \]
    \[ = k + \frac{C}{5}k + \frac{7C}{10}k \]
    \[ \leq Ck ?? \]
  - (aka, want to show that IH holds for \( n = k \)).
- **Conclusion:**
  - There is some \( C \) so that for all \( n \geq 1, T(n) \leq Cn \)
  - By the definition of big-Oh, \( T(n) = O(n) \).
Step 3: Profit
(Aka, pretend we knew this all along).

**Theorem:** $T(n) = O(n)$

**Proof:**

- Inductive Hypothesis: $T(n) \leq 10n$.
- Base case: $1 = T(n) \leq 10n$ for all $1 \leq n \leq 10$
- Inductive step:
  - Let $k > 10$. Assume that the IH holds for all $n$ so that $1 \leq n < k$.
  - $T(k) \leq k + T\left(\frac{k}{5}\right) + T\left(\frac{7k}{10}\right)$
    \[\leq k + 10 \cdot \left(\frac{k}{5}\right) + 10 \cdot \left(\frac{7k}{10}\right)\]
    \[= k + 2k + 7k = 10k\]
  - Thus, IH holds for $n=k$.

- Conclusion:
  - For all $n \geq 1$, $T(n) \leq 10n$
  - Then, $T(n) = O(n)$, using the definition of big-Oh with $n_0 = 1$, $c = 10$. 

$T(n) \leq n + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)$ for $n > 10$.

Base case: $T(n) = 1$ when $1 \leq n \leq 10$
What have we learned?

• The substitution method can work when the master theorem doesn’t.
  • For example, with different-sized sub-problems.

• Step 1: generate a guess
  • Throw the kitchen sink at it.

• Step 2: try to prove that your guess is correct
  • You may have to leave some constants unspecified till the end – then see what they need to be for the proof to work!!

• Step 3: profit
  • Pretend you didn’t do Steps 1 and 2 and write down a nice proof.
The Plan

1. More practice with the Substitution Method.
2. k-SELECT problem
3. k-SELECT solution
4. Return of the Substitution Method.
The k-SELECT problem
from your pre-lecture exercise

A is an array of size n, k is in \{1,\ldots,n\}

• **SELECT**(A, k):
  • Return the k-th smallest element of A.

<table>
<thead>
<tr>
<th>7</th>
<th>4</th>
<th>3</th>
<th>8</th>
<th>1</th>
<th>5</th>
<th>9</th>
<th>14</th>
</tr>
</thead>
</table>

• **SELECT**(A, 1) = 1
• **SELECT**(A, 2) = 3
• **SELECT**(A, 3) = 4
• **SELECT**(A, 8) = 14

• **SELECT**(A, 1) = MIN(A)
• **SELECT**(A, n/2) = MEDIAN(A)
• **SELECT**(A, n) = MAX(A)

Note that the definition of Select is 1-indexed...

For today, assume all arrays have distinct elements.
On your pre-lecture exercise...

An $O(n \log(n))$-time algorithm

- **SELECT**(A, k):
  - A = **MergeSort**(A)
  - return A[k-1]

- Running time is $O(n \log(n))$.
- So that’s the benchmark....

Can we do better?

We’re hoping to get $O(n)$
Goal: An $O(n)$-time algorithm

• On your pre-lecture exercise: SELECT($A$, 1).
  • (aka, $\text{MIN}(A)$)

• $\text{MIN}(A)$:
  • $\text{ret} = \infty$
  • For $i=0, ..., n-1$:
    • If $A[i] < \text{ret}$:
      • $\text{ret} = A[i]$
  • Return $\text{ret}$

• Time $O(n)$.  Yay!
Also on your pre-lecture exercise

How about $\text{SELECT}(A, 2)$?

- $\text{SELECT2}(A)$:
  - $\text{ret} = \infty$
  - $\text{minSoFar} = \infty$
  - For $i=0, \ldots, n-1$:
    - If $A[i] < \text{ret}$ and $A[i] < \text{minSoFar}$:
      - $\text{ret} = \text{minSoFar}$
      - $\text{minSoFar} = A[i]$
      - $\text{minSoFar} = A[i]$
    - Else if $A[i] < \text{ret}$ and $A[i] \geq \text{minSoFar}$:
      - $\text{ret} = A[i]$
  - Return $\text{ret}$

(The actual algorithm here is not very important because this won’t end up being a very good idea...)

Still $O(n)$

SO FAR SO GOOD.
SELECT(A, n/2) aka MEDIAN(A)?

• MEDIAN(A):
  • ret = ∞
  • minSoFar = ∞
  • secondMinSoFar = ∞
  • thirdMinSoFar = ∞
  • fourthMinSoFar = ∞
  • ....

• This is not a good idea for large k (like n/2 or n).
• Basically, this is just going to turn into something like INSERTIONSORT...and that was O(n²).
The Plan

1. More practice with the Substitution Method.
2. k-SELECT problem
3. k-SELECT solution
4. Return of the Substitution Method.
Idea: divide and conquer!

Say we want to find \( \text{SELECT}(A, k) \)

First, pick a “pivot.” We’ll see how to do this later.

Next, partition the array into “bigger than 6” or “less than 6”

\[
L = \text{array with things smaller than } A[\text{pivot}]
\]

\[
R = \text{array with things larger than } A[\text{pivot}]
\]
Idea: divide and conquer!

Say we want to find $\text{SELECT}(A, k)$

First, pick a "pivot." We’ll see how to do this later.

Next, partition the array into "bigger than 6" or "less than 6"

$L = \text{array with things smaller than } A[\text{pivot}]$

$R = \text{array with things larger than } A[\text{pivot}]$
Idea continued...

Say we want to find \( \text{SELECT}(A, k) \)

- If \( k = 5 = \text{len}(L) + 1 \):
  - We should return \( A[\text{pivot}] \)
- If \( k < 5 \):
  - We should return \( \text{SELECT}(L, k) \)
- If \( k > 5 \):
  - We should return \( \text{SELECT}(R, k - 5) \)

This suggests a recursive algorithm

(Still need to figure out how to pick the pivot...)
Pseudocode

- **Select**$(A,k)$:
  - **If** $\text{len}(A) \leq 50$:
    - $A = \text{MergeSort}(A)$
    - **Return** $A[k-1]$
  - $p = \text{getPivot}(A)$
  - $L$, pivotVal, $R = \text{Partition}(A,p)$
  - **If** $\text{len}(L) == k-1$:
    - return pivotVal
  - **Else if** $\text{len}(L) > k-1$:
    - return **Select**$(L,k)$
  - **Else if** $\text{len}(L) < k-1$:
    - return **Select**$(R,k - \text{len}(L) - 1)$

**Base Case:** If $\text{len}(A) = O(1)$, then any sorting algorithm runs in time $O(1)$.

**Case 1:** We got lucky and found exactly the $k$’th smallest value!

**Case 2:** The $k$’th smallest value is in the first part of the list

**Case 3:** The $k$’th smallest value is in the second part of the list

- **getPivot**$(A)$ returns some pivot for us.
  - **How??** We’ll see later...
  - See Lecture 4 Python notebook for code
Does it work?

• Check out the Python notebook for Lecture 4, which implements this with a bunch of different pivot-selection methods.
  • Seems to work!

• Check out the lecture notes for a rigorous proof based on induction that this works, with any pivot-choosing mechanism.
  • It provably works!
  • Also, this is a good example of proving that a recursive algorithm is correct.
What is the running time?

Assuming we pick the pivot in time $O(n)$…

$$T(n) = \begin{cases} 
T(len(L)) + O(n) & \text{len}(L) > k - 1 \\
T(len(R)) + O(n) & \text{len}(L) < k - 1 \\
O(n) & \text{len}(L) = k - 1 
\end{cases}$$

• What are $len(L)$ and $len(R)$?
• That depends on how we pick the pivot…

Think: one minute
Share: (wait) one minute

What would be a “good” pivot?
What would be a “bad” pivot?

The best way would be to always pick the pivot so that $len(L) = k-1$. But say we don’t have control over $k$, just over how we pick the pivot.
The ideal pivot

- We split the input exactly in half:
  - \( \text{len}(L) = \text{len}(R) = (n-1)/2 \)

What happens in that case?

Think: one minute
Share: (wait) one minute

In case it’s helpful...
- Suppose \( T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d) \). Then

\[
T(n) = \begin{cases} 
0(n^d \log(n)) & \text{if } a = b^d \\
0(n^d) & \text{if } a < b^d \\
0(n^{\log_b(a)}) & \text{if } a > b^d 
\end{cases}
\]
The ideal pivot

- We split the input exactly in half:
  - \( \text{len}(L) = \text{len}(R) = \frac{n-1}{2} \)

- Let’s pretend that’s the case and use the Master Theorem!

\[
T(n) \leq T\left(\frac{n}{2}\right) + O(n)
\]

- So \( a = 1, b = 2, d = 1 \)

\[
T(n) \leq O\left(n^d\right) = O(n)
\]

That would be great!
The worst pivot

• Say our choice of pivot doesn’t depend on A.
• A bad guy who knows what pivots we will choose gets to come up with A.
The distinction matters!

For this one I chose the worst possible pivot. Looks like $O(n^2)$.  

This one is a random pivot, so it splits the array about in half. Looks fast!

See Lecture 4 Python notebook for code that generated this picture.
How do we pick a good pivot?

• Randomly?
  • That works well if there’s no bad guy.
  • But if there is a bad guy who gets to see our pivot choices, that’s just as bad as the worst-case pivot.

Aside:
• In practice, there is often no bad guy. In that case, just pick a random pivot and it works really well!
• (More on this next lecture)
How do we pick a good pivot?

• For today, let’s assume there’s this bad guy.
• Reasons:
  • This gives us a very strong guarantee
  • We’ll get to see a really clever algorithm.
    • Necessarily it will look at A to pick the pivot.
  • We’ll get to use the substitution method.
The Plan

1. More practice with the Substitution Method.
2. k-SELECT problem
3. k-SELECT solution
   a) The outline of the algorithm.
   b) How to pick the pivot.
4. Return of the Substitution Method.
Approach

• First, we’ll figure out what the ideal pivot would be.
  • But we won’t be able to get it.

• Then, we’ll figure out what a pretty good pivot would be.
  • But we still won’t know how to get it.

• Finally, we will see how to get our pretty good pivot!
  • And then we will celebrate.
How do we pick our ideal pivot?

• We’d like to live in the ideal world.

• Pick the pivot to divide the input in half.

• Aka, pick the median!

• Aka, pick $\text{SELECT}(A, n/2)$!
How about a good enough pivot?

• We’d like to approximate the ideal world.

• Pick the pivot to divide the input about in half!
• Maybe this is easier!
A good enough pivot

• We split the input not quite in half:
  • $3n/10 < \text{len}(L) < 7n/10$
  • $3n/10 < \text{len}(R) < 7n/10$

• If we could do that (let’s say, in time $O(n)$), the Master Theorem would say:
  • $T(n) \leq T\left(\frac{7n}{10}\right) + O(n)$
  • So $a = 1$, $b = 10/7$, $d = 1$
  • $T(n) \leq O\left(n^d\right) = O(n)$

We still don’t know that we can get such a pivot, but at least it gives us a goal and a direction to pursue!

Lucky the lackadaisical lemur

STILL GOOD!
Goal

- Efficiently pick the pivot so that

\[
\frac{3n}{10} < \text{len}(L) < \frac{7n}{10}
\]

\[
\frac{3n}{10} < \text{len}(R) < \frac{7n}{10}
\]
Another divide-and-conquer alg!

• We can’t solve $\text{SELECT}(A, n/2)$ (yet)

• But we can divide and conquer and solve $\text{SELECT}(B, m/2)$ for smaller values of $m$ (where $\text{len}(B) = m$).

• **Lemma**: The median of sub-medians is close to the median.

*we will make this a bit more precise.*
How to pick the pivot

- **CHOOSEPIVOT(A):**
  - Split A into \( m = \left\lfloor \frac{n}{5} \right\rfloor \) groups, of size \( \leq 5 \) each.
  - For \( i = 1, \ldots, m \):
    - Find the median within the \( i \)'th group, call it \( p_i \)
    - \( p = \text{SELECT}( [ p_1, p_2, p_3, \ldots, p_m ] , m/2 ) \)
  - return the index of \( p \) in A

This takes time \( O(1) \) for each group, since each group has size 5. So that’s \( O(m) = O(n) \) total in the for loop.

Why 5 and not 3? See the concept check questions!

Pivot is \( \text{SELECT}( 8, 4, 5, 6, 12 , 3 ) = 6 \):

PARTITION around that 6:
CLAIM: this works divides the array *approximately* in half

- Empirically (see Lecture 4 Python Notebook):
CLAIM: this works
divides the array \textit{approximately} in half

• Formally, we will prove (later):

\textbf{Lemma}: If we choose the pivots like this, then

\[ |L| \leq \frac{7n}{10} + 5 \]

and

\[ |R| \leq \frac{7n}{10} + 5 \]
Sanity Check

\[ |L| \leq \frac{7n}{10} + 5 \text{ and } |R| \leq \frac{7n}{10} + 5 \]

In practice (on randomly chosen arrays) it looks even better!

But this is a worst-case bound.
How about the running time?

• Suppose the Lemma is true. (It is).
  • $|L| \leq \frac{7n}{10} + 5$ and $|R| \leq \frac{7n}{10} + 5$

• Recurrence relation:

  $$T(n) \leq ?$$

Think: 1 minute
Share: (wait) 1 minute
Pseudocode

• **Select**\( (A,k) \):
  • **If** \( \text{len}(A) \leq 50 \):
    • \( A = \text{MergeSort}(A) \)
    • **Return** \( A[k-1] \)
  • \( p = \text{choosePivot}(A) \)
  • \( L, \text{pivotVal}, R = \text{Partition}(A,p) \)
  • **if** \( \text{len}(L) == k-1 \):
    • return \( \text{pivotVal} \)
  • **Else if** \( \text{len}(L) > k-1 \):
    • return \( \text{Select}(L, k) \)
  • **Else if** \( \text{len}(L) < k-1 \):
    • return \( \text{Select}(R, k - \text{len}(L) - 1) \)

**Base Case:** If \( \text{len}(A) = O(1) \), then any sorting algorithm runs in time \( O(1) \).

**Case 1:** We got lucky and found exactly the \( k \)'th smallest value!

**Case 2:** The \( k \)'th smallest value is in the first part of the list

**Case 3:** The \( k \)'th smallest value is in the second part of the list

• Lemma says that \( |L| \leq \frac{7n}{10} + 5 \) and \( |R| \leq \frac{7n}{10} + 5 \)
• Suppose **Partition** runs in time \( O(n) \)
• Come up with a recurrence relation for \( T(n) \), the running time of **Select**, using the **choosePivot** algorithm we just described.
How about the running time?

• Suppose the Lemma is true. (It is).
  • $|L| \leq \frac{7n}{10} + 5$ and $|R| \leq \frac{7n}{10} + 5$

• Recurrence relation:

$$T(n) \leq T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + O(n)$$

The call to CHOOSEPIVOT makes one further recursive call to SELECT on an array of size $n/5$.

Outside of CHOOSEPIVOT, there’s at most one recursive call to SELECT on an array of size $7n/10 + 5$.

We’re going to drop the “+5” for convenience, but it does not change the final answer. Why?

Hint: Define $T'(n) := T(n+1000)$ and write recurrence for $T'$

Siggi the Studious Stork
The Plan

1. More practice with the Substitution Method.
2. k-SELECT problem
3. k-SELECT solution
   a) The outline of the algorithm.
   b) How to pick the pivot.
4. Return of the Substitution Method.
This sounds like a job for...

The Substitution Method!

Step 1: generate a guess
Step 2: try to prove that your guess is correct
Step 3: profit

\[ T(n) \leq T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + O(n) \]

That’s convenient! We did this at the beginning of lecture!

Conclusion: \( T(n) = O(n) \)

Technically we only did it for \( T(n) \leq T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + n \), not when the last term has a big-Oh...

Plucky the Pedantic Penguin
Recap of approach

• First, we figured out what the ideal pivot would be.
  • Find the median

• Then, we figured out what a pretty good pivot would be.
  • An approximate median

• Finally, we saw how to get our pretty good pivot!
  • Median of medians and divide and conquer!
  • Hooray!
In practice?

- With not-very-slick implementation, our fancy version of SELECT is worse than the MergeSort-based SELECT 😞
  - But $O(n)$ is better than $O(n \log(n))$! How can that be?
  - What’s the constant in front of the $n$ in our proof? 20? 30?

- On non-adversarial inputs, random pivot choice is much better.

**Moral:**
Just pick a random pivot if you don’t expect nefarious arrays.

Optimize the implementation of SELECT (with the fancy pivot). Can you beat MergeSort?

Siggi the Studious Stork
What have we learned?

Pending the Lemma

• It is possible to solve SELECT in time $O(n)$.
  • Divide and conquer!

• If you want a deterministic algorithm or expect that a bad guy will be picking the list, **choose a pivot cleverly**.
  • More divide and conquer!

• If you don’t expect that a bad guy will be picking the list, in practice it’s better just to **pick a random pivot**.
The Plan

1. More practice with the Substitution Method.
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   a) The outline of the algorithm.
   b) How to pick the pivot.
4. Return of the Substitution Method.
5. (If time) Proof of that Lemma.
If time, back to the Lemma

• **Lemma**: If $L$ and $R$ are as in the algorithm SELECT given above, then

\[ |L| \leq \frac{7n}{10} + 5 \]

and

\[ |R| \leq \frac{7n}{10} + 5 \]

• We will see a proof by picture.

• See lecture notes for proof by proof.
Proof by picture

Say these are our $m = \lfloor n/5 \rfloor$ sub-arrays of size at most 5.
In our head, let’s sort them.
Then find medians.
Proof by picture

Then let’s sort them by the median
The median of the medians is 7. That’s our pivot!
Proof by picture

We will show that lots of elements are smaller than the pivot, hence not too many are larger than the pivot.

How many elements are SMALLER than the pivot?
Proof by picture

At least these ones: everything above and to the left.
Proof by picture

\( \frac{m}{2} \) of these, but then one of them could have been the “leftovers” group.

How many of those are there?

\[ \text{at least } 3 \cdot \left( \left\lfloor \frac{m}{2} \right\rfloor - 2 \right) \]
So how many are LARGER than the pivot? At most...

\[ n - 1 - 3 \left( \left\lfloor \frac{m}{2} \right\rfloor - 2 \right) \leq \frac{7n}{10} + 5 \]

Remember \( m = \left\lfloor \frac{n}{5} \right\rfloor \)

(derivation on board)
That was one part of the lemma

• **Lemma:** If L and R are as in the algorithm SELECT given above, then

\[ |L| \leq \frac{7n}{10} + 5 \]

and

\[ |R| \leq \frac{7n}{10} + 5 \]

The other part is exactly the same.
The Plan

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5. (If time) Proof of that Lemma.

Recap
Recap

• Substitution method can work when the master theorem doesn’t.

• One place we needed it was for SELECT.
  • Which we can do in time $O(n)$!
Next time

• Randomized algorithms and QuickSort!

BEFORE next time

• Pre-lecture 5 exercise
  • Remember probability theory?
  • The pre-lecture exercise will jog your memory.