Induction

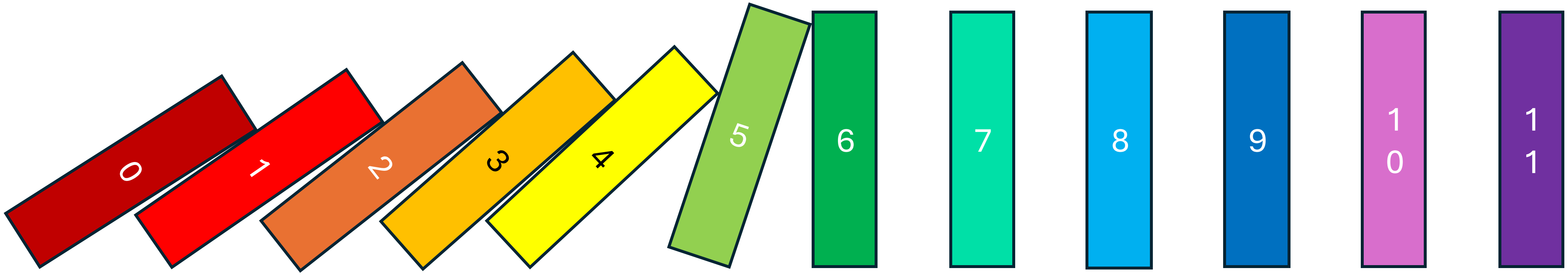
CS 161

Prereqs Review

January 10, 2025

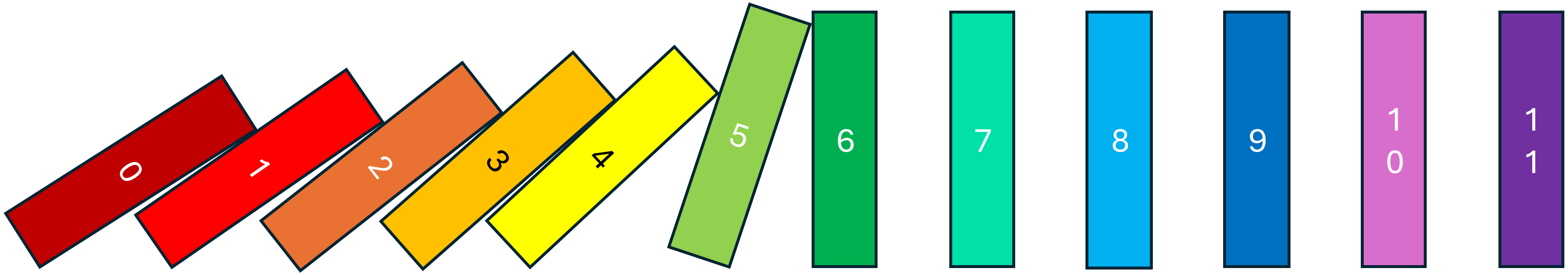
What is induction?

- Proof technique for showing something is true for all natural numbers
- Step-by-step logical structure
- Like knocking over an infinite line of dominoes



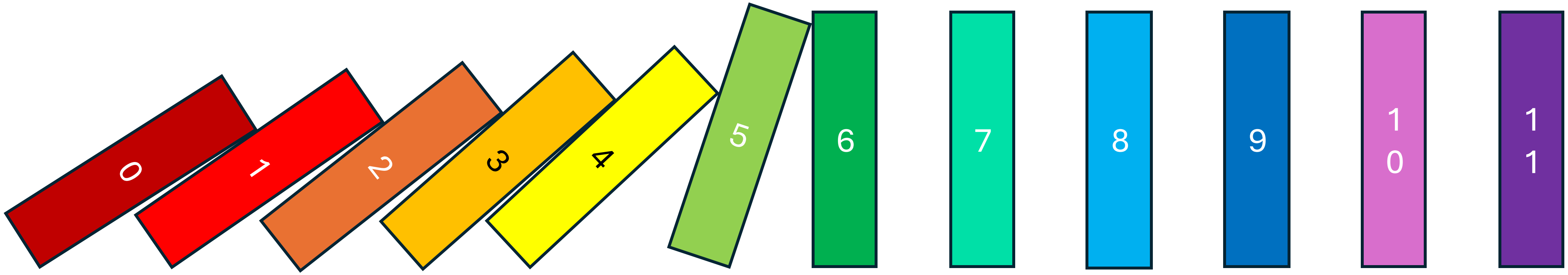
Proof structure

- **Claim**
- **Base case(s)**
- **Inductive hypothesis**
- **Inductive step**
- **Conclusion**



Proof structure

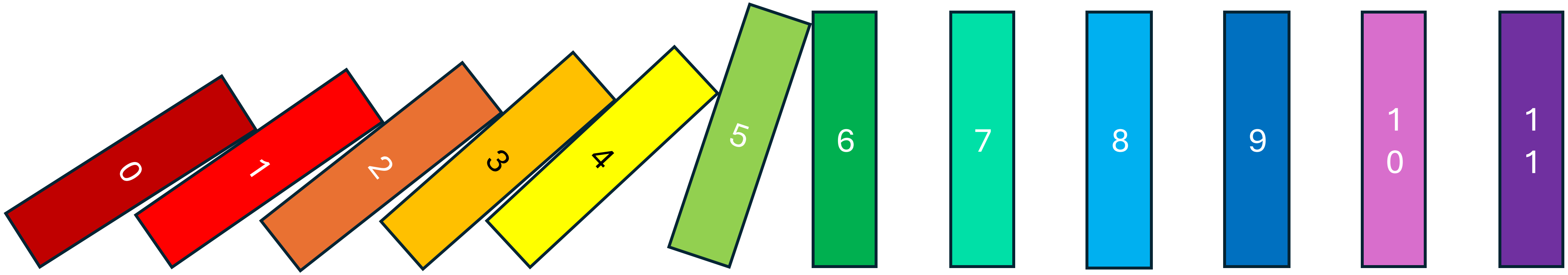
- **Claim:** All dominoes will fall
- **Base case(s):** The first domino falls because I knocked it down
- **Inductive hypothesis:** Assume the *k*th domino falls
- **Inductive step:** Then the *(k+1)*th domino is knocked down
- **Conclusion:** By induction, all dominoes will fall



Proof structure

- **Claim:** $\forall n: P(n)$ is true
- **Base case(s):** $P(0)$ is true
- **Inductive hypothesis:** Assume $P(k)$ is true
- **Inductive step:** Then we prove that $P(k+1)$ is true
- **Conclusion:** Therefore $\forall n: P(n)$ is true

$$P(0), \forall k : P(k) \rightarrow P(k+1) \Rightarrow \forall n : P(n)$$



Strong induction

- When we get to the *k*th domino, we know that **all** dominoes up to this point have fallen
- Sometimes this extra information is useful
- What changes?
- Old inductive hypothesis: ~~Assume $P(k)$ is true~~
- New: Assume $P(k')$ is true for all $k' \leq k$
- Why not always use strong induction?

Tips

- Make sure you have enough base cases
- Make sure you use $P(k)$ to prove $P(k+1)$
  - If you don't need $P(k)$, you might not need induction at all!
  - If you use $P(k)$ to prove $P(k)$, there's a problem
- Look for problems suited to proof by induction
  - Key words: recursion, subproblems, divide and conquer

Questions?

Example 1: warmup

Prove that for all $n \geq 1$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

- We want to prove something holds for all natural numbers
- There's a structural relationship between one case and the next:

$$\underbrace{1^2 + 2^2 + \cdots + n^2}_{\text{previous case}} + (n+1)^2$$

- So let's try induction!

Example 1: Warmup

- **Claim:** for all $n \geq 1$, $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
- **Base case:** $1^2 = \frac{1 \cdot 2 \cdot 3}{6} = 1$ ✅
- **Inductive hypothesis:** Assume that

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Example 1: Warmup

- **Inductive step:**

$$\underbrace{1^2 + \cdots + k^2} + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{(2k^3 + 3k^2 + k) + (6k^2 + 12k + 6)}{6}$$

$$= \frac{2k^3 + 9k^2 + 13k + 6}{6}$$

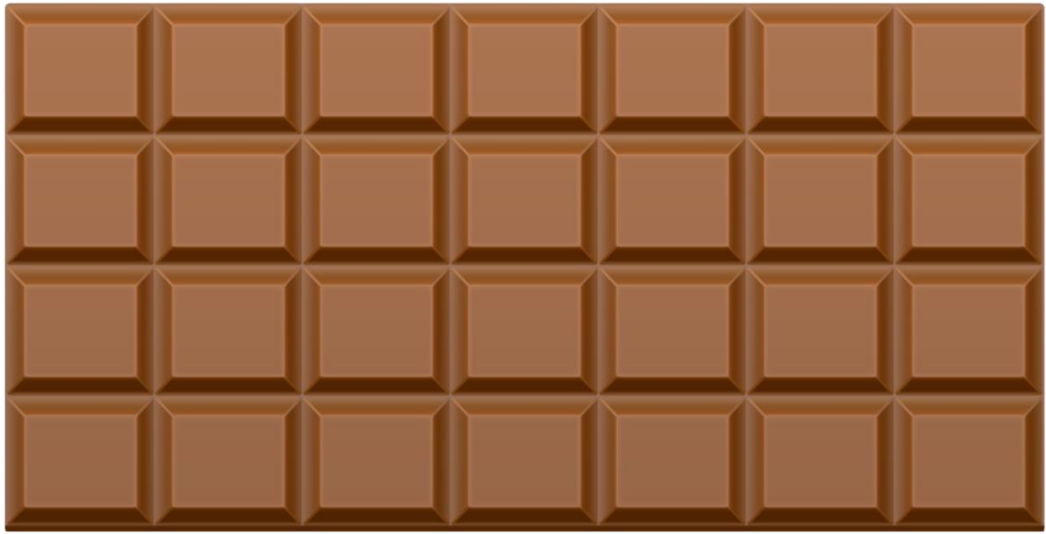
$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

- **Conclusion:** The claim holds for all $n \geq 1$.

Example 2: Strong induction

Prove that an $m \times n$ chocolate bar can be divided into 1×1 pieces using $mn - 1$ breaks

- Recursive structure: breaking chocolate bar produces a smaller chocolate bar
- Many possible sizes of smaller chocolate bar
- Strong induction!

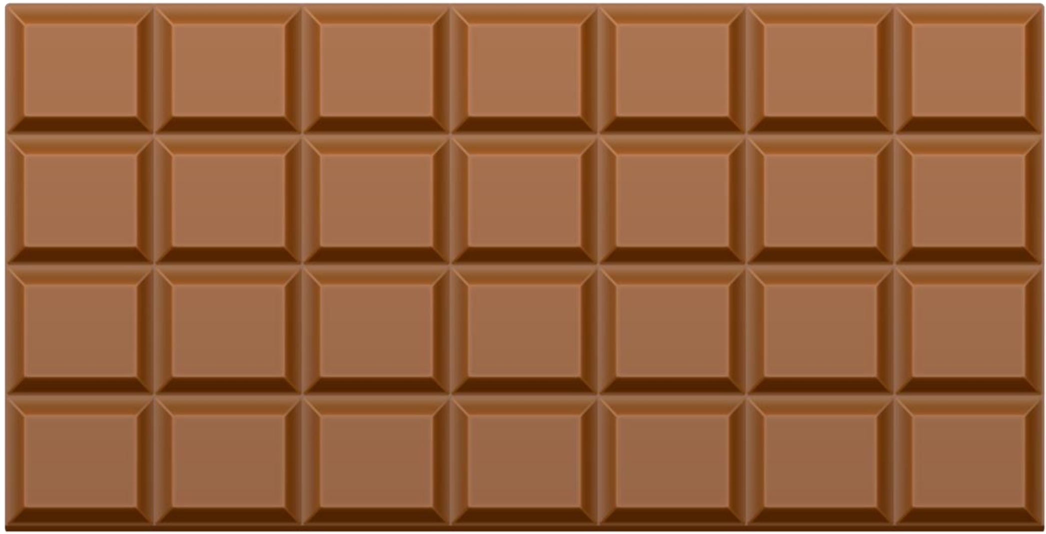


Example 2: Strong induction

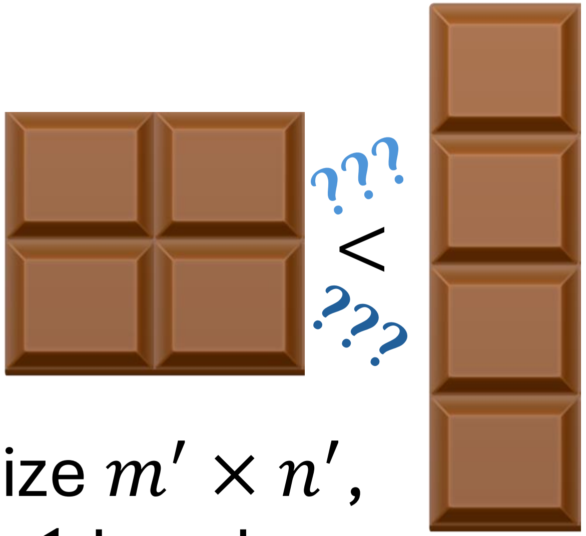
- **Base case:** A 1×1 chocolate bar requires zero breaks
- **Inductive hypothesis:** Assume all chocolate bars smaller than $m \times n$ require the stated number of breaks.
- **Inductive step:** Break our $m \times n$ bar somewhere
  - We get two bars of size $m' \times n$ and $(m - m') \times n$
  - Total breaks:

$$1 + (m'n - 1) + ((m - m')n - 1) \\ = mn - 1$$

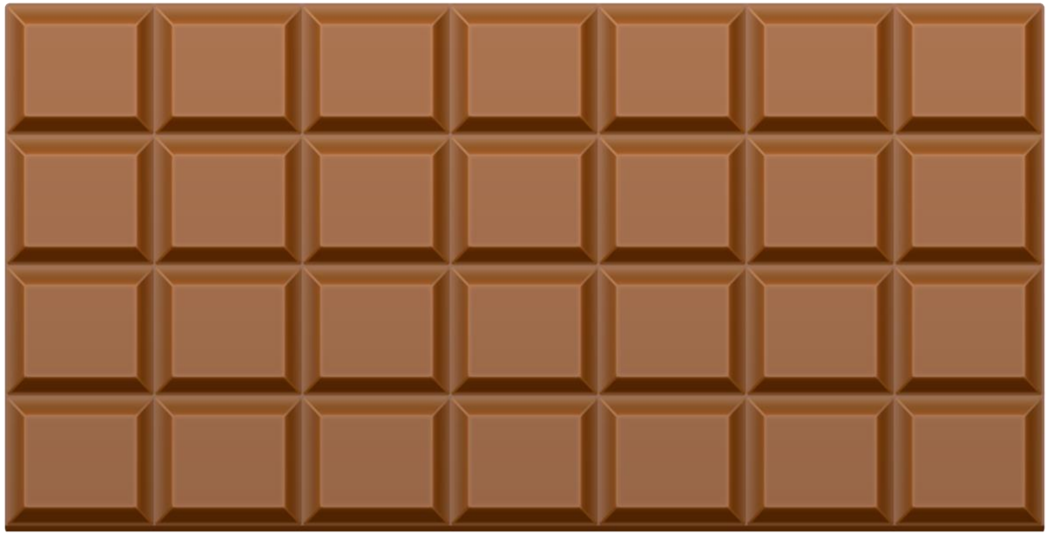
- **Conclusion:** Claim holds by strong induction



Example 2: Strong induction

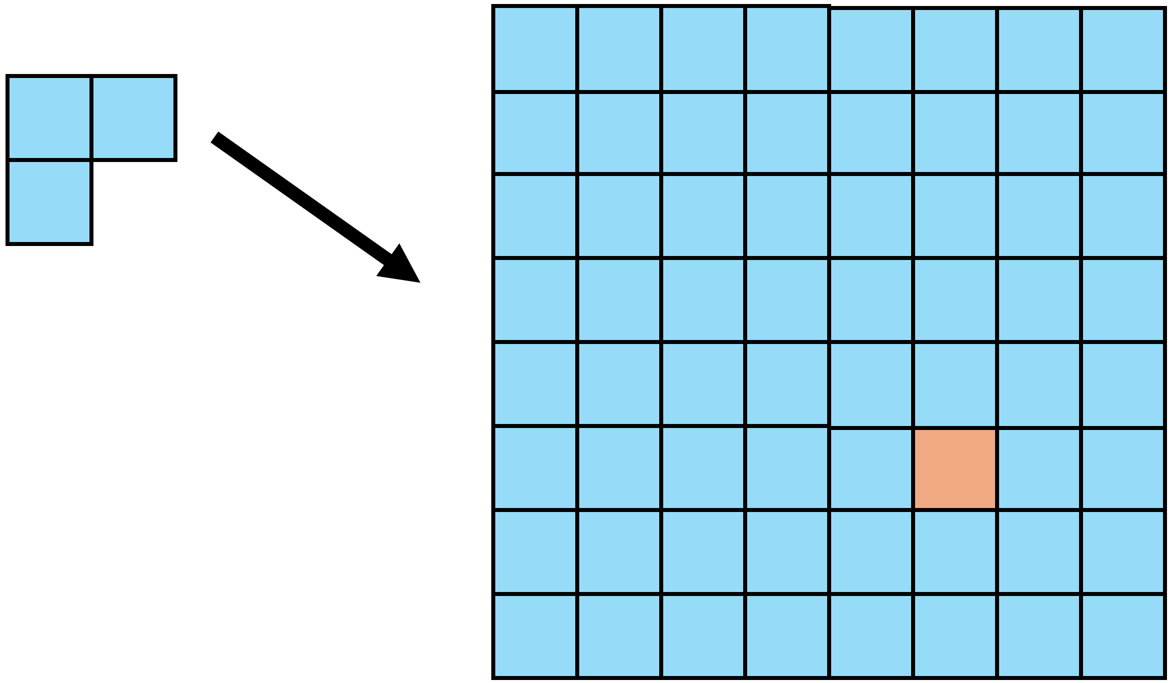


- What does “smaller” mean?
  - Needs to be strong enough to support our inductive step
  - But not so strong that the domino chain breaks
- **Revised hypothesis:** Assume all chocolate bars of size $m' \times n'$, where $m' < m$ or $n' < n$, can be divided using $m'n' - 1$ breaks
  - Works for our inductive step ✅
  - Can we trace the logic down to base case?
  - Yes! Number of chocolate squares gets smaller each step



Example 3: Tiling a square

Prove that for any $n \geq 1$, you can tile a $2^n \times 2^n$ grid with L shapes such that all squares are covered except for one of my choice



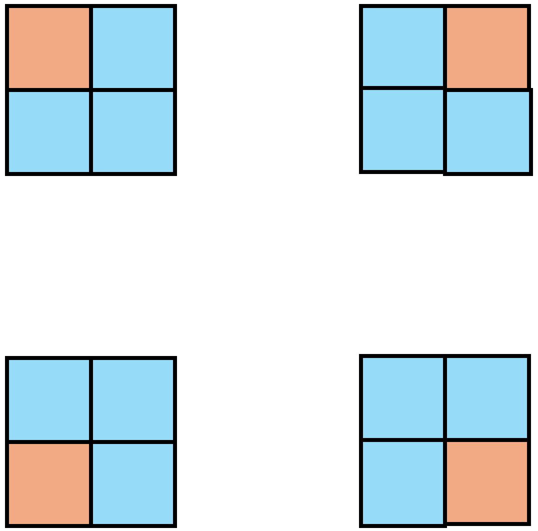
Inductive proof a tiling exists

$\Leftrightarrow$

Recursive algorithm to construct a tiling

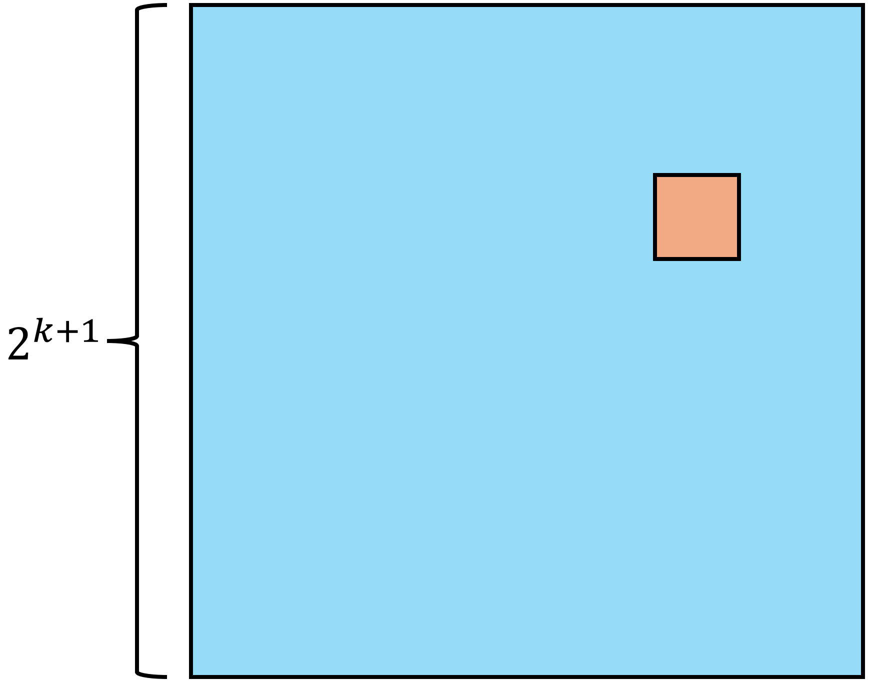
Example 3: Tiling a square

- **Base case:**



Example 3: Tiling a square

- **Inductive hypothesis:** Assume we can tile a $2^k \times 2^k$ square
- **Inductive step:**



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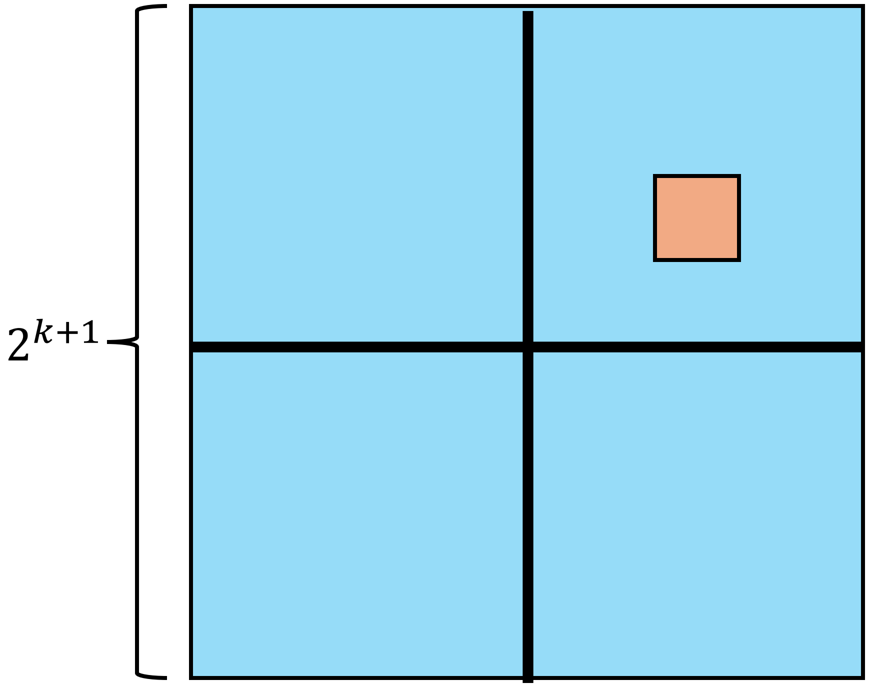

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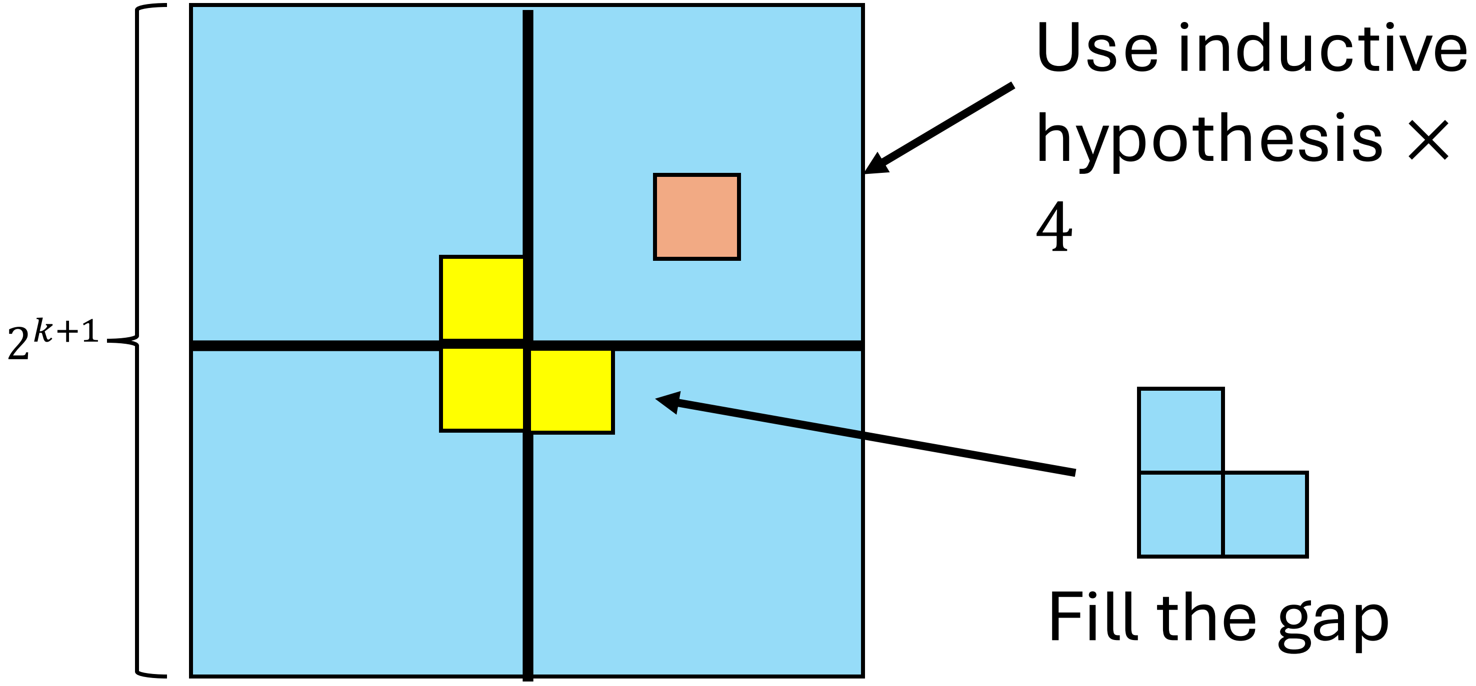

Questions?

Next: Big O