Lecture 13

More dynamic programming!

Longest Common Subsequences, Knapsack, and (if time) independent sets in trees.

Last time

Dynamic Programming!

- Not coding in an action movie.

T se programs dynamically
in Mission Impossible

Last time

Dynamic Programming!

- Dynamic programming is an algorithm design paradigm.
- Basic idea:
  - Identify **optimal sub-structure**
    - Optimum to the big problem is built out of optima of small sub-problems
  - Take advantage of **overlapping sub-problems**
    - Only solve each sub-problem once, then use it again and again
  - Keep track of the solutions to sub-problems in a table as you build to the final solution.

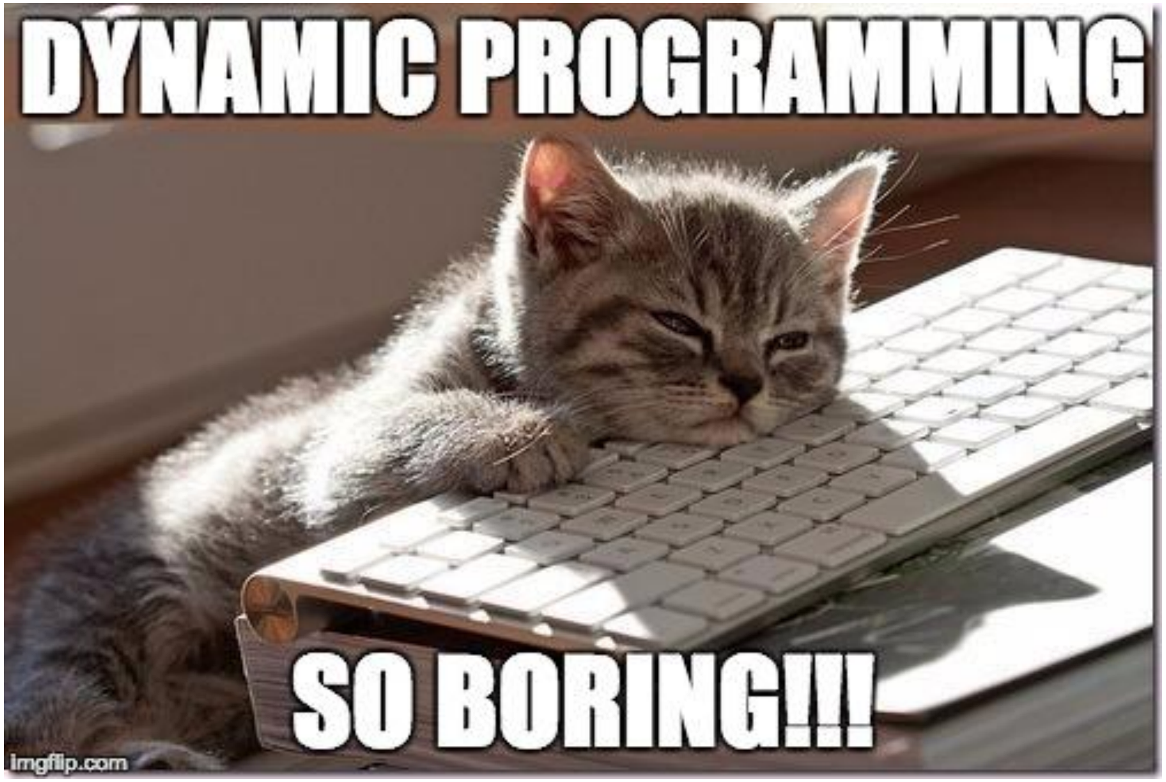
Today

- Examples of dynamic programming:
  1. Longest common subsequence
  2. Knapsack problem
     - Two versions!
  3. Independent sets in trees
     - If we have time…
     - (If not the slides will be there as a reference)

- Yet more examples of DP in CLRS!
  - Optimal order of matrix multiplications
  - Optimal binary search trees
  - Longest paths in DAGs, …

The goal of this lecture

- For you to get really bored of dynamic programming



Longest Common Subsequence

- How similar are these two species?

DNA:

AGCCCTAAGGGCTACCTAGCTT

DNA:

GACAGCCTACAAGCGTTAGCTTG

Longest Common Subsequence

- How similar are these two species?

DNA:

AGCCCTAAGGGCTACCTAGCTT

DNA:

GACAGCCTACAAGCGTTAGCTTG

- Pretty similar, their DNA has a long common subsequence:

AGCCTAAGCTTAGCTT

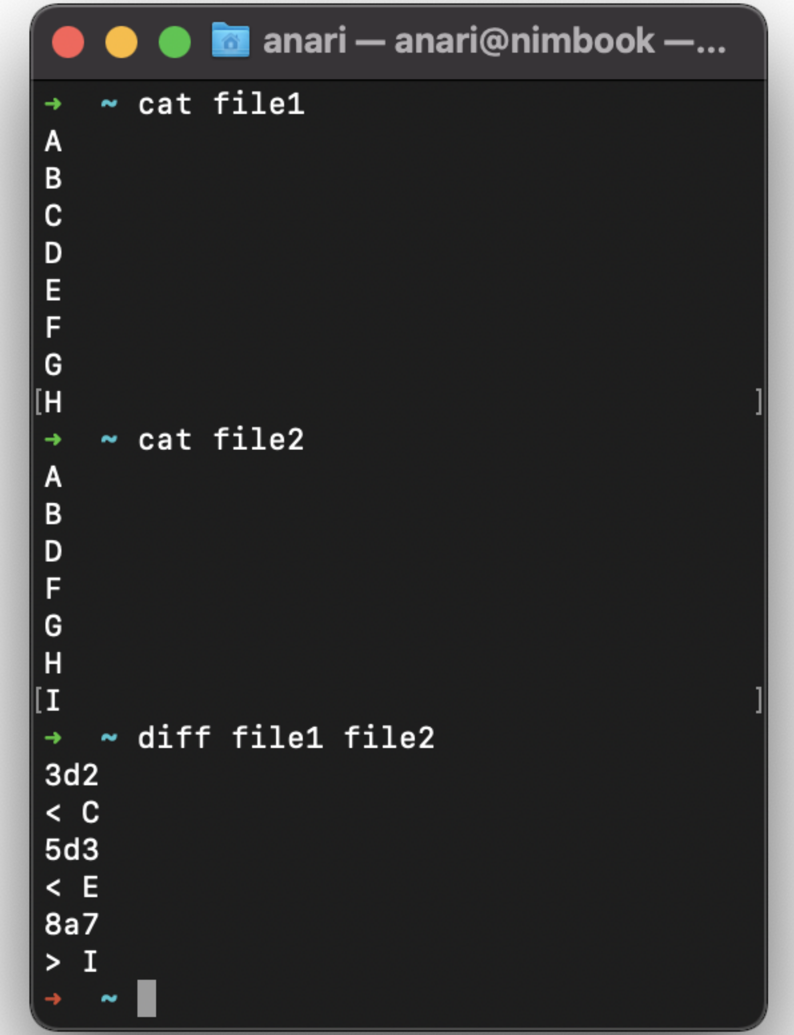
Longest Common Subsequence

- Subsequence:
  - BDFH is a **subsequence** of ABCDEFGH
- If X and Y are sequences, a **common subsequence** is a sequence which is a subsequence of both.
  - BDFH is a **common subsequence** of ABCDEFGH and of ABDFGHI
- A **longest common subsequence**…
  - …is a common subsequence that is longest.
  - The **longest common subsequence** of ABCDEFGH and ABDFGHI is ABDFGH.

We sometimes want to find these

- Applications in bioinformatics

- The unix command `diff`
- Merging in version control
  - svn, git, etc…



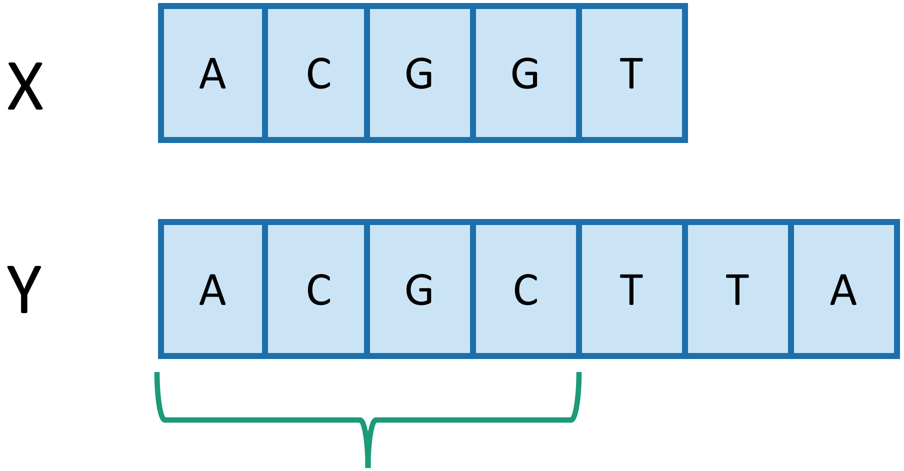
```
➜  ~ cat file1
A
B
C
D
E
F
G
[H                                                     ]
➜  ~ cat file2
A
B
D
F
G
H
[I                                                     ]
➜  ~ diff file1 file2
3d2
< C
5d3
< E
8a7
> I
➜  ~ █
```

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the length of the longest common subsequence.
- **Step 3:** Use dynamic programming to find the length of the longest common subsequence.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual LCS.
- **Step 5:** If needed, code this up like a reasonable person.

Step 1: Optimal substructure

Prefixes:



X: A C G G T

Y: A C G C T T A

Notation: denote this prefix **ACGC** by Y_4

- Our sub-problems will be finding LCS's of prefixes to X and Y.
- Let $C[i,j] = \text{length\_of\_LCS}(X_i, Y_j)$

Examples:
$C[2,3] = 2$
$C[4,4] = 3$

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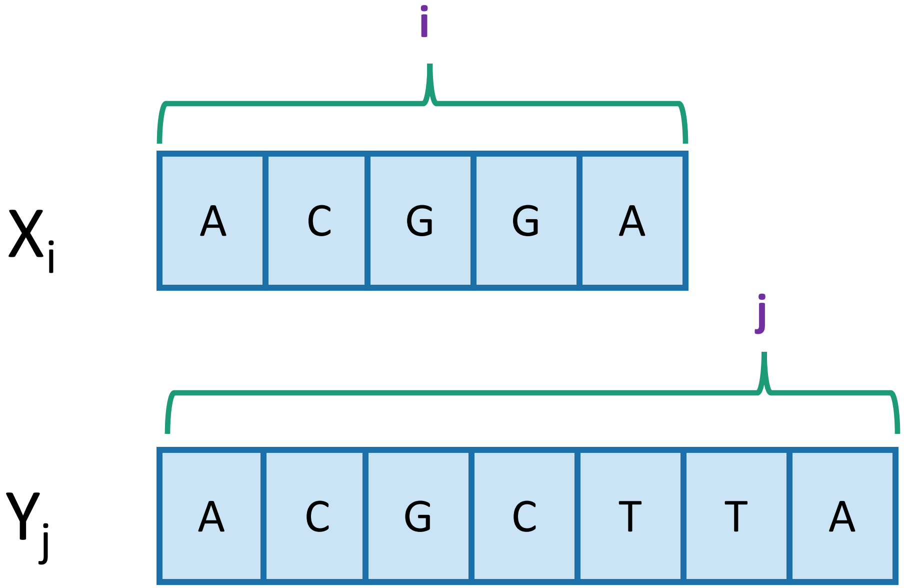
Goal

- Write C[i,j] in terms of the solutions to smaller sub-problems



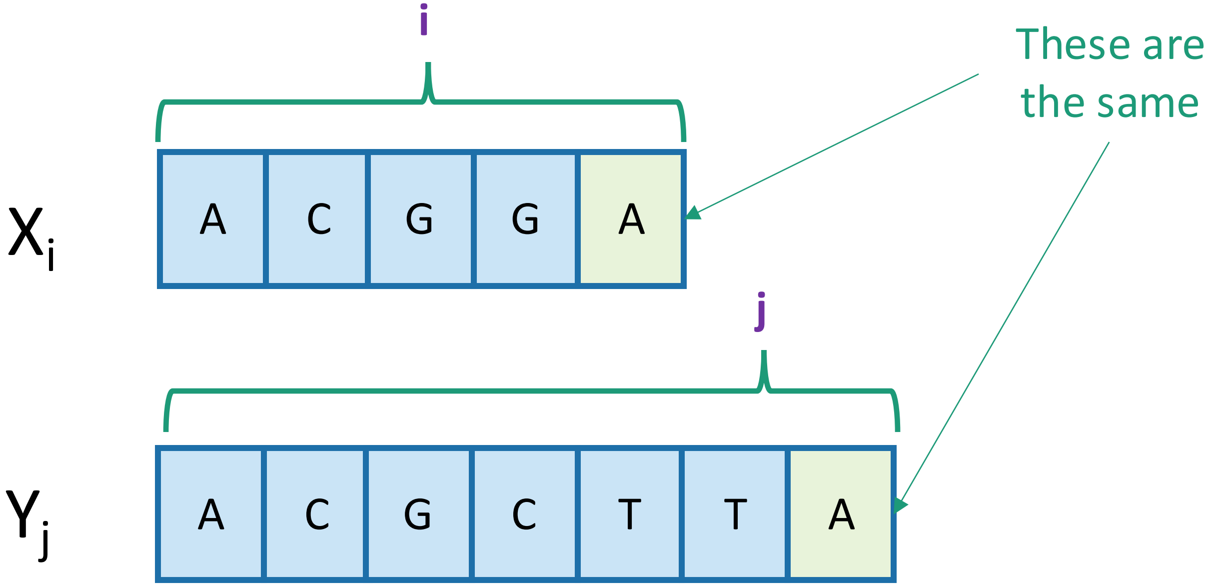
X_i	A	C	G	G	A		
Y_j	A	C	G	C	T	T	A

$$C[i,j] = \text{length\_of\_LCS}(\ X_i,\ Y_j\ )$$

Two cases

- Our sub-problems will be finding LCS's of prefixes to X and Y.
- Let C[i,j] = length_of_LCS(X_i, Y_j)

Case 1: X[i] = Y[j]



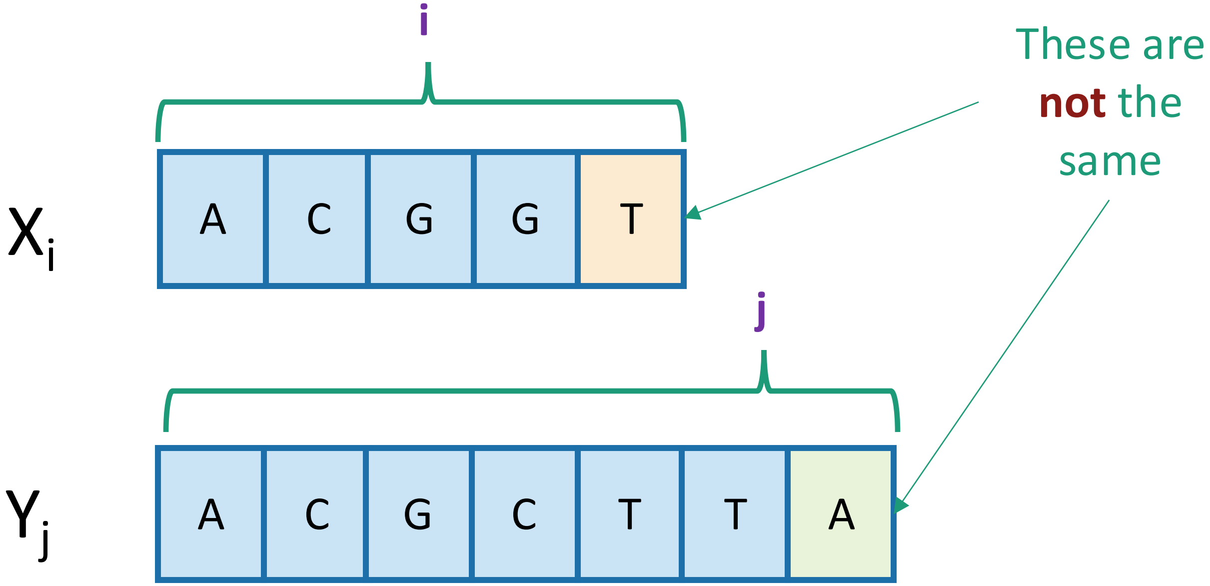
These are the same

- Then C[i,j] = 1 + C[i-1,j-1].
  - because LCS(X_i,Y_j) = LCS(X_{i-1},Y_{j-1}) followed by A

Two cases

Case 2: X[i] != Y[j]

- Our sub-problems will be finding LCS's of prefixes to X and Y.
- Let $C[i,j] = \text{length\_of\_LCS}(X_i, Y_j)$

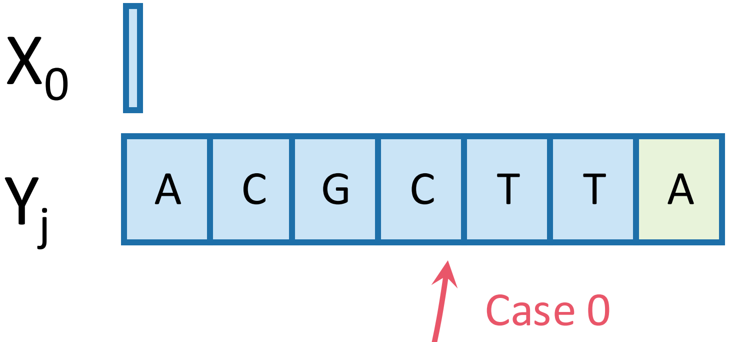


These are **not** the same

- Then $C[i,j] = \max\{ C[i-1,j], C[i,j-1] \}$.
    - either $\text{LCS}(X_i,Y_j) = \text{LCS}(X_{i-1},Y_j)$ and $\boxed{\text{T}}$ is not involved,
    - or $\text{LCS}(X_i,Y_j) = \text{LCS}(X_i,Y_{j-1})$ and $\boxed{\text{A}}$ is not involved,
    - (maybe both are not involved, that's covered by the "or").

Recursive formulation

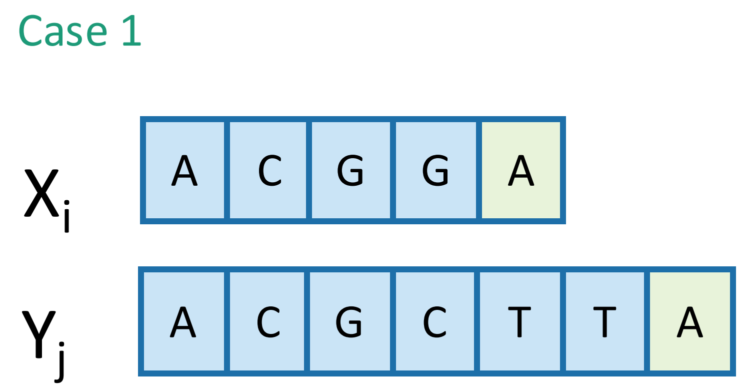
of the optimal solution



X_0							
Y_j	A	C	G	C	T	T	A

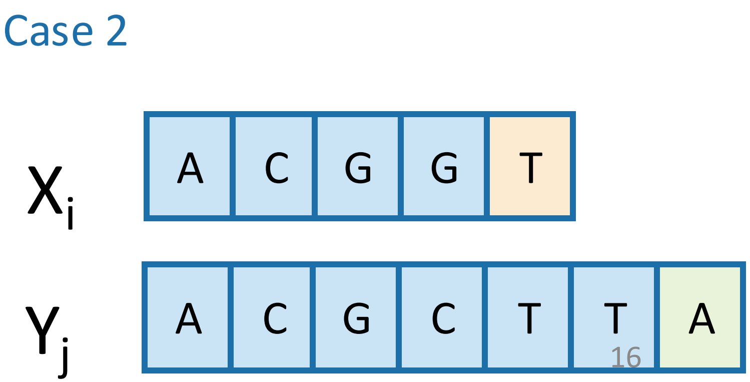
Case 0

- $$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$



Case 1

X_i	A	C	G	G	A		
Y_j	A	C	G	C	T	T	A



Case 2

X_i	A	C	G	G	T		
Y_j	A	C	G	C	T	T	A

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LCS DP

```
• LCS(X, Y):
    • C[i,0] = C[0,j] = 0 for all i = 0,…,m, j=0,…n.
    • For i = 1,…,m and j = 1,…,n:
        • If X[i] = Y[j]:
            • C[i,j] = C[i-1,j-1] + 1
        • Else:
            • C[i,j] = max{ C[i,j-1], C[i-1,j] }
    • Return C[m,n]
```

Running time: O(nm)

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X	A	C	G	G	A

Y	A	C	T	G

Y

		A	C	T	G
	0	0	0	0	0
A	0				
C	0				
G	0				
G	0				
A	0				

X

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1,j-1]+1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{\, C[i,j-1], C[i-1,j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X	A	C	G	G	A
Y	A	C	T	G	

Y

X		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

So the LCM of X and Y has length 3.

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{ C[i,j-1], C[i-1,j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

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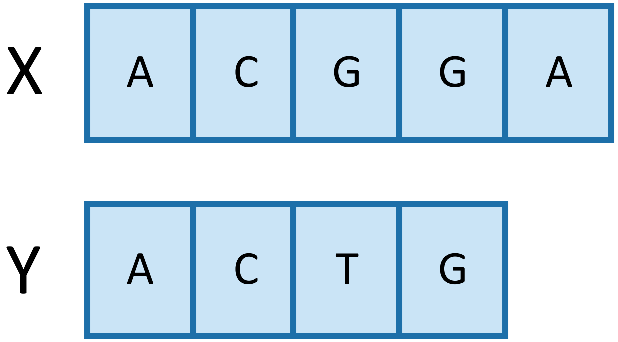
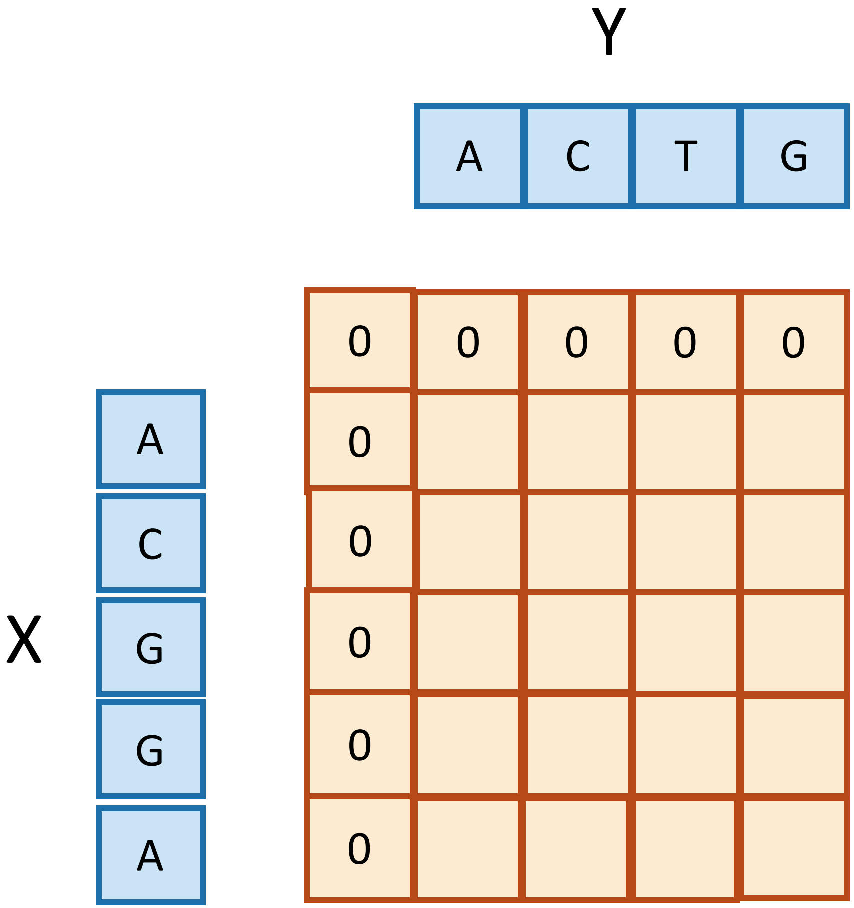
Example



X	A	C	G	G	A
Y	A	C	T	G	



		Y			
		A	C	T	G
		0	0	0	0	0
X	A	0				
	C	0				
	G	0				
	G	0				
	A	0				

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{\, C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X	A	C	G	G	A
Y	A	C	T	G	

Y

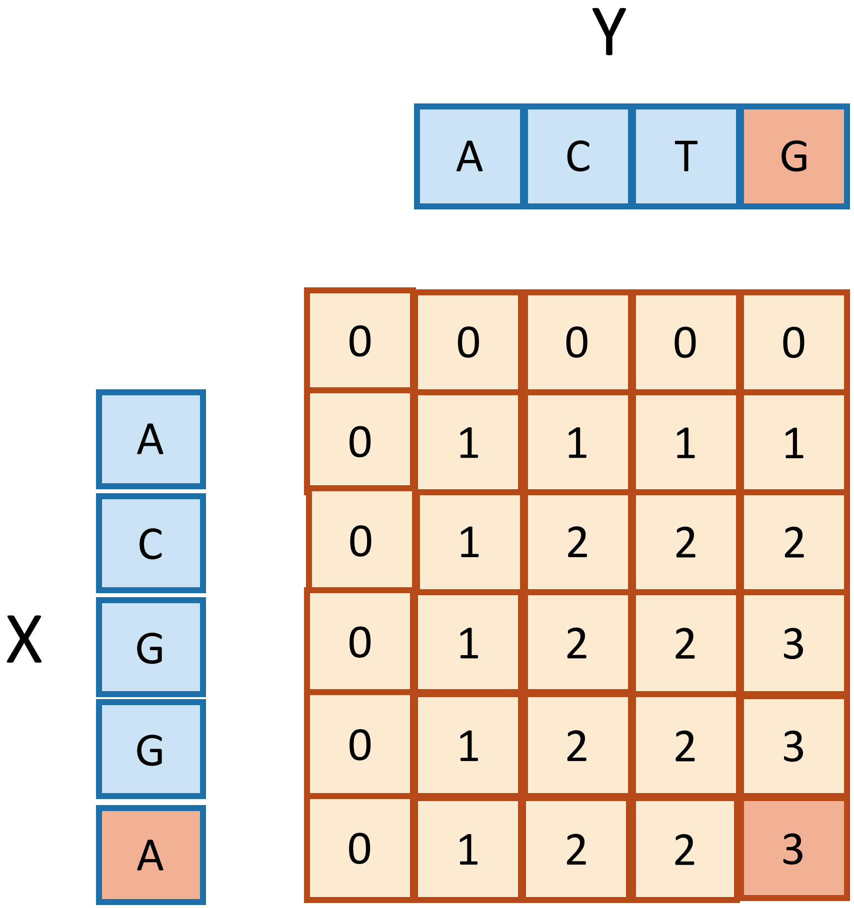
		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

X

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X	A	C	G	G	A
Y	A	C	T	G	



			Y			
			A	C	T	G
		0	0	0	0	0
	A	0	1	1	1	1
	C	0	1	2	2	2
X	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	3

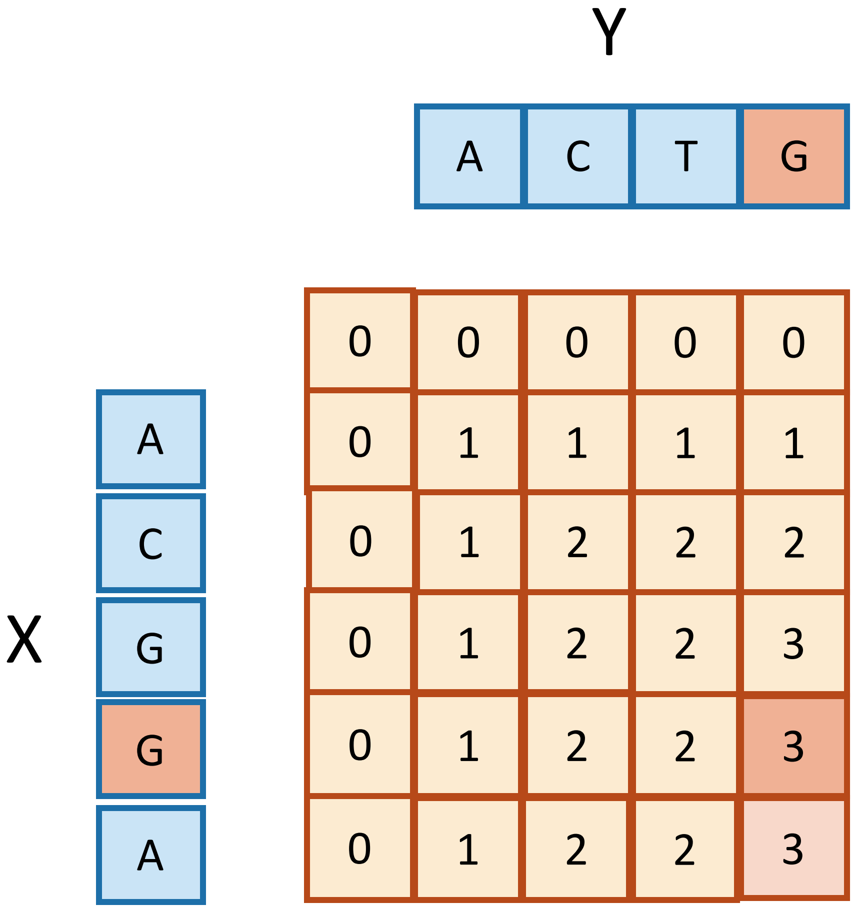
- Once we've filled this in, we can work backwards.

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X: A C G G A

Y: A C T G



		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

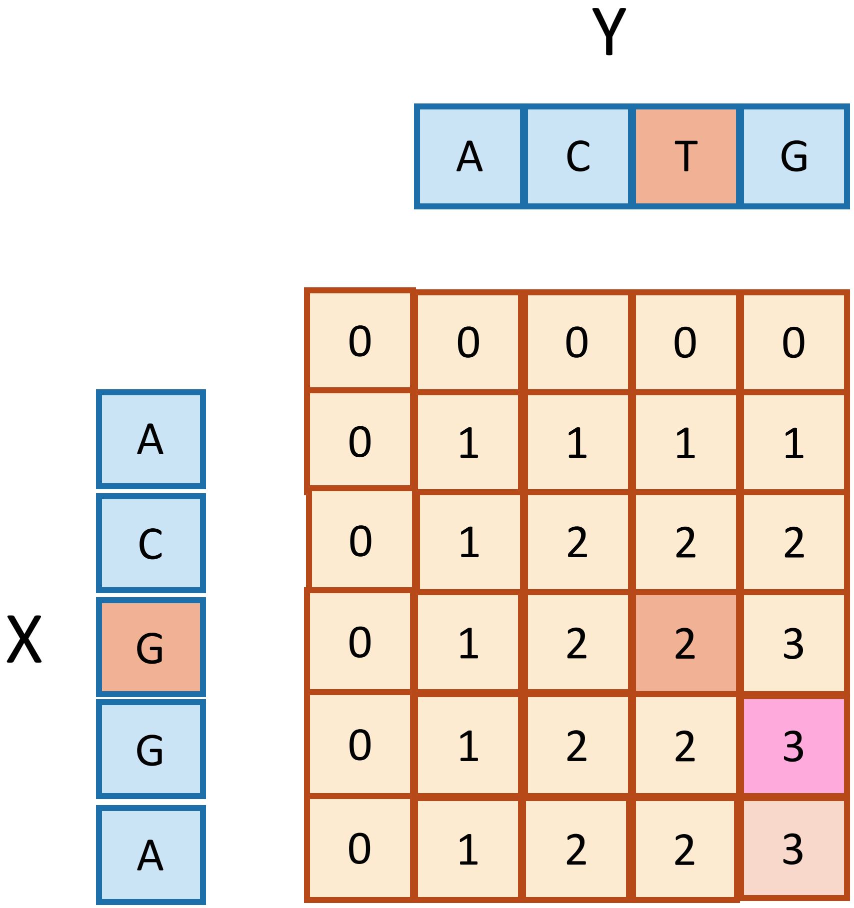
- Once we've filled this in, we can work backwards.

That 3 must have come from the 3 above it.

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X	A	C	G	G	A
Y	A	C	T	G	



	Y	A	C	T	G
X	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

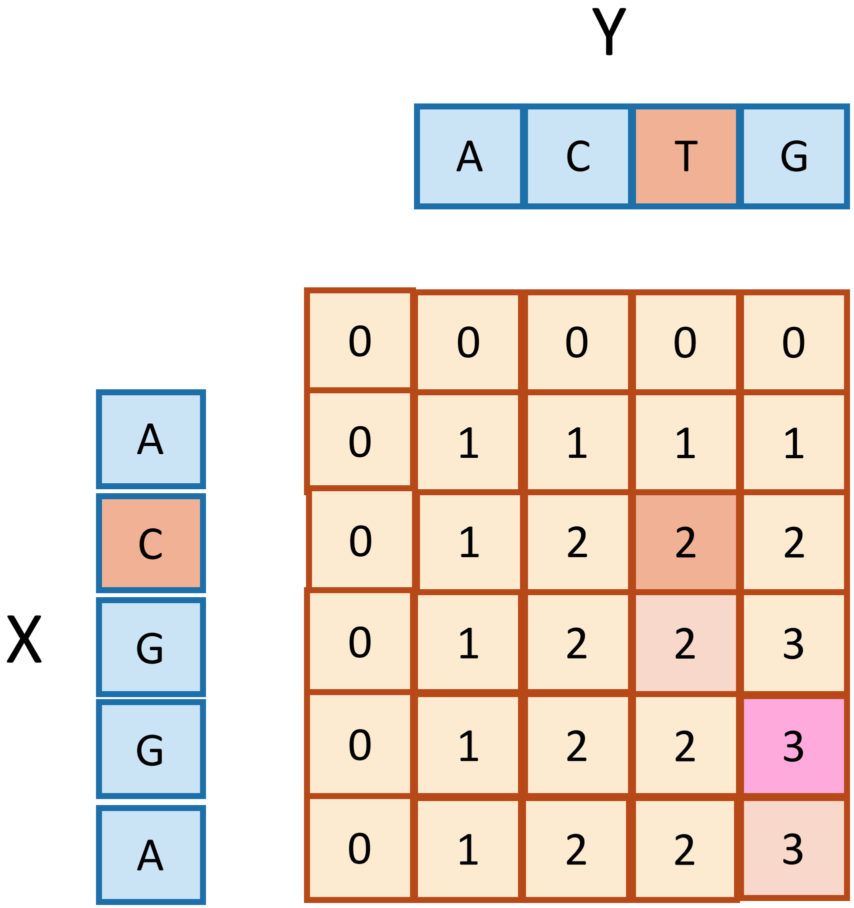
This 3 came from that 2 – we found a match!

G

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 \end{cases}$$

Example

X	A	C	G	G	A
Y	A	C	T	G	



Y

		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

X

- Once we’ve filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

That 2 may as well have come from this other 2.

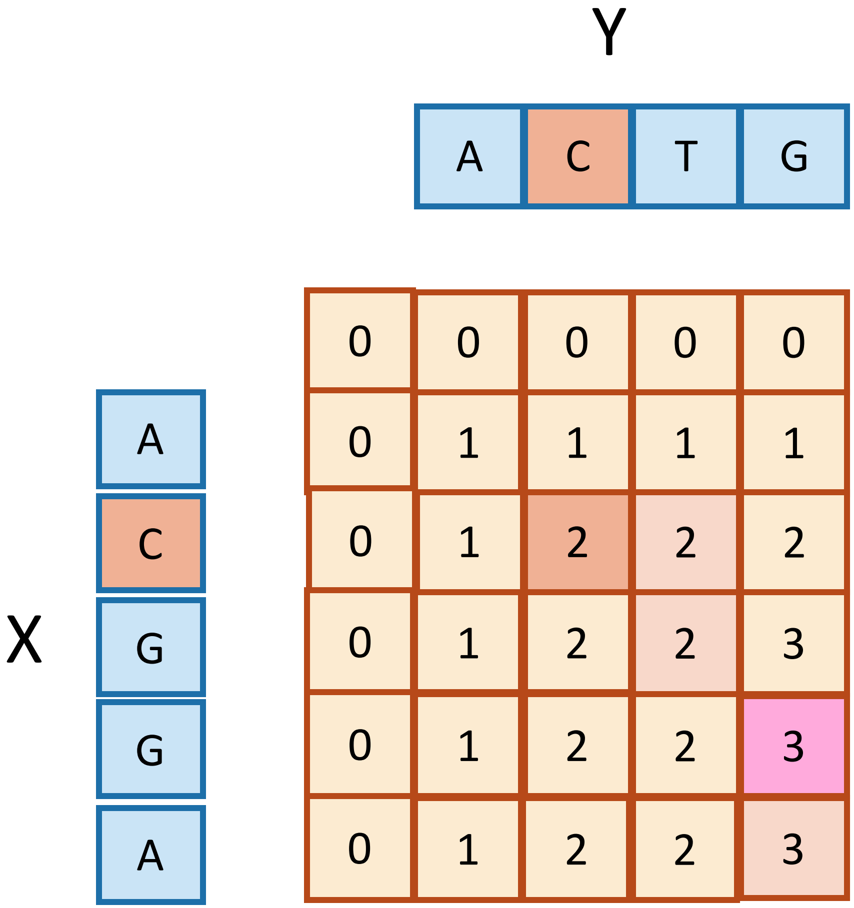
G

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1,j-1]+1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{\, C[i,j-1], C[i-1,j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X: A C G G A

Y: A C T G



Y

X

		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

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G

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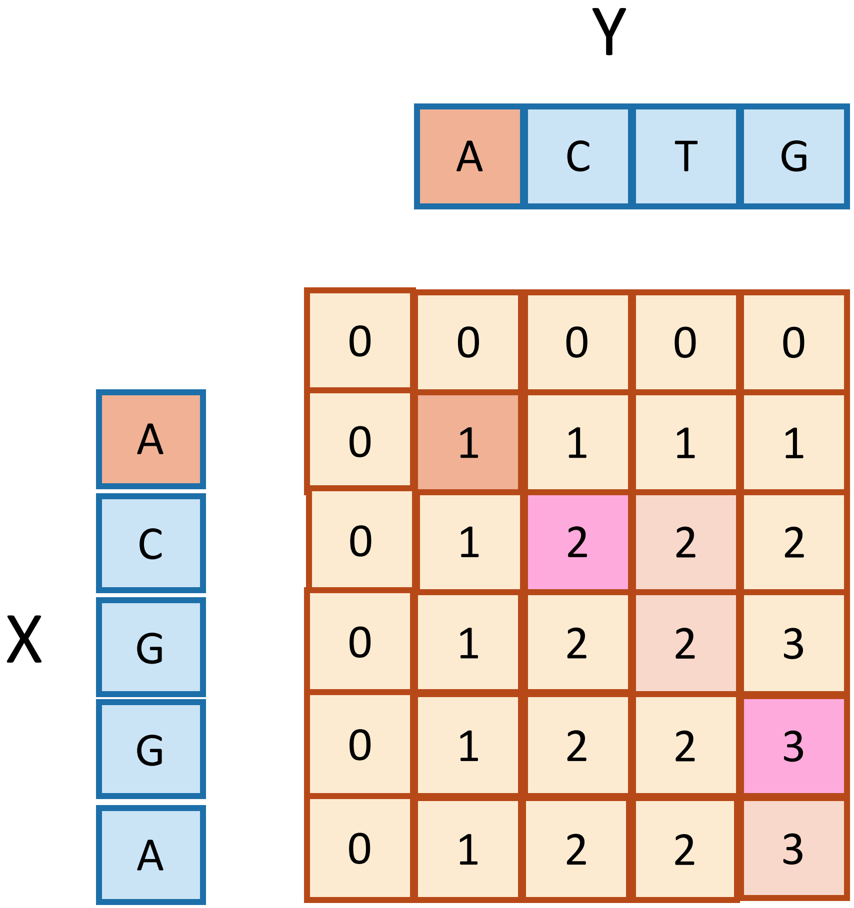
Example

X A C G G A

Y A C T G



			A	C	T	G
		0	0	0	0	0
X	A	0	1	1	1	1
	C	0	1	2	2	2
	G	0	1	2	2	3
	G	0	1	2	2	3
	A	0	1	2	2	3

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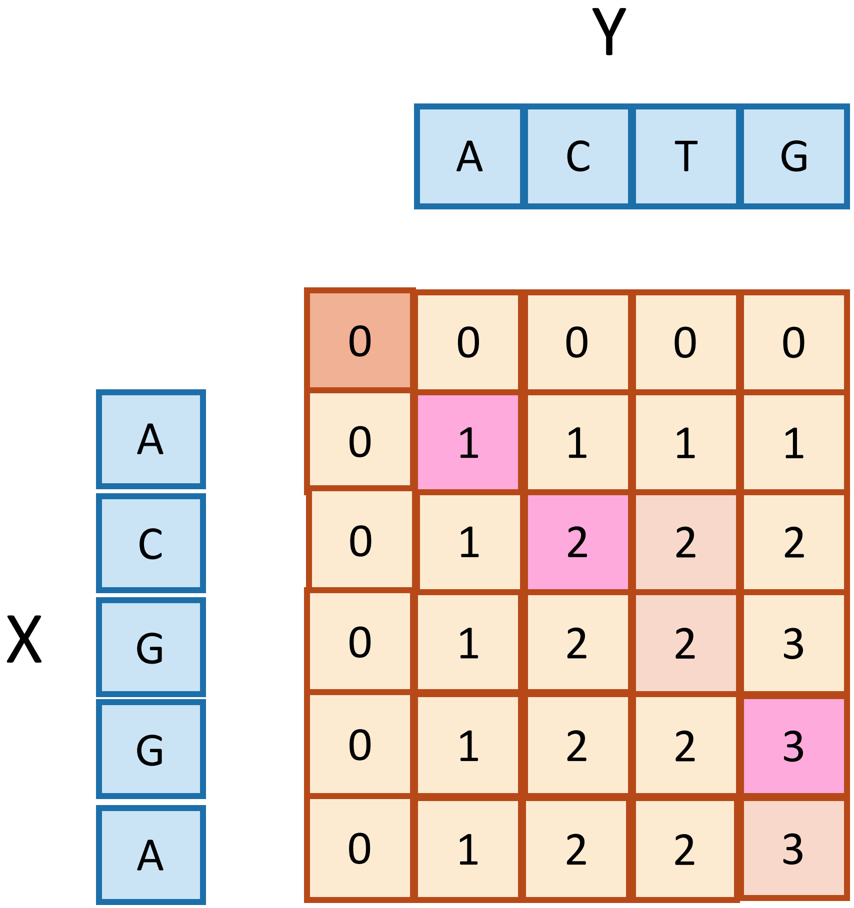
C G

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{ C[i, j-1], C[i-1, j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Example

X: A C G G A

Y: A C T G



		A	C	T	G
	0	0	0	0	0
A	0	1	1	1	1
C	0	1	2	2	2
G	0	1	2	2	3
G	0	1	2	2	3
A	0	1	2	2	3

- Once we've filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

A C G

This is the LCS!

$$C[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\ \max\{\, C[i, j-1], C[i-1, j] \} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 \end{cases}$$

Finding an LCS

- Good exercise to write out pseudocode for what we just saw!
  - Or you can find it in lecture notes.
- Takes time O(mn) to fill the table
- Takes time O(n + m) on top of that to recover the LCS
  - We walk up and left in an n-by-m array
  - We can only do that for n + m steps.
- Altogether, we can find LCS(X,Y) in time O(mn).

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the length of the longest common subsequence.
- **Step 3:** Use dynamic programming to find the length of the longest common subsequence.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual LCS.
- **Step 5:** If needed, code this up like a reasonable person.

Our approach actually isn't so bad

- If we are only interested in the length of the LCS we can do a bit better on space:
  - Since we go across the table one-row-at-a-time, we can only keep two rows if we want.
- If we want to recover the LCS, we need to keep the whole table.

- Can we do better than O(mn) time?
  - A bit better.
    - By a log factor or so.
  - But doing much better (polynomially better) is an open problem!

What have we learned?

- We can find LCS(X,Y) in time O(nm)
    - if |Y|=n, |X|=m
- We went through the steps of coming up with a dynamic programming algorithm.
    - We kept a 2-dimensional table, breaking down the problem by decrementing the length of X and Y.

Example 2: Knapsack Problem

- We have n items with weights and values:

Item:	🐢	💡	🍉	🌮	🚒
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- And we have a knapsack:
  - it can only carry so much weight:

Capacity: 10

Capacity: 10

Item:	🐢	💡	🍉	🌮	🚒
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- Unbounded Knapsack:
  - Suppose I have infinite copies of all items.
  - What’s the most valuable way to fill the knapsack?

🌮 🌮 💡 💡

Total weight: 10
Total value: 42

- 0/1 Knapsack:
  - Suppose I have only one copy of each item.
  - What’s the most valuable way to fill the knapsack?

💡 🍉 🌮

Total weight: 9
Total value: 35

Some notation

Item:					
Weight:	w_1	w_2	w_3	…	w_n
Value:	v_1	v_2	v_3		v_n

Capacity: W

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure. ⬅
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

Optimal substructure

- Sub-problems:
  - Unbounded Knapsack with a smaller knapsack.
  - K[x] = value you can fit in a knapsack of capacity x

First solve the
problem for
small knapsacks

Then larger
knapsacks

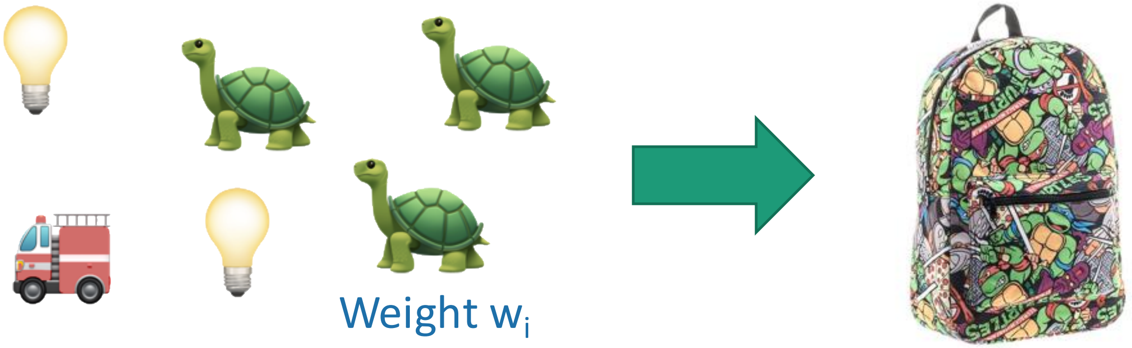
Then larger
knapsacks

Optimal substructure

item i

- Suppose this is an optimal solution for capacity x:

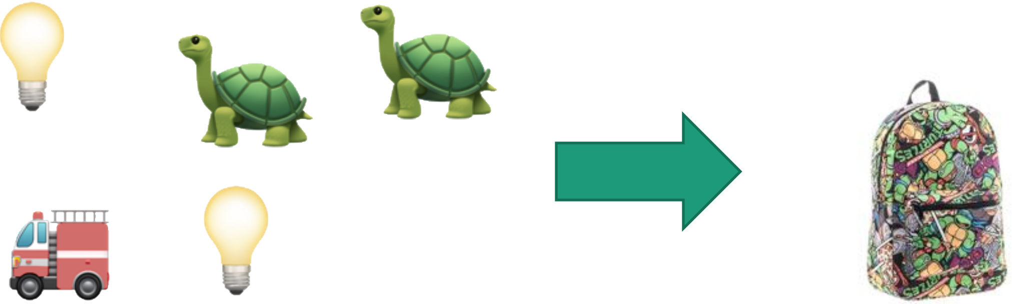
Say that the optimal solution contains at least one copy of item i.



Weight w_i
Value v_i

Capacity x
Value V

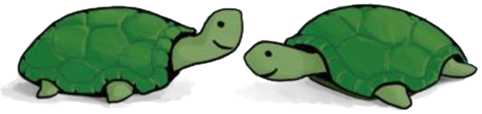
- Then this is optimal for capacity x - w_i:



Capacity x – w_i
Value V - v_i

Why?

1 minute think
(wait) 1 minute share

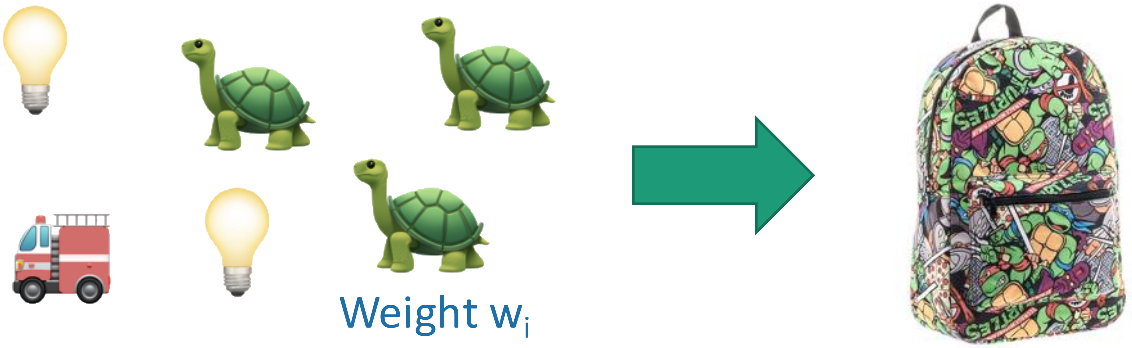


Optimal substructure


item i

- Suppose this is an optimal solution for capacity x:

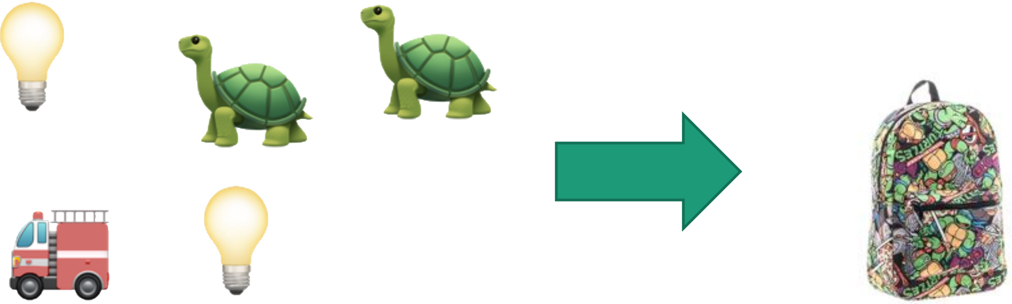
Say that the optimal solution contains at least one copy of item i.



Weight w_i
Value v_i

Capacity x
Value V

- Then this is optimal for capacity x - w_i:



If I could do better than the second solution, then adding a turtle to that improvement would improve the first solution.

Capacity $x - w_i$
Value $V - v_i$

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

Recursive relationship

- Let K[x] be the optimal value for capacity x.

$K[x] = \max_i \{$  $+$ 🐢 $\}$

The maximum is over all i so that $w_i \leq x$.

Optimal way to fill the smaller knapsack

The value of item i.

$$K[x] = \max_i \{ K[x - w_i] + v_i \}$$

- (And K[x] = 0 if the maximum is empty).
  - That is, if there are no i so that $w_i \leq x$

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

Let’s write a bottom-up DP algorithm

```
UnboundedKnapsack(W, n, weights, values):
    K[0] = 0
    for x = 1, …, W:
        K[x] = 0
        for i = 1, …, n:
            if $w_i \leq x$:
                $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
    return K[W]
```

Running time: O(nW)

$K[x] = \max_i \{$  $+$  $\}$

$= \max_i \{ K[x – w_i] + v_i \}$

Can we do better?

- Writing down W takes log(W) bits.
- Writing down all n weights takes at most nlog(W) bits.
- Input size: nlog(W).
  - Maybe we could have an algorithm that runs in time O(nlog(W)) instead of O(nW)?
  - Or even $O(\ n^{1000000} \log^{1000000}(W)\ )$?

- Open problem!
  - (But probably the answer is **no**…otherwise P = NP)

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

Let’s write a bottom-up DP algorithm

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        - for i = 1, …, n:
            - if $w_i \leq x$:
                - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
    - return K[W]
```

K[x] = $\max_i$ {  +  }

= $\max_i$ { $K[x - w_i] + v_i$ }

Let’s write a bottom-up DP algorithm

```
UnboundedKnapsack(W, n, weights, values):
    K[0] = 0
    ITEMS[0] = Ø
    for x = 1, …, W:
        K[x] = 0
        for i = 1, …, n:
            if $w_i \leq x$:
                $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
                If K[x] was updated:
                    ITEMS[x] = ITEMS[x – $w_i$] U { item i }
    return ITEMS[W]
```

$K[x] = \max_i \{$  $+$  $\}$

$= \max_i \{ K[x - w_i] + v_i \}$

Example

```
UnboundedKnapsack(W, n, weights, values):
    K[0] = 0
    ITEMS[0] = ∅
    for x = 1, …, W:
        K[x] = 0
        for i = 1, …, n:
            if $w_i \leq x$:
                $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
                If K[x] was updated:
                    ITEMS[x] = ITEMS[x – $w_i$] ∪ { item i }
    return ITEMS[W]
```

	0	1	2	3	4
K	0				
ITEMS					

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

	0	1	2	3	4
K	0	1			
ITEMS					

ITEMS[1] = ITEMS[0] + 

```
- UnboundedKnapsack(W, n, weights, values):
    - K[0] = 0
    - ITEMS[0] = Ø
    - for x = 1, …, W:
        - K[x] = 0
        - for i = 1, …, n:
            - if $w_i \leq x$:
                - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
                - If K[x] was updated:
                    - ITEMS[x] = ITEMS[x – $w_i$] ∪ { item i }
    - return ITEMS[W]
```

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

```
• UnboundedKnapsack(W, n, weights, values):
    • K[0] = 0
    • ITEMS[0] = ∅
    • for x = 1, …, W:
        • K[x] = 0
        • for i = 1, …, n:
            • if $w_i \leq x$:
                • $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
                • If K[x] was updated:
                    • ITEMS[x] = ITEMS[x – $w_i$] ∪ { item i }
    • return ITEMS[W]
```

	0	1	2	3	4
K	0	1	2		
ITEMS		🐢	🐢 🐢		

ITEMS[2] = ITEMS[1] + 🐢

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

```
UnboundedKnapsack(W, n, weights, values):
    K[0] = 0
    ITEMS[0] = Ø
    for x = 1, …, W:
        K[x] = 0
        for i = 1, …, n:
            if $w_i \leq x$:
                $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
                If K[x] was updated:
                    ITEMS[x] = ITEMS[x – w_i] ∪ { item i }
    return ITEMS[W]
```

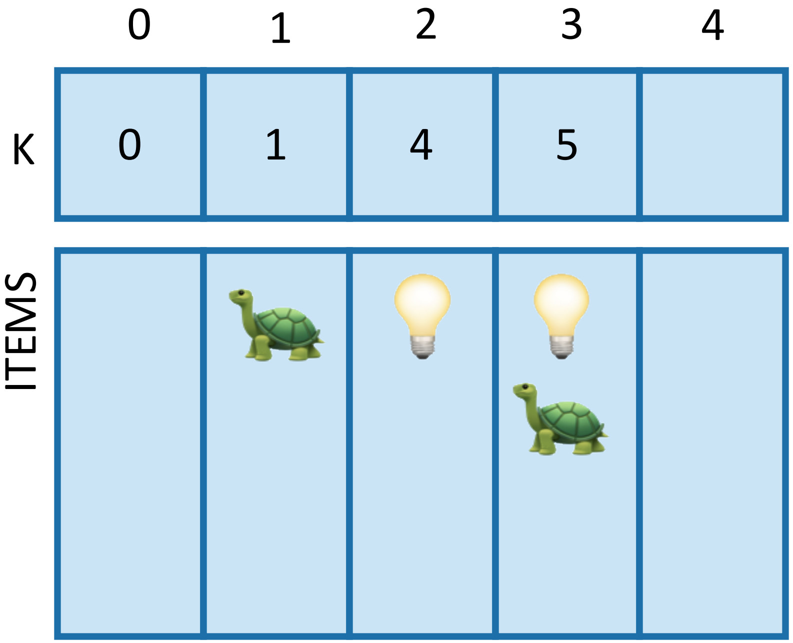
	0	1	2	3	4
K	0	1	4		
ITEMS		🐢	💡		

ITEMS[2] = ITEMS[0] + 💡

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example



	0	1	2	3	4
K	0	1	4	5	
ITEMS		🐢	💡	💡 🐢	

ITEMS[3] = ITEMS[2] + 🐢

```
• UnboundedKnapsack(W, n, weights, values):
    • K[0] = 0
    • ITEMS[0] = Ø
    • for x = 1, …, W:
        • K[x] = 0
        • for i = 1, …, n:
            • if $w_i \leq x$:
                • $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
                • If K[x] was updated:
                    • ITEMS[x] = ITEMS[x – $w_i$] ∪ { item i }
    • return ITEMS[W]
```

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

```
UnboundedKnapsack(W, n, weights, values):
    K[0] = 0
    ITEMS[0] = Ø
    for x = 1, …, W:
        K[x] = 0
        for i = 1, …, n:
            if $w_i \leq x$:
                $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
                If K[x] was updated:
                    ITEMS[x] = ITEMS[x – $w_i$] ∪ { item i }
    return ITEMS[W]
```

	0	1	2	3	4
K	0	1	4	6	
ITEMS		🐢	💡	🍉	

ITEMS[3] = ITEMS[0] + 🍉

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

```
- UnboundedKnapsack(W, n, weights, values):
    - K[0] = 0
    - ITEMS[0] = Ø
    - for x = 1, …, W:
        - K[x] = 0
        - for i = 1, …, n:
            - if $w_i \leq x$:
                - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
                - If K[x] was updated:
                    - ITEMS[x] = ITEMS[x – $w_i$] ∪ { item i }
    - return ITEMS[W]
```

	0	1	2	3	4
K	0	1	4	6	7
ITEMS		🐢	💡	🍉	🍉 🐢

ITEMS[4] = ITEMS[3] + 🐢

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Example

```
• UnboundedKnapsack(W, n, weights, values):
    • K[0] = 0
    • ITEMS[0] = ∅
    • for x = 1, …, W:
        • K[x] = 0
        • for i = 1, …, n:
            • if $w_i \leq x$:
                • $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
                • If K[x] was updated:
                    • ITEMS[x] = ITEMS[x – $w_i$] ∪ { item i }
    • return ITEMS[W]
```

	0	1	2	3	4
K	0	1	4	6	8
ITEMS		🐢	💡	🍉	💡 💡

ITEMS[4] = ITEMS[2] + 💡

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 4

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
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- **Step 5:** If needed, code this up like a reasonable person.

(Pass)

What have we learned?

- We can solve unbounded knapsack in time O(nW).
    - If there are n items and our knapsack has capacity W.

- We again went through the steps to create DP solution:
    - We kept a one-dimensional table, creating smaller problems by making the knapsack smaller.

Capacity: 10

Item:	🐢	💡	🍉	🌮	🚒
Weight:	6	2	4	3	11
Value:	20	8	14	13	35

- Unbounded Knapsack:
  - Suppose I have infinite copies of all of the items.
  - What’s the most valuable way to fill the knapsack?

🌮 🌮 💡 💡

Total weight: 10
Total value: 42

- 0/1 Knapsack:
  - Suppose I have only one copy of each item.
  - What’s the most valuable way to fill the knapsack?

💡 🍉 🌮

Total weight: 9
Total value: 35

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
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- **Step 5:** If needed, code this up like a reasonable person.

Optimal substructure: try 1

- Sub-problems:
  - Unbounded Knapsack with a smaller knapsack.

First solve the problem for small knapsacks

Then larger knapsacks

Then larger knapsacks

This won’t quite work…

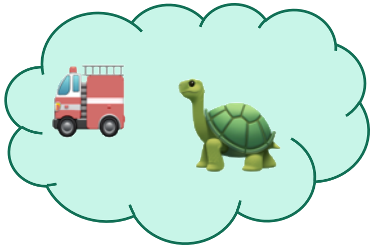
- We are only allowed **one copy of each item**.
- The sub-problem needs to “know” what items we’ve used and what we haven’t.

Optimal substructure: try 2

- Sub-problems:
  - 0/1 Knapsack with fewer items.

First solve the problem with few items

Then more items



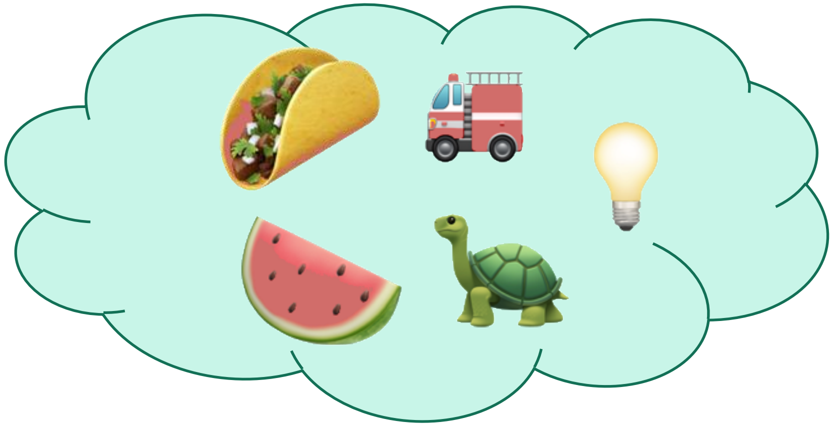
Then yet more items

We’ll still increase the size of the knapsacks.

(We’ll keep a two-dimensional table).

Our sub-problems:

- Indexed by x and j



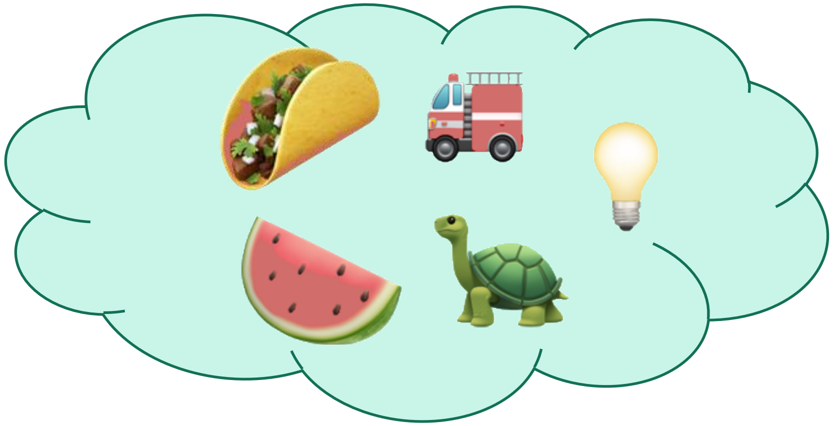
First j items

Capacity x

K[x,j] = optimal solution for a knapsack of size x using only the first j items.

Relationship between sub-problems

- Want to write K[x,j] in terms of smaller sub-problems.



First j items

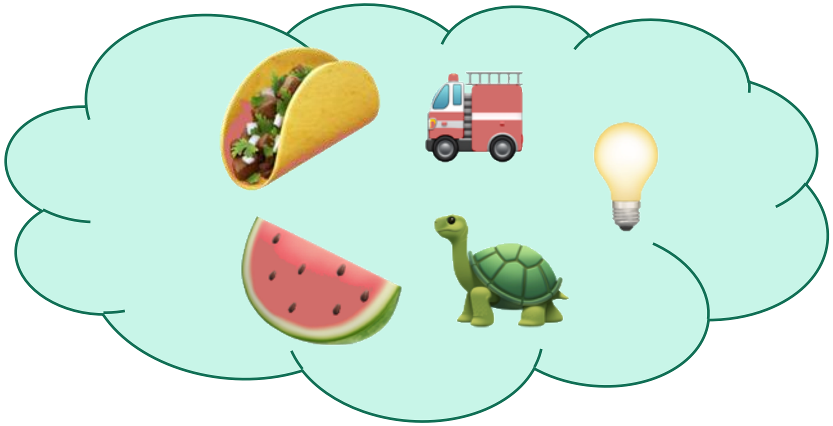
Capacity x

K[x,j] = optimal solution for a knapsack of size x using only the first j items.

Two cases


item j

- **Case 1**: Optimal solution for j items does not use item j.
- **Case 2**: Optimal solution for j items does use item j.


First j items

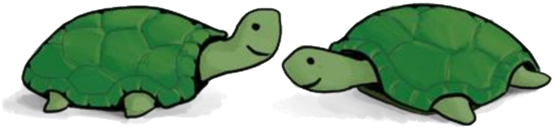

Capacity x

K[x,j] = optimal solution for a knapsack of size x using only the first j items.

Two cases

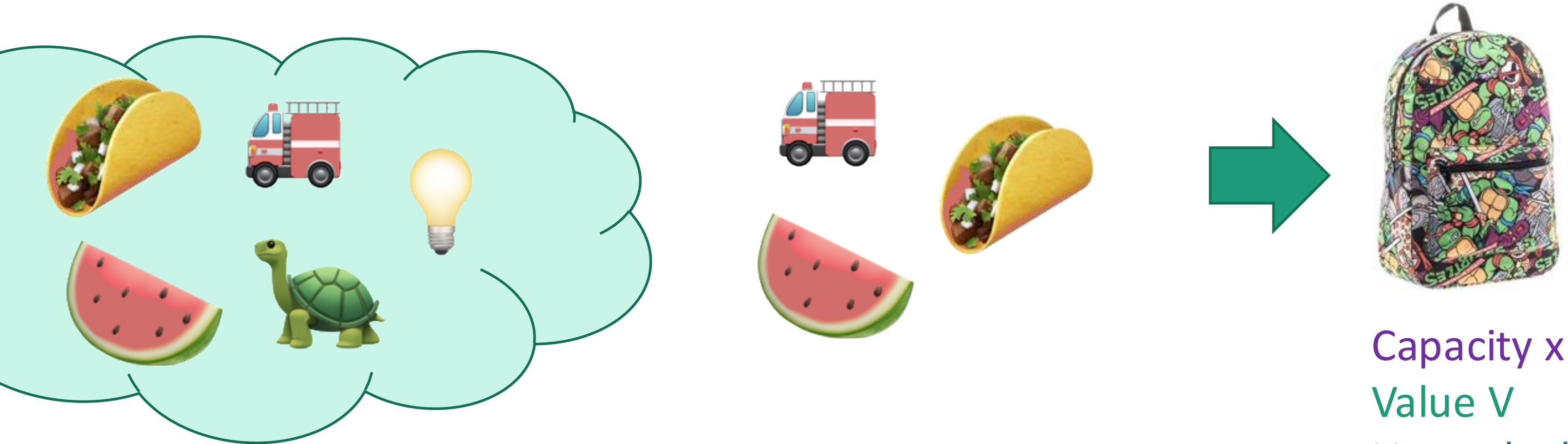
- **Case 1**: Optimal solution for j items does not use item j.

What lower-indexed problem should we solve to solve this problem?
1 min think; (wait) 1 min share

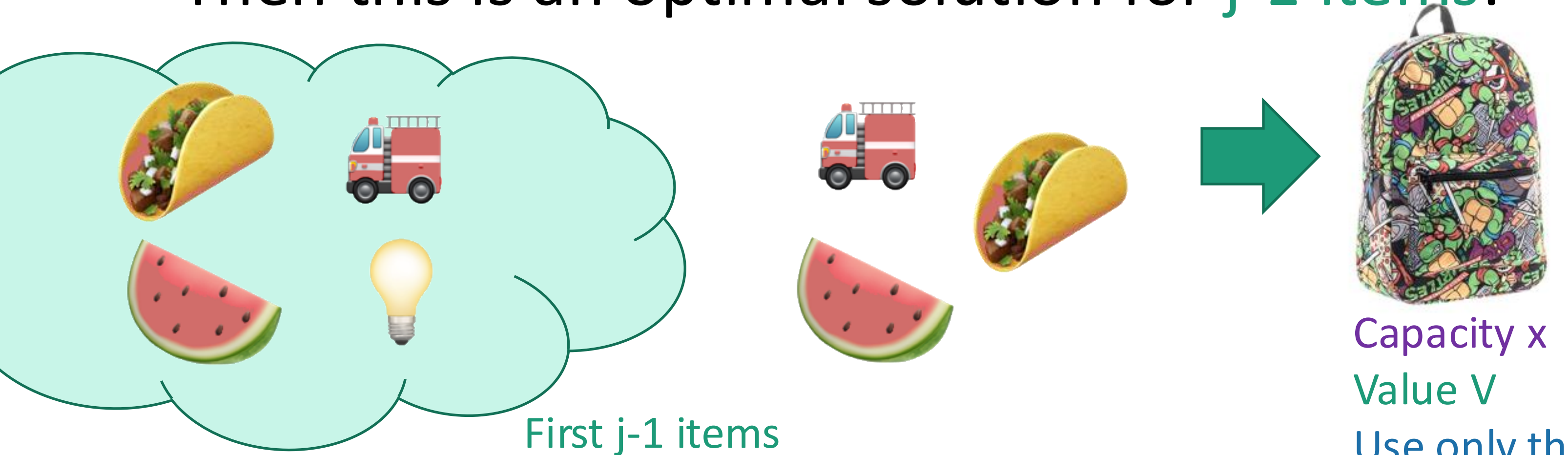


Two cases

- **Case 1**: Optimal solution for j items does not use item j.



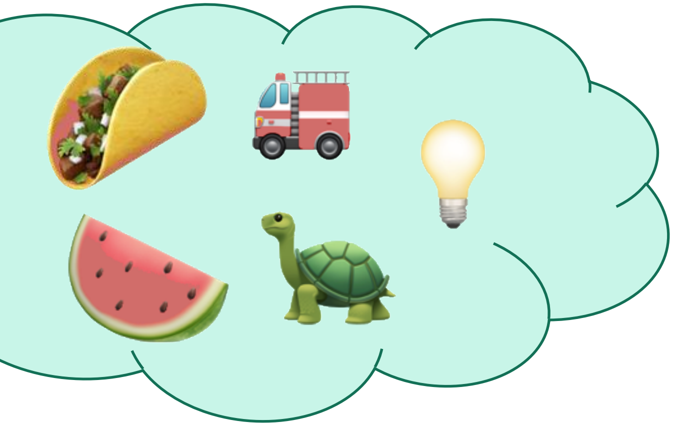
- Then this is an optimal solution for j-1 items:



Two cases

item j

- **Case 2**: Optimal solution for j items uses item j.



First j items

Weight w_j

Value v_j

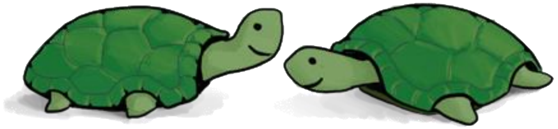
Capacity x

Value V

Use only the first j items

What lower-indexed problem should we solve to solve this problem?

1 min think; (wait) 1 min share



Two cases

item j

- **Case 2**: Optimal solution for j items uses item j.

Weight w_j
Value v_j

Capacity x
Value V
Use only the first j items

First j items

- Then this is an optimal solution for j-1 items and a smaller knapsack:

Capacity $x - w_j$
Value $V - v_j$
Use only the first j-1 items.

First j-1 items

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
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- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

Recursive relationship

- Let K[x,j] be the optimal value for:
  - capacity x,
  - with j items.

$$K[x,j] = \max\{ \underbrace{K[x, j-1]}_{\text{Case 1}} , \underbrace{K[x - w_{j,} j-1] + v_j}_{\text{Case 2}} \}$$

- (And K[x,0] = 0 and K[0,j] = 0).

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
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- **Step 5:** If needed, code this up like a reasonable person.

Bottom-up DP algorithm

```
Zero-One-Knapsack(W, n, w, v):
    K[x,0] = 0 for all x = 0,…,W
    K[0,i] = 0 for all i = 0,…,n
    for x = 1,…,W:
        for j = 1,…,n:
                                      Case 1
            K[x,j] = K[x, j-1]
            if $w_j \leq x$:
                                                          Case 2
                K[x,j] = max{ K[x,j], $K[x - w_j, j-1] + v_j$ }
    return K[W,n]
```

Running time O(nW)

Example

```
• Zero-One-Knapsack(W, n, w, v):
    • K[x,0] = 0 for all x = 0,…,W
    • K[0,i] = 0 for all i = 0,…,n
    • for x = 1,…,W:
        • for j = 1,…,n:
            • K[x,j] = K[x, j-1]
            • if $w_j \leq x$:
                • K[x,j] = max{ K[x,j],
                      $K[x - w_j, j-1] + v_j$ }
    • return K[W,n]
```

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
🐢 j=1	0			
💡 🐢 j=2	0			
🍉 💡 🐢 j=3	0			

🟥 current entry

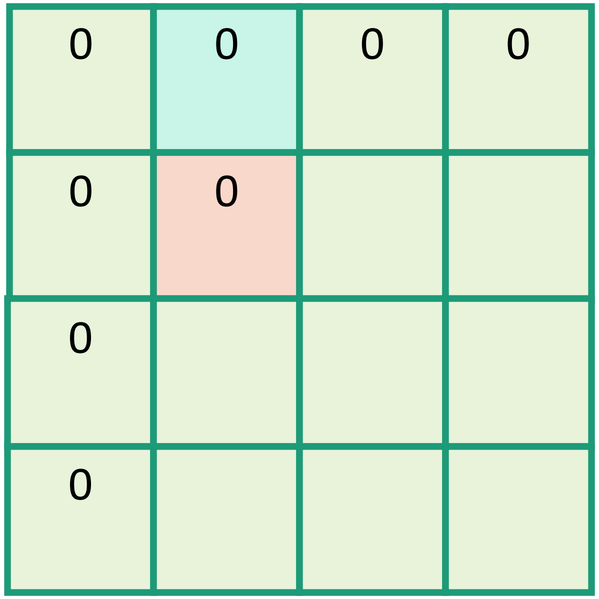
🟩 relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

```
• Zero-One-Knapsack(W, n, w, v):
    • K[x,0] = 0 for all x = 0,…,W
    • K[0,i] = 0 for all i = 0,…,n
    • for x = 1,…,W:
        • for j = 1,…,n:
            • K[x,j] = K[x, j-1]
            • if $w_j \leq x$:
                • K[x,j] = max{ K[x,j],
                      $K[x – w_j, j-1] + v_j$ }
    • return K[W,n]
```



	x=0	x=1	x=2	x=3
j=0	0	0	0	0
🐢 j=1	0	0		
💡 🐢 j=2	0			
🍉 💡 🐢 j=3	0			

🟧 current entry

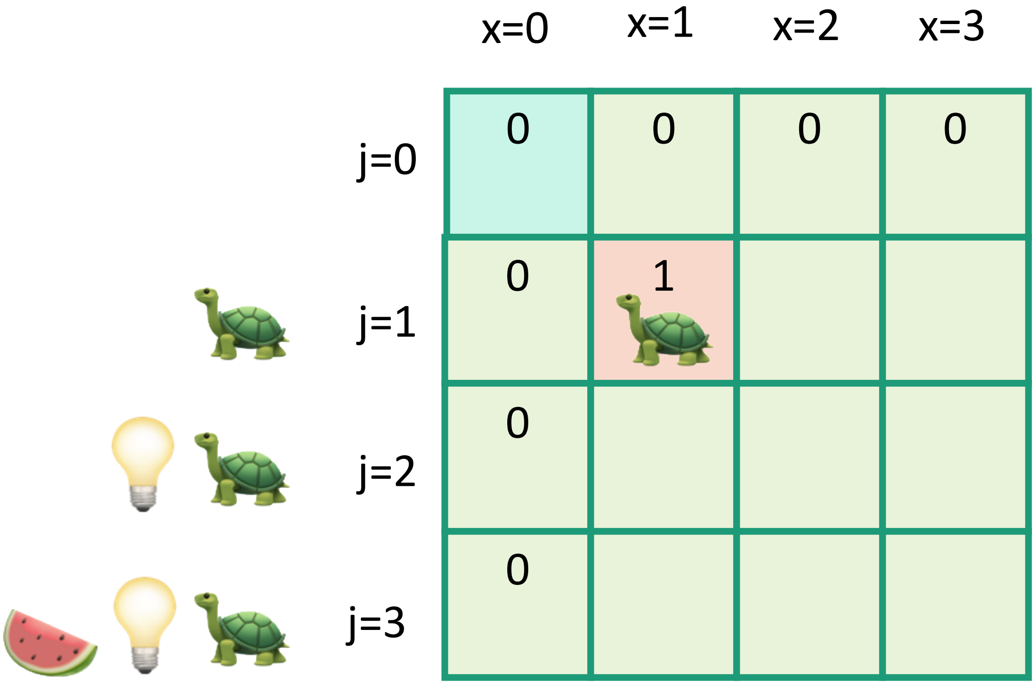
🟩 relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

```
• Zero-One-Knapsack(W, n, w, v):
    • K[x,0] = 0 for all x = 0,…,W
    • K[0,i] = 0 for all i = 0,…,n
    • for x = 1,…,W:
        • for j = 1,…,n:
            • K[x,j] = K[x, j-1]
            • if $w_j \leq x$:
                • K[x,j] = max{ K[x,j],
                      K[x – $w_j$, j-1] + $v_j$ }
    • return K[W,n]
```



	x=0	x=1	x=2	x=3
j=0	0	0	0	0
🐢 j=1	0	1 🐢		
💡 🐢 j=2	0			
🍉 💡 🐢 j=3	0			

🟥 current entry

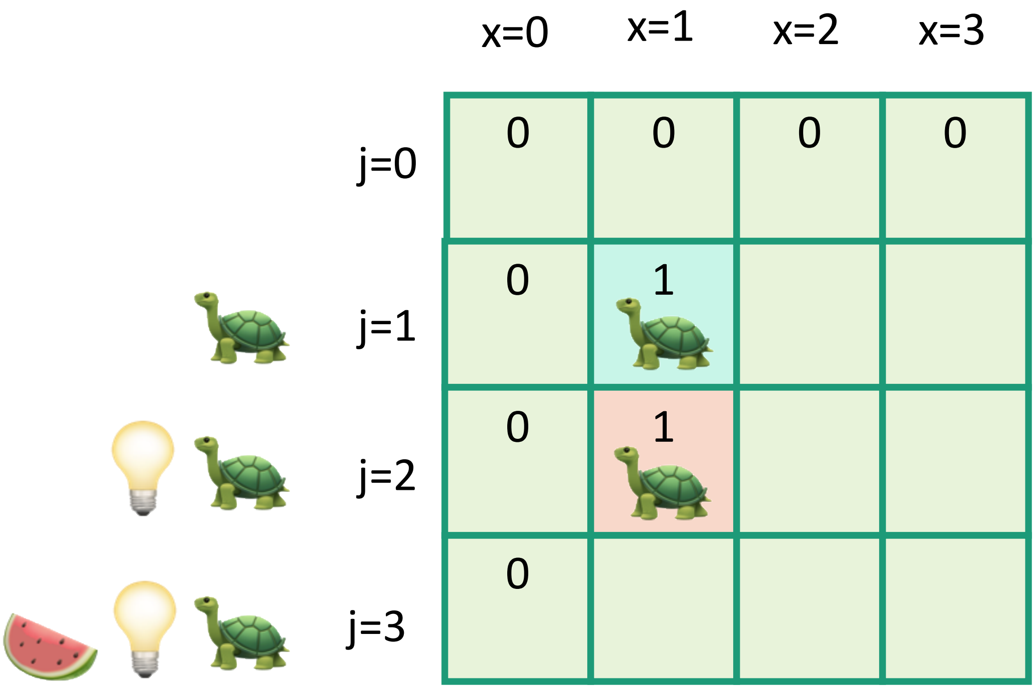
🟩 relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

```
• Zero-One-Knapsack(W, n, w, v):
    • K[x,0] = 0 for all x = 0,…,W
    • K[0,i] = 0 for all i = 0,…,n
    • for x = 1,…,W:
        • for j = 1,…,n:
            • K[x,j] = K[x, j-1]
            • if $w_j \leq x$:
                • K[x,j] = max{ K[x,j],
                      K[x – $w_j$, j-1] + $v_j$ }
    • return K[W,n]
```



		x=0	x=1	x=2	x=3
	j=0	0	0	0	0
🐢	j=1	0	1 🐢		
💡 🐢	j=2	0	1 🐢		
🍉 💡 🐢	j=3	0			

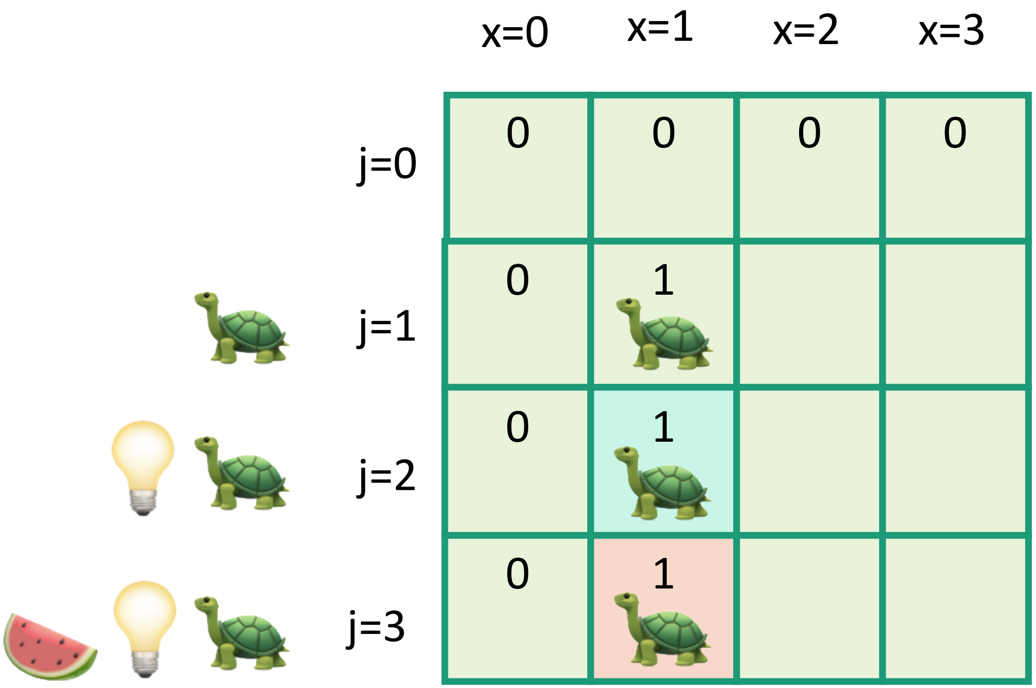
🟥	current entry
🟩	relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

```
• Zero-One-Knapsack(W, n, w, v):
    • K[x,0] = 0 for all x = 0,…,W
    • K[0,i] = 0 for all i = 0,…,n
    • for x = 1,…,W:
        • for j = 1,…,n:
            • K[x,j] = K[x, j-1]
            • if $w_j \leq x$:
                • K[x,j] = max{ K[x,j],
                      K[x – $w_j$, j-1] + $v_j$ }
    • return K[W,n]
```



	x=0	x=1	x=2	x=3
j=0	0	0	0	0
🐢 j=1	0	1 🐢		
💡 🐢 j=2	0	1 🐢		
🍉 💡 🐢 j=3	0	1 🐢		

🟥 current entry

🟩 relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
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Capacity: 3

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```
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    • for x = 1,…,W:
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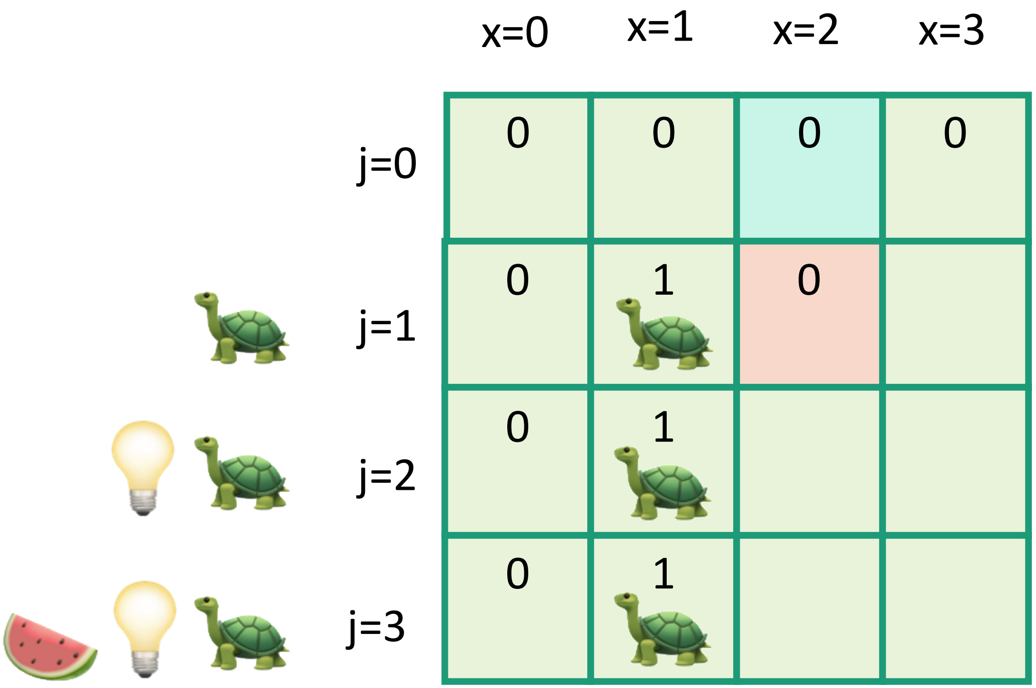
    • return K[W,n]
```



	x=0	x=1	x=2	x=3
j=0	0	0	0	0
🐢 j=1	0	1 🐢	0	
💡🐢 j=2	0	1 🐢		
🍉💡🐢 j=3	0	1 🐢		

🟧 current entry

🟩 relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

```
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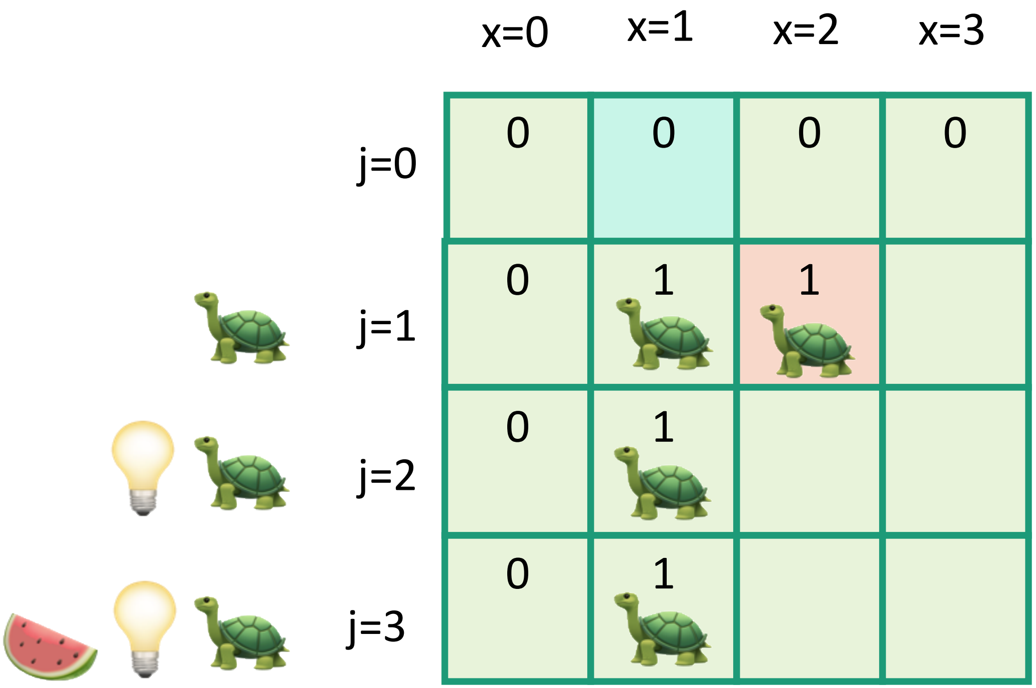
        - for j = 1,…,n:
            - K[x,j] = K[x, j-1]
            - if $w_j \leq x$:
                - K[x,j] = max{ K[x,j],
                      K[x – $w_j$, j-1] + $v_j$ }
    - return K[W,n]
```



		x=0	x=1	x=2	x=3
	j=0	0	0	0	0
🐢	j=1	0	1 🐢	1 🐢	
💡 🐢	j=2	0	1 🐢		
🍉 💡 🐢	j=3	0	1 🐢		

🟧 current entry

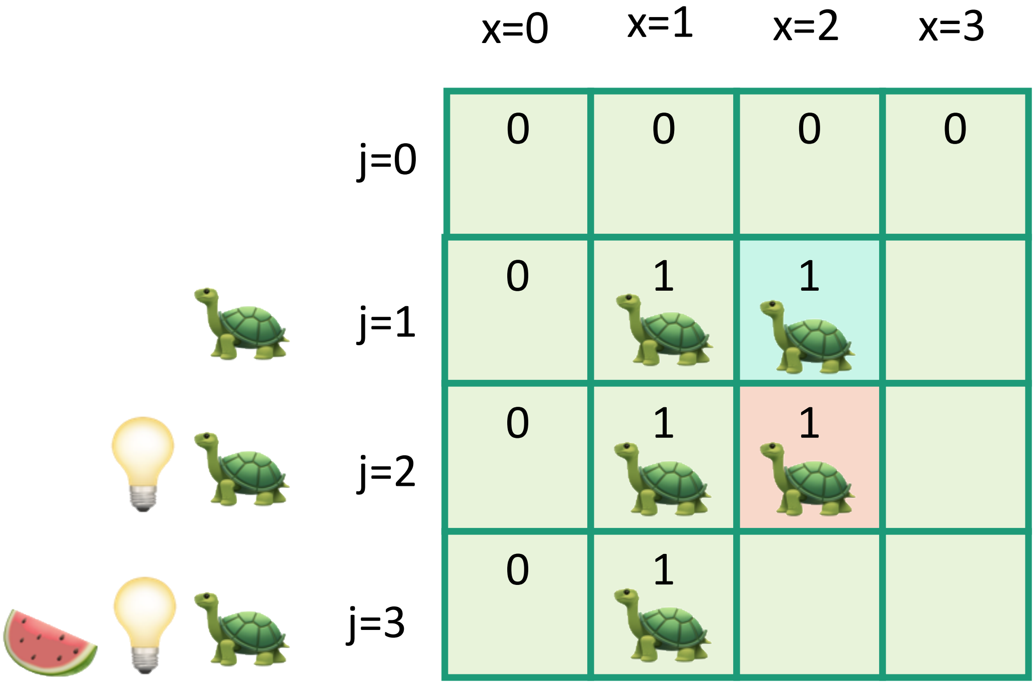
🟦 relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

```
• Zero-One-Knapsack(W, n, w, v):
    • K[x,0] = 0 for all x = 0,…,W
    • K[0,i] = 0 for all i = 0,…,n
    • for x = 1,…,W:
        • for j = 1,…,n:
            • K[x,j] = K[x, j-1]
            • if $w_j \leq x$:
                • K[x,j] = max{ K[x,j],
                      K[x – $w_j$, j-1] + $v_j$ }
    • return K[W,n]
```



		x=0	x=1	x=2	x=3
	j=0	0	0	0	0
🐢	j=1	0	1 🐢	1 🐢	
💡🐢	j=2	0	1 🐢	1 🐢	
🍉💡🐢	j=3	0	1 🐢		

🟥 current entry

🟩 relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
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Capacity: 3

Example

```
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    - K[x,0] = 0 for all x = 0,…,W
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            - if $w_j \leq x$:
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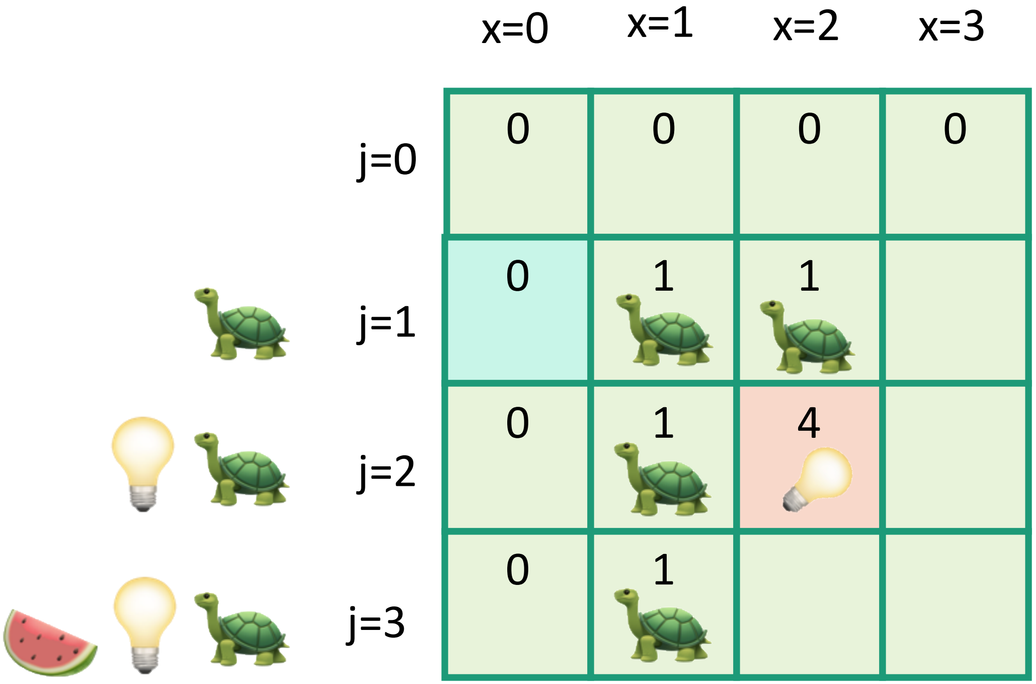

		x=0	x=1	x=2	x=3
	j=0	0	0	0	0
🐢	j=1	0	1 🐢	1 🐢	
💡 🐢	j=2	0	1 🐢	4 💡	
🍉 💡 🐢	j=3	0	1 🐢		

🟥 current entry

🟩 relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

- Zero-One-Knapsack(W, n, w, v):

```
• K[x,0] = 0 for all x = 0,…,W
• K[0,i] = 0 for all i = 0,…,n
• for x = 1,…,W:
    • for j = 1,…,n:
        • K[x,j] = K[x, j-1]
        • if $w_j \leq x$:
            • K[x,j] = max{ K[x,j],
                  K[x – $w_j$, j-1] + $v_j$ }
• return K[W,n]
```

Example

		x=0	x=1	x=2	x=3
	j=0	0	0	0	0
🐢	j=1	0	1 🐢	1 🐢	
💡 🐢	j=2	0	1 🐢	4 💡	
🍉 💡 🐢	j=3	0	1 🐢	4 💡	

🟥 current entry

🟩 relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

```
Zero-One-Knapsack(W, n, w, v):
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    K[0,i] = 0 for all i = 0,…,n
    for x = 1,…,W:
        for j = 1,…,n:
            K[x,j] = K[x, j-1]
            if $w_j \leq x$:
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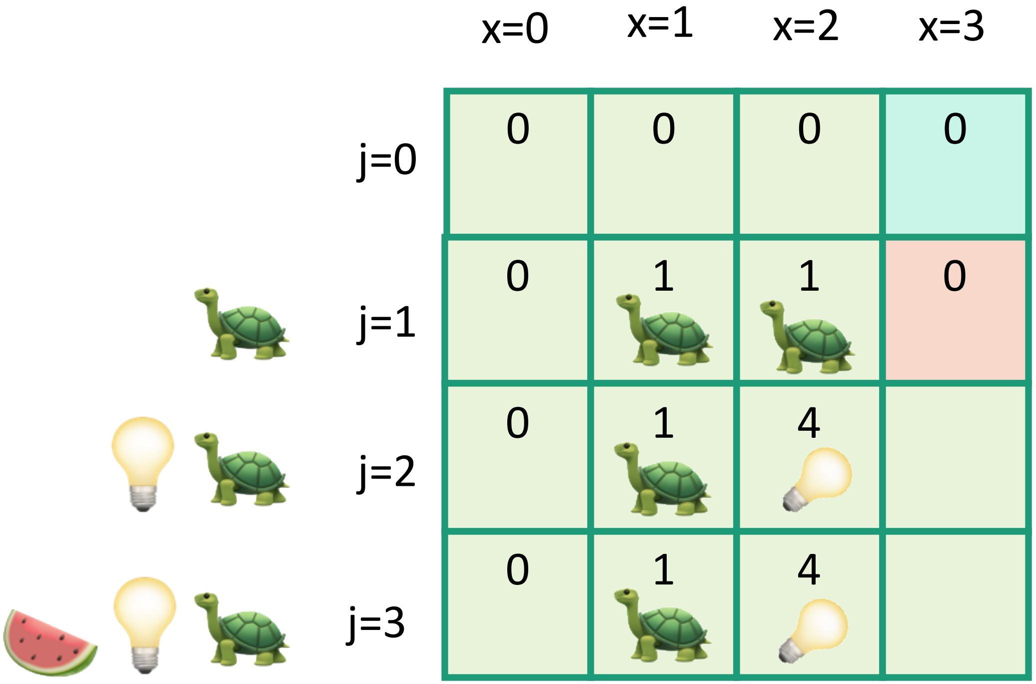
    return K[W,n]
```



		x=0	x=1	x=2	x=3
	j=0	0	0	0	0
🐢	j=1	0	1 🐢	1 🐢	0
💡 🐢	j=2	0	1 🐢	4 💡	
🍉 💡 🐢	j=3	0	1 🐢	4 💡	

current entry

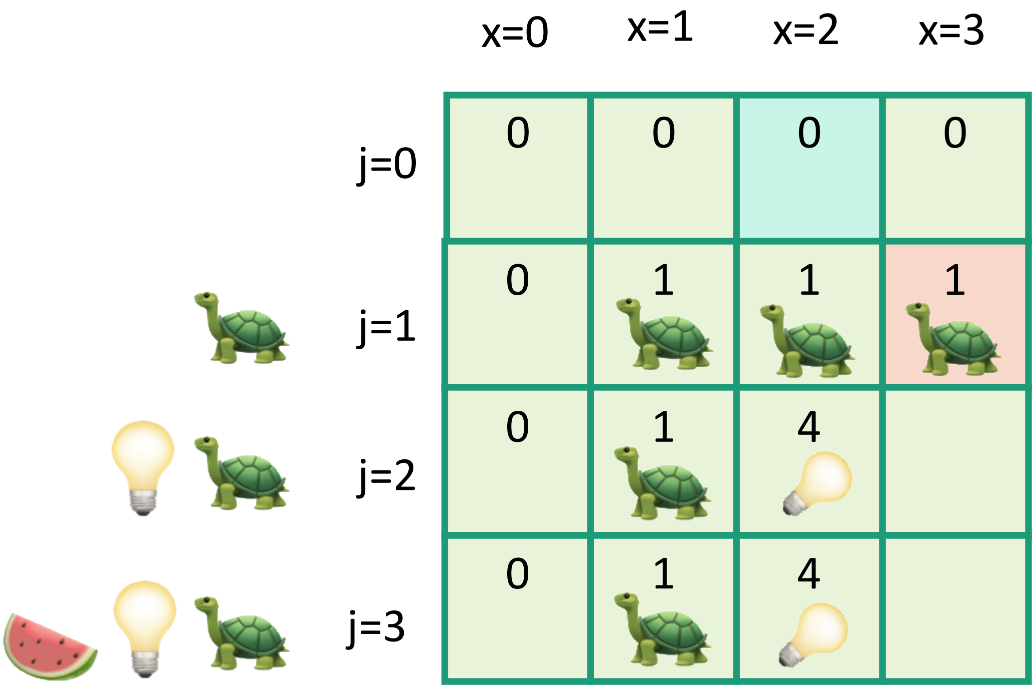
relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

```
Zero-One-Knapsack(W, n, w, v):
    K[x,0] = 0 for all x = 0,…,W
    K[0,i] = 0 for all i = 0,…,n
    for x = 1,…,W:
        for j = 1,…,n:
            K[x,j] = K[x, j-1]
            if $w_j \leq x$:
                K[x,j] = max{ K[x,j], K[x – $w_j$, j-1] + $v_j$ }
    return K[W,n]
```



		x=0	x=1	x=2	x=3
	j=0	0	0	0	0
🐢	j=1	0	1 🐢	1 🐢	1 🐢
💡 🐢	j=2	0	1 🐢	4 💡	
🍉 💡 🐢	j=3	0	1 🐢	4 💡	

🟥 current entry

🟩 relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

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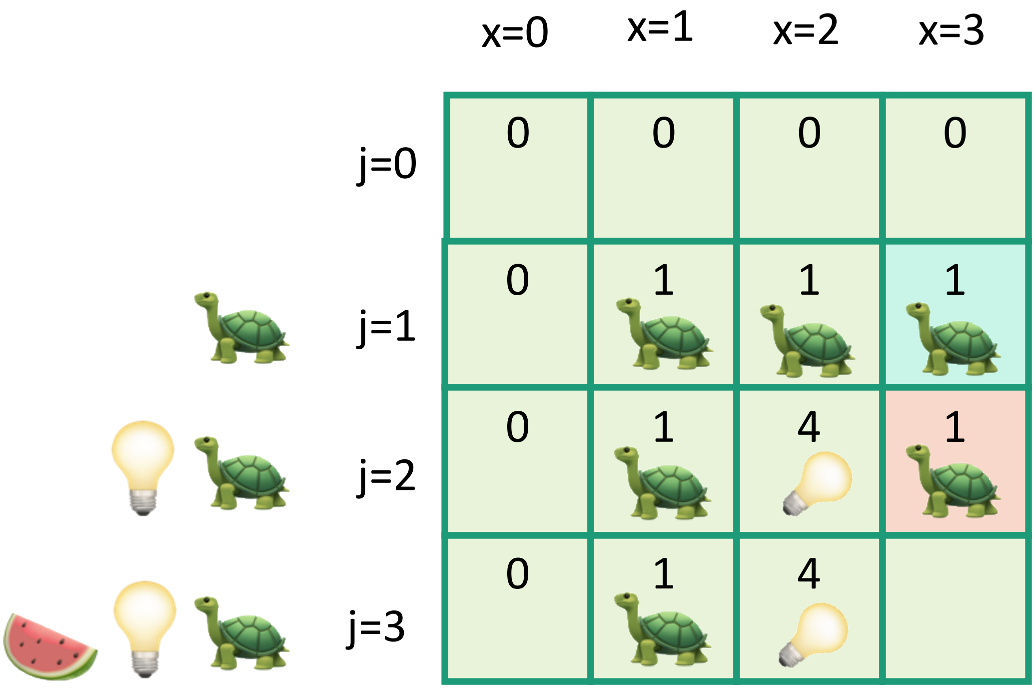
                      K[x – $w_j$, j-1] + $v_j$ }
    - return K[W,n]
```



		x=0	x=1	x=2	x=3
	j=0	0	0	0	0
🐢	j=1	0	1 🐢	1 🐢	1 🐢
💡 🐢	j=2	0	1 🐢	4 💡	1 🐢
🍉 💡 🐢	j=3	0	1 🐢	4 💡	

🟥 current entry

🟩 relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

```
Zero-One-Knapsack(W, n, w, v):
    K[x,0] = 0 for all x = 0,…,W
    K[0,i] = 0 for all i = 0,…,n
    for x = 1,…,W:
        for j = 1,…,n:
            K[x,j] = K[x, j-1]
            if $w_j \leq x$:
                K[x,j] = max{ K[x,j],
                    K[x – $w_j$, j-1] + $v_j$ }
    return K[W,n]
```

	x=0	x=1	x=2	x=3
j=0	0	0	0	0
🐢 j=1	0	1 🐢	1 🐢	1 🐢
💡 🐢 j=2	0	1 🐢	4 💡	5 🐢 💡
🍉 💡 🐢 j=3	0	1 🐢	4 💡	

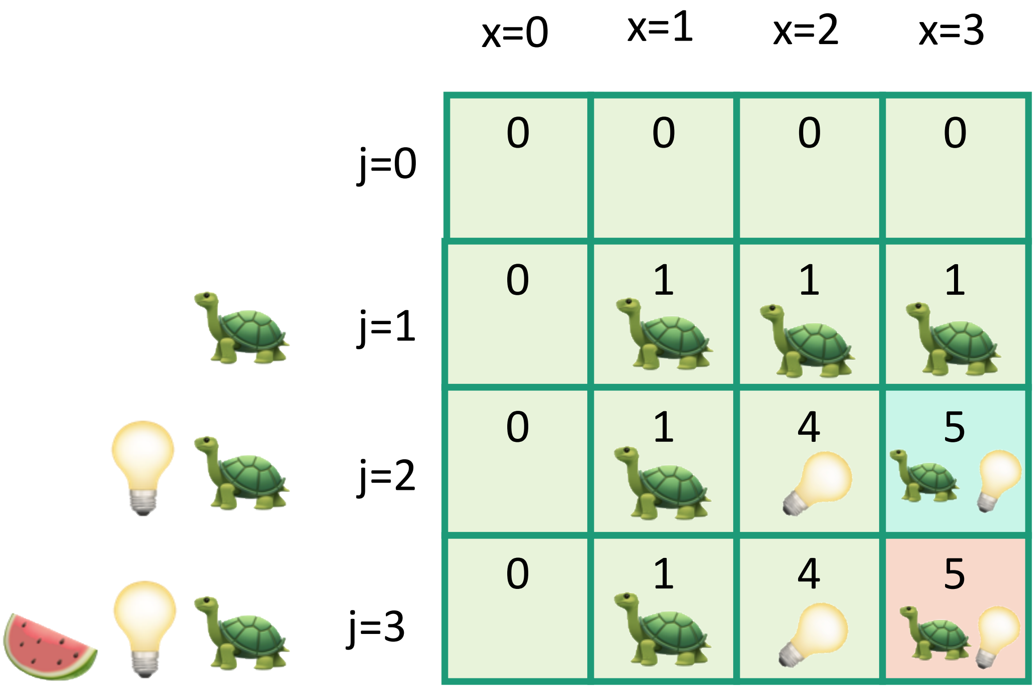
🟥 current entry

🟩 relevant previous entry

Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example



	x=0	x=1	x=2	x=3
j=0	0	0	0	0
🐢 j=1	0	1 🐢	1 🐢	1 🐢
💡 🐢 j=2	0	1 🐢	4 💡	5 🐢 💡
🍉 💡 🐢 j=3	0	1 🐢	4 💡	5 🐢 💡

```
• Zero-One-Knapsack(W, n, w, v):
    • $K[x,0] = 0$ for all $x = 0,…,W$
    • $K[0,i] = 0$ for all $i = 0,…,n$
    • for $x = 1,…,W$:
        • for $j = 1,…,n$:
            • $K[x,j] = K[x, j-1]$
            • if $w_j \leq x$:
                • $K[x,j] = \max\{ K[x,j],$
                      $K[x – w_j, j-1] + v_j \}$
    • return $K[W,n]$
```

🟥 current entry

🟩 relevant previous entry

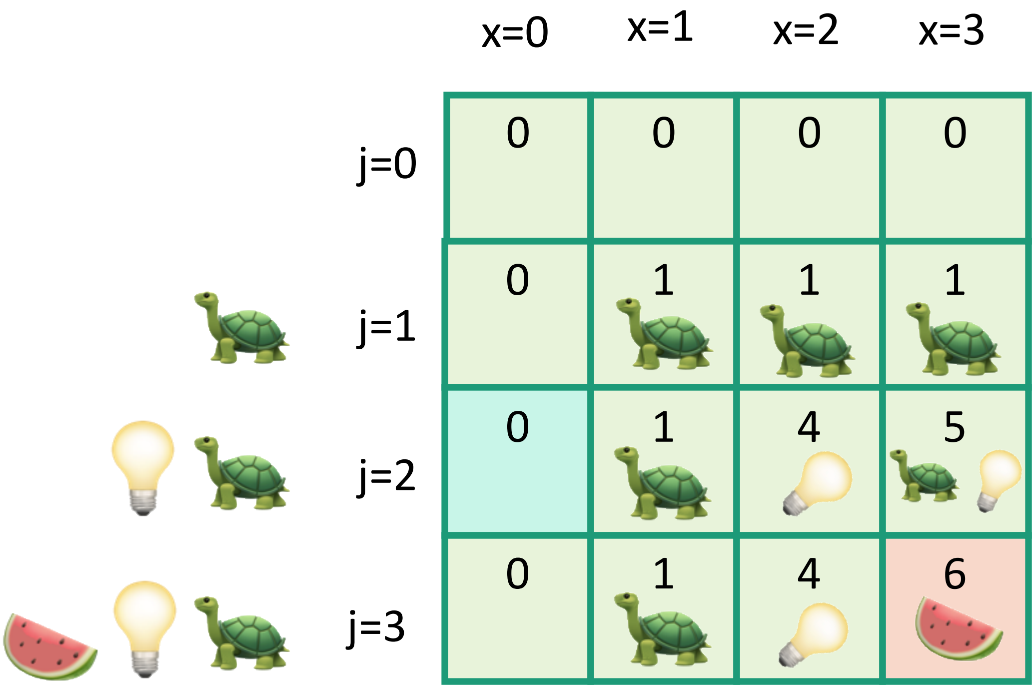
Item:	🐢	💡	🍉
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

- Zero-One-Knapsack(W, n, w, v):

```
K[x,0] = 0 for all x = 0,…,W
K[0,i] = 0 for all i = 0,…,n
for x = 1,…,W:
    for j = 1,…,n:
        K[x,j] = K[x, j-1]
        if $w_j \leq x$:
            K[x,j] = max{ K[x,j],
                          $K[x - w_j, j-1] + v_j$ }
return K[W,n]
```



	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	1
j=2	0	1	4	5
j=3	0	1	4	6

🟥 current entry

🟩 relevant previous entry

Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Example

```
Zero-One-Knapsack(W, n, w, v):
    K[x,0] = 0 for all x = 0,…,W
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    for x = 1,…,W:
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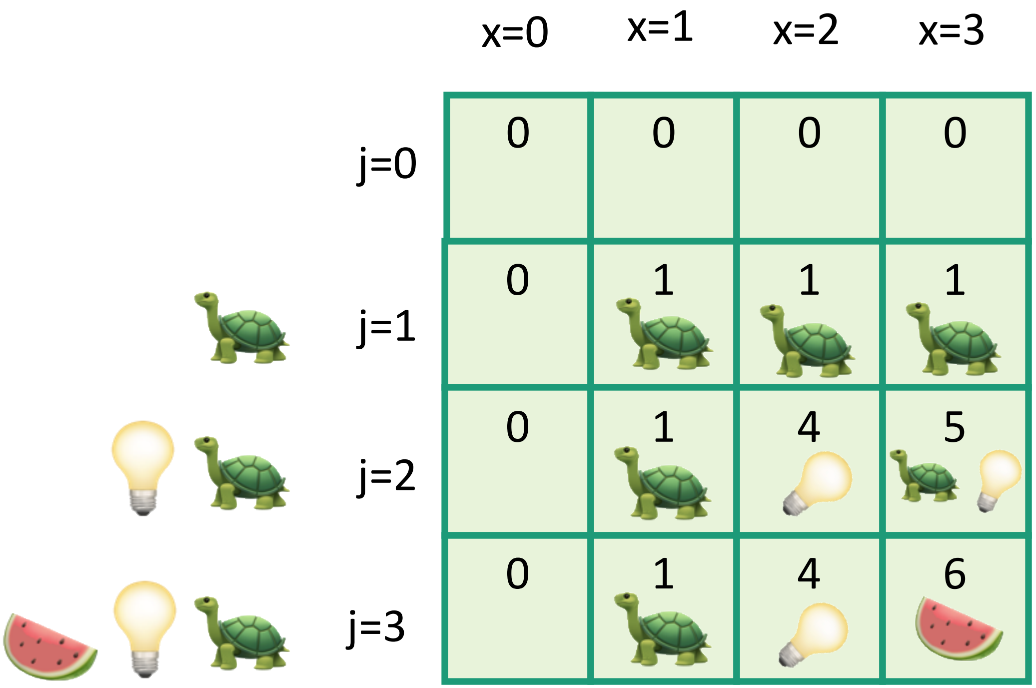
```



	x=0	x=1	x=2	x=3
j=0	0	0	0	0
j=1	0	1	1	1
j=2	0	1	4	5
j=3	0	1	4	6

So the optimal solution is to put one watermelon in your knapsack!

🟧 current entry

🟩 relevant previous entry

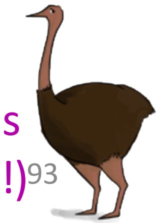
Item:			
Weight:	1	2	3
Value:	1	4	6

Capacity: 3

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

You do this one!
(We did it on the slide in the previous example, just not in the pseudocode!)



What have we learned?

- We can solve 0/1 knapsack in time O(nW).
  - If there are n items and our knapsack has capacity W.
- We again went through the steps to create DP solution:
  - We kept a two-dimensional table, creating smaller problems by restricting the set of allowable items.

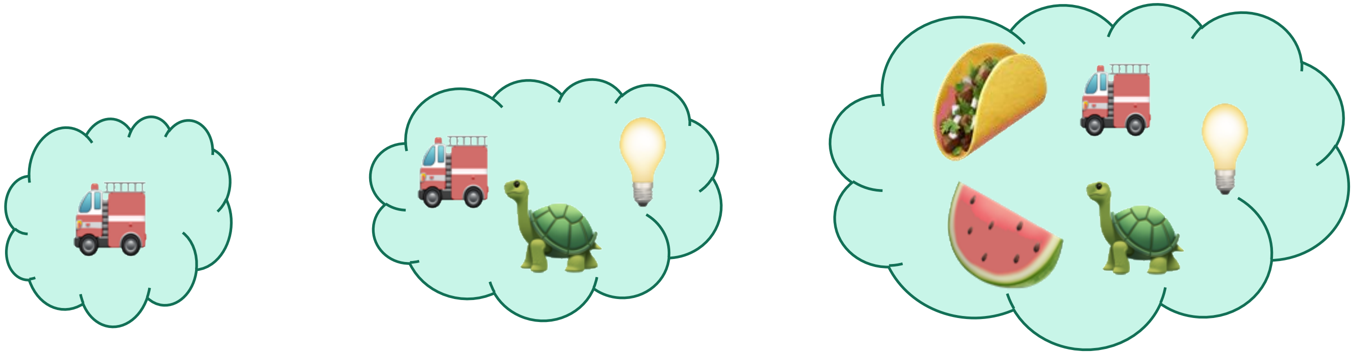
Question

- How did we know which substructure to use in which variant of knapsack?

Answer in retrospect:

This one made sense for unbounded knapsack because it doesn't have any memory of what items have been used.

vs.

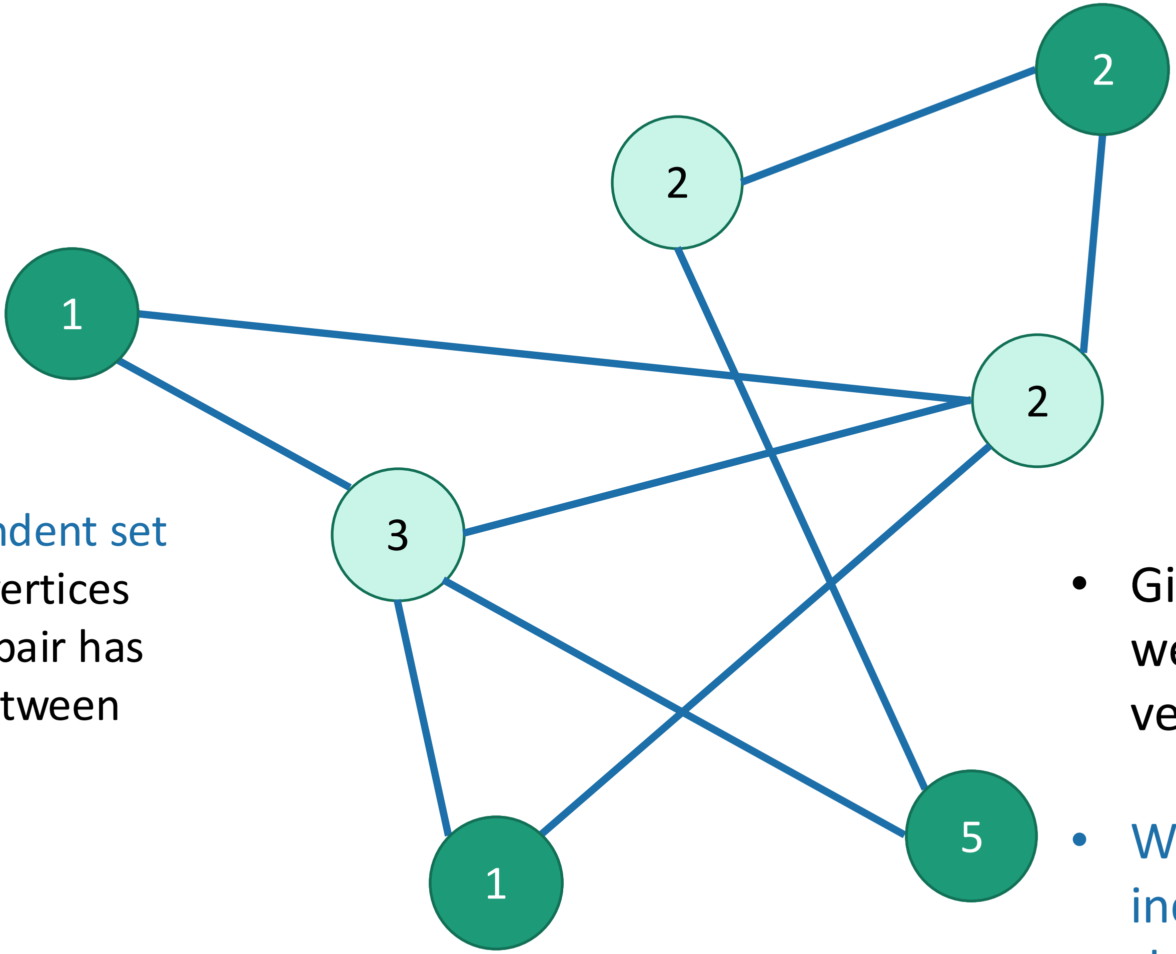


In 0/1 knapsack, we can only use each item once, so it makes sense to leave out one item at a time.

Operational Answer: try some stuff, see what works!

Example 3: Independent Set

if we still have time



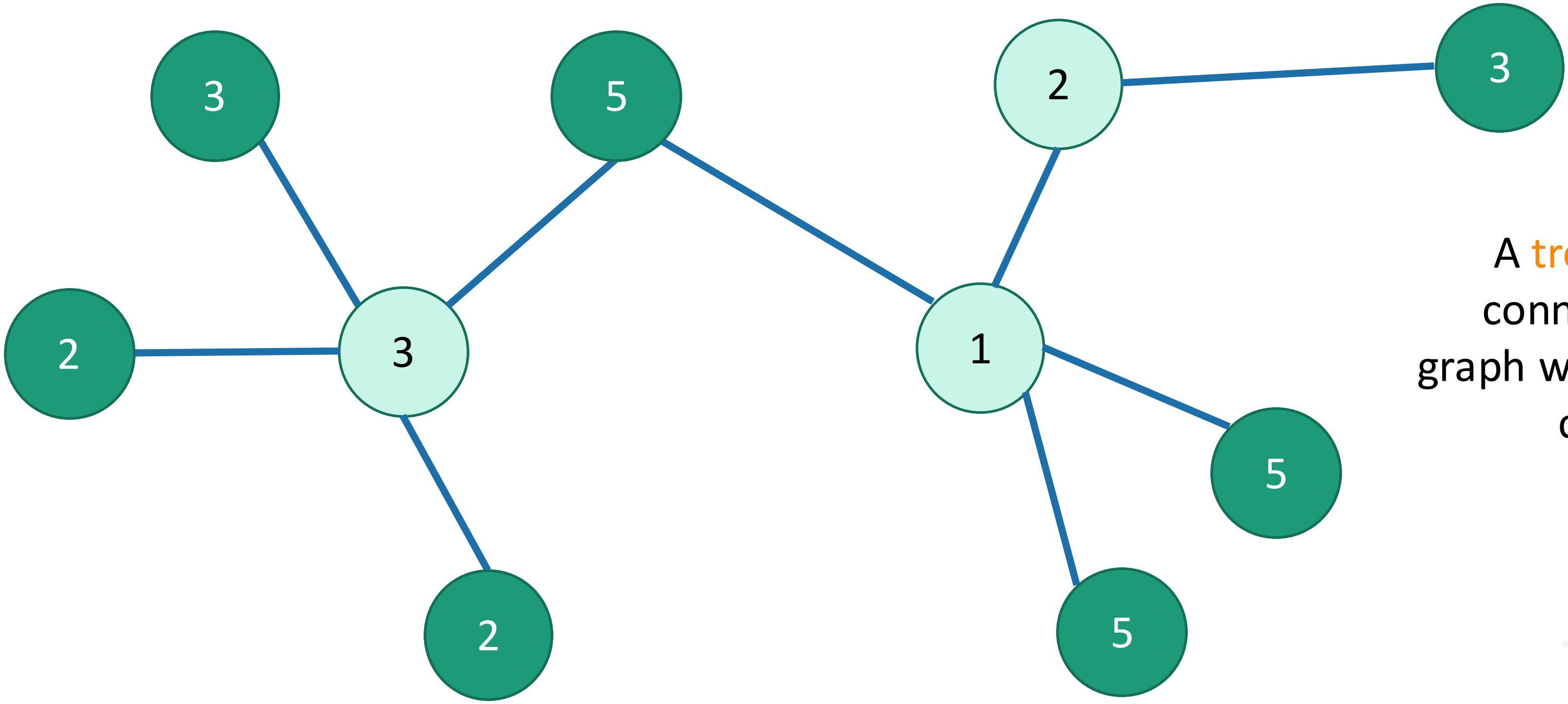
An independent set is a set of vertices so that no pair has an edge between them.

- Given a graph with weights on the vertices…
- What is the independent set with the largest weight?

Actually, this problem is NP-complete.

So, we are unlikely to find an efficient algorithm.

- But if we also assume that the graph is a tree…



A tree is a connected graph with no cycles.

Problem:

find a maximal independent set in a tree (with vertex weights).

Recipe for applying Dynamic Programming

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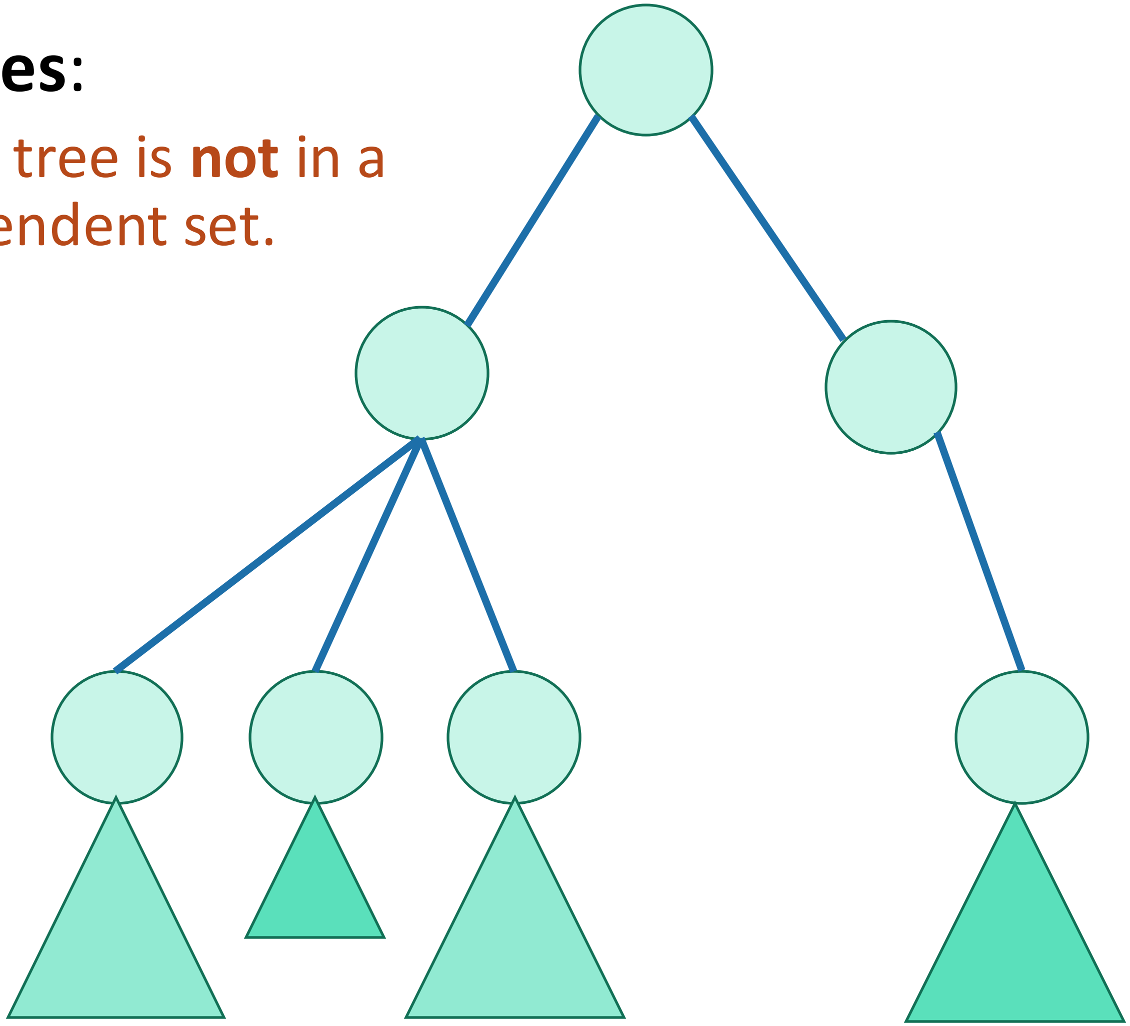
- **Step 5:** If needed, code this up like a reasonable person.

Optimal substructure

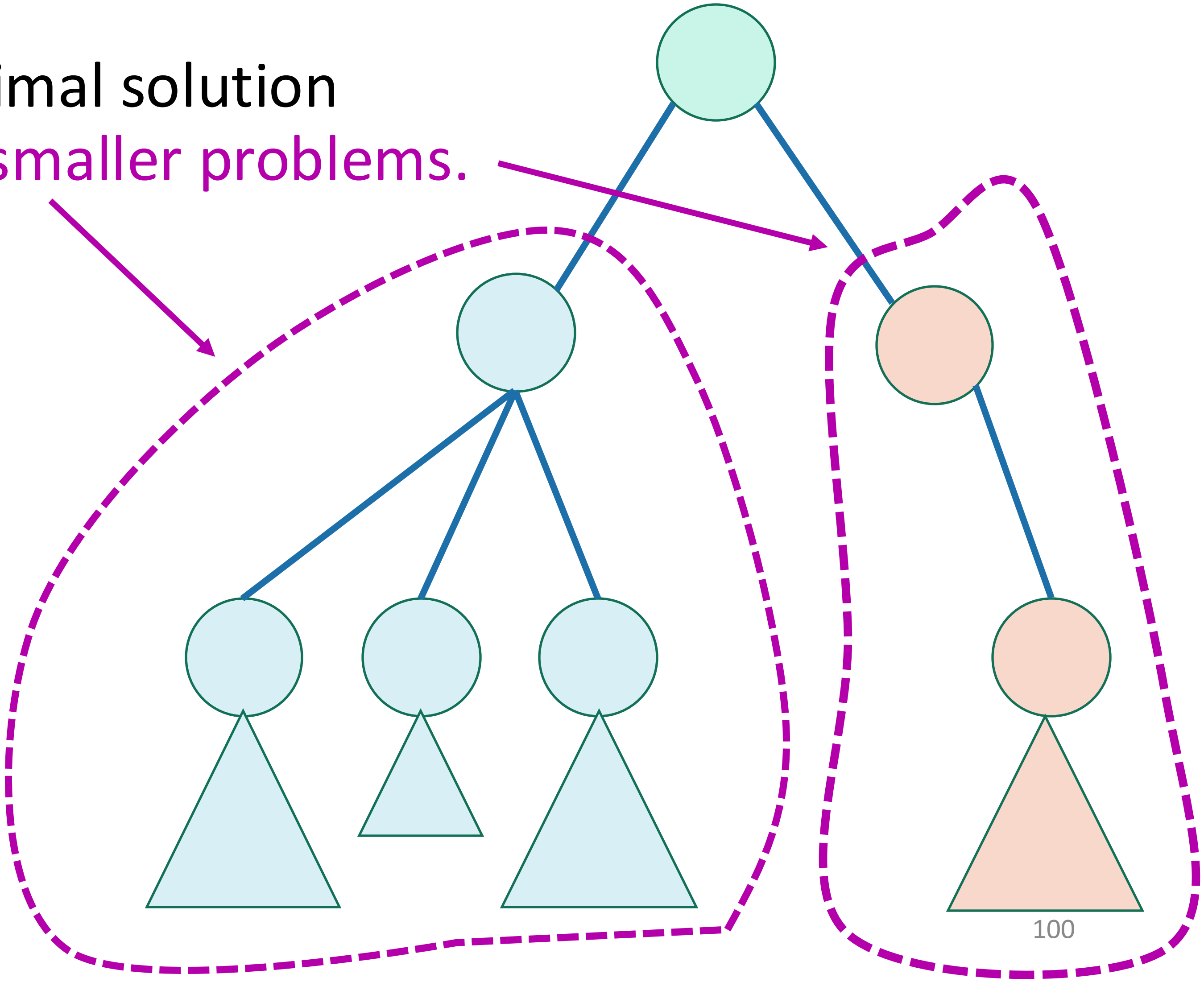
- **Subtrees** are a natural candidate.
- There are **two cases**:
    1. The root of this tree is **not** in a maximal independent set.
    2. Or it is.



Case 1:

the root is **not** in a maximal independent set

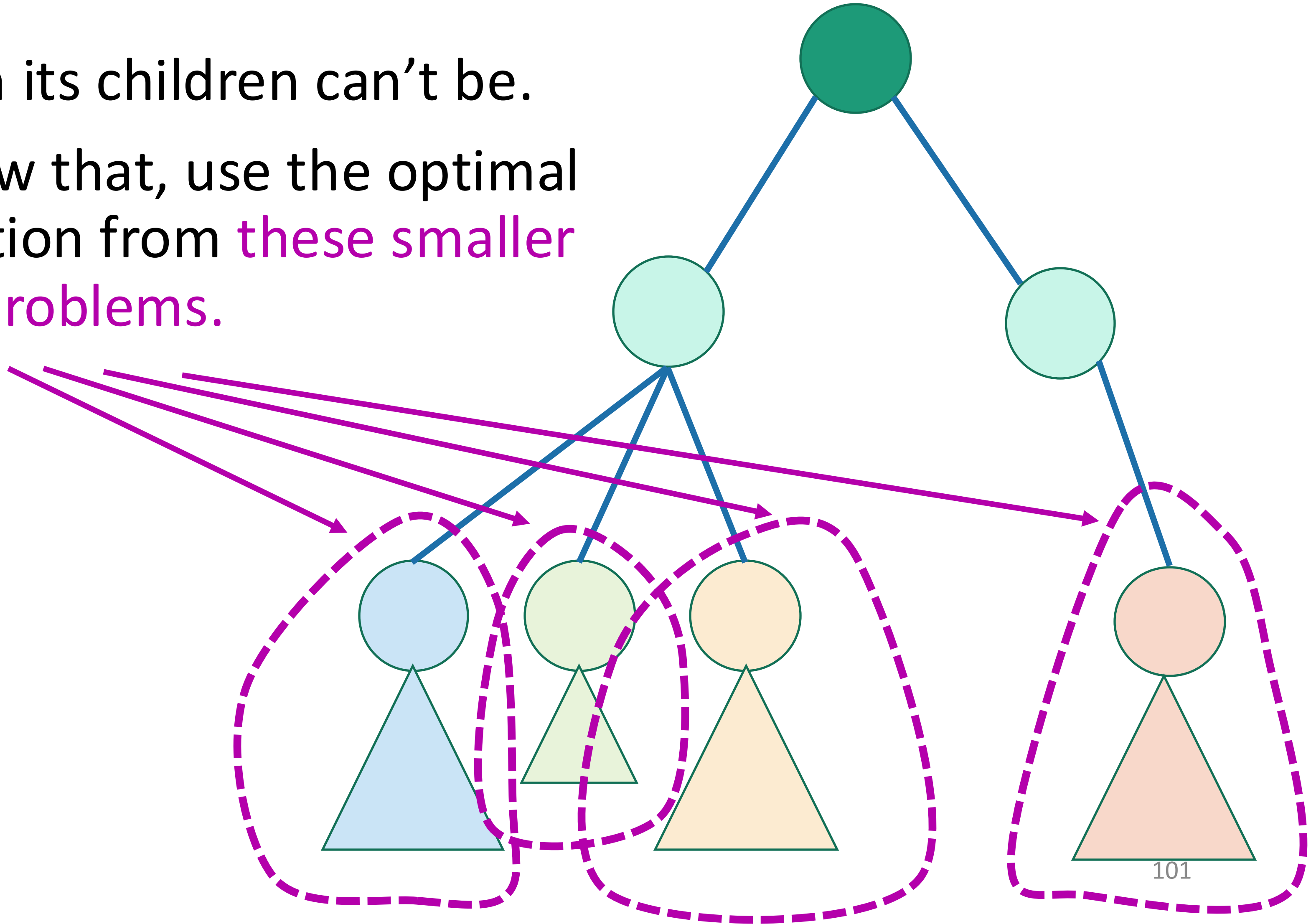
- Use the optimal solution from these smaller problems.



Case 2:

the root is in an maximal independent set

- Then its children can't be.
- Below that, use the optimal solution from these smaller subproblems.



Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
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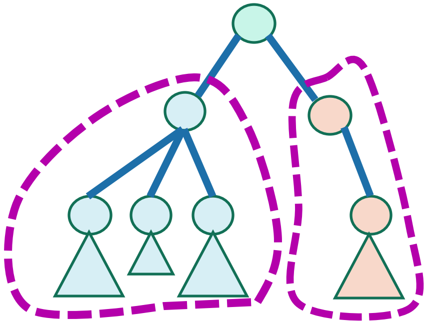
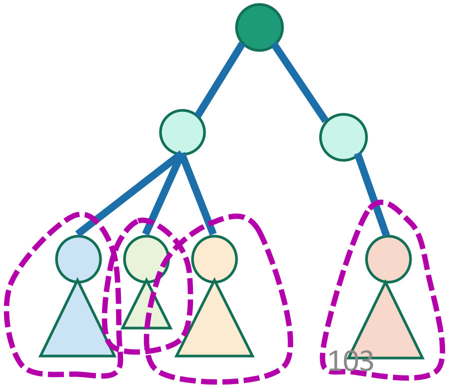
Recursive formulation: try 1

- Let A[u] be the weight of a maximal independent set in the tree rooted at u.



- $A[u] =$

$$\max\begin{cases} \sum_{v \in u.\mathrm{children}} A[v] \\ \mathrm{weight}(u) + \sum_{v \in u.\mathrm{grandchildren}} A[v] \end{cases}$$

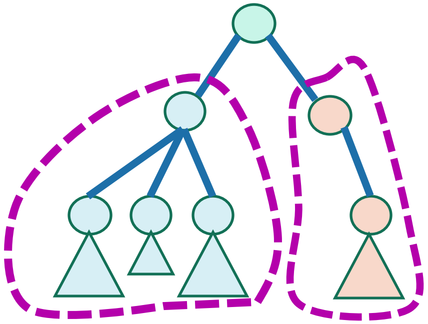


When we implement this, how do we keep track of this term?

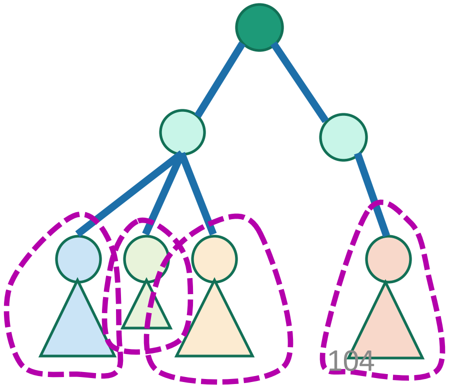
Recursive formulation: try 2

Keep two arrays!

- Let A[u] be the weight of a maximal independent set in the tree rooted at u.
- Let B[u] = $\sum_{v \in u.\text{children}} A[v]$



- $A[u] = \max \begin{cases} \sum_{v \in u.\text{children}} A[v] \\ \text{weight}(u) + \sum_{v \in u.\text{children}} B[v] \end{cases}$



Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
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- **Step 5:** If needed, code this up like a reasonable person.

A top-down DP algorithm

- MIS_subtree(u):
  - **if** u is a leaf:
    - A[u] = weight(u)
    - B[u] = 0
  - **else**:
    - **for** v in u.children:
      - MIS_subtree(v)
    - $A[u] = \max\{ \sum_{v \in u.\text{children}} A[v] , \text{weight}(u) + \sum_{v \in u.\text{children}} B[v] \}$
    - $\text{B}[u] = \sum_{v \in u.\text{children}} A[v]$

Initialize global arrays A, B that we will use in all of the recursive calls.

- MIS(T):
  - MIS_subtree(T.root)
  - **return** A[T.root]

Running time?

- We visit each vertex once, and for every vertex we do O(1) work:
  - Make a recursive call
  - Participate in summations of parent node
- Running time is O(|V|)

Why is this different from divide-and-conquer?

That's always worked for us with tree problems before…

```
• MIS_subtree(u):
    • if u is a leaf:
        • return weight(u)
    • else:
        • return max{ $\sum_{v \in u.\text{children}} \text{MIS\_subtree}(v)$ ,
                      $\text{weight}(u) + \sum_{v \in u.\text{grandchildren}} \text{MIS\_subtree}(v)$ }

• MIS(T):
    • return MIS_subtree(T.root)
```

This is exactly the same pseudocode, except we've ditched the table and are just calling MIS_subtree(v) instead of looking up A[v] or B[v].

Why is this different from divide-and-conquer?

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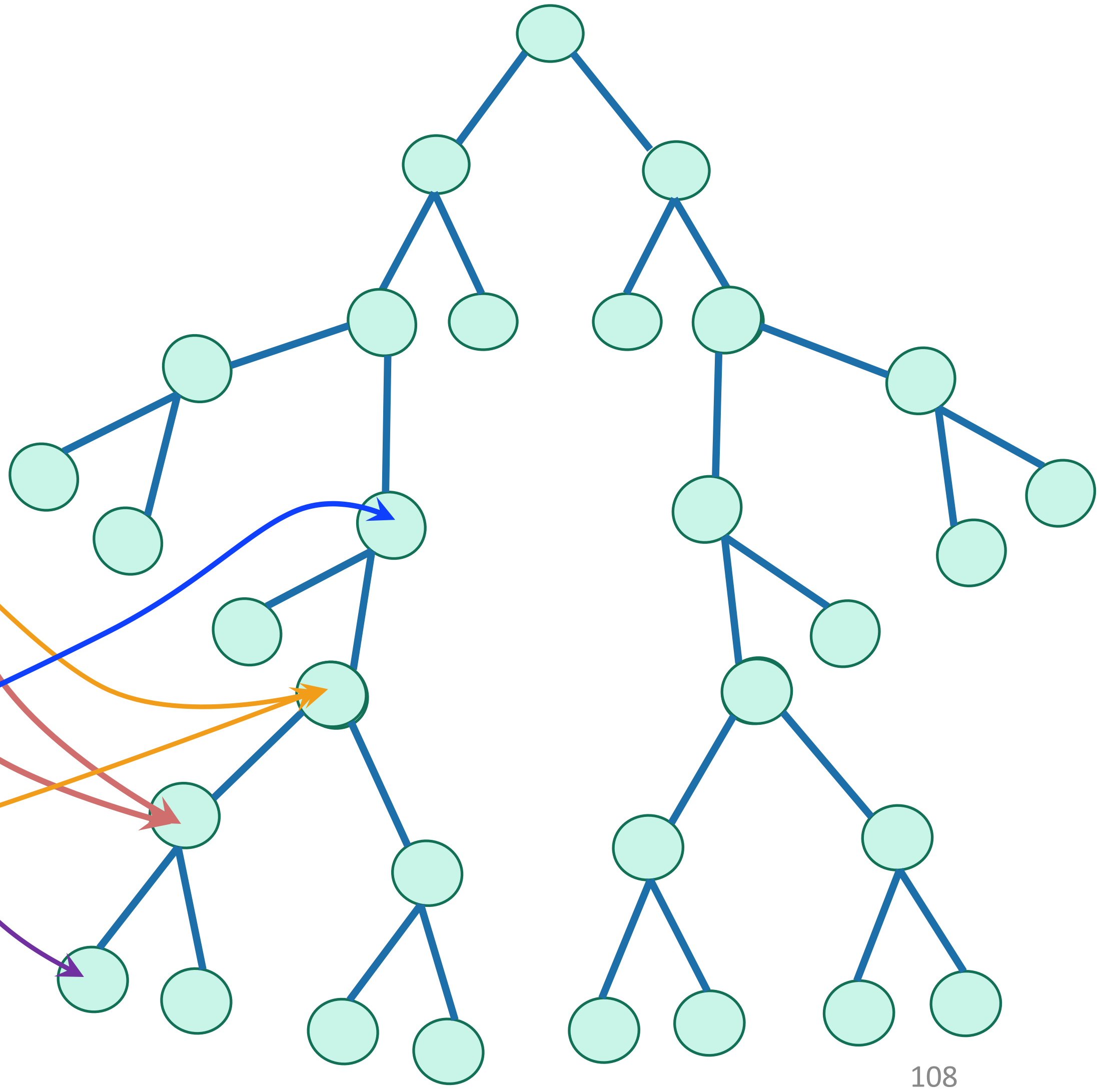

How often would we ask about the subtree rooted **here**?

Once for **this node** and once for **this one**.

But we then ask about **this node** twice, **here** and **here**.

This will blow up exponentially without using dynamic programming to take advantage of **overlapping subproblems**.

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
- **Step 5:** If needed, code this up like a reasonable person.

You do this one!

What have we learned?

- We can find maximal independent sets in trees in time O(|V|) using dynamic programming!
- For this example, it was natural to implement our DP algorithm in a top-down way.

Recap

- Today we saw examples of how to come up with dynamic programming algorithms.
  - Longest Common Subsequence
  - Knapsack two ways
  - (If time) maximal independent set in trees.
- There is a **recipe** for dynamic programming algorithms.

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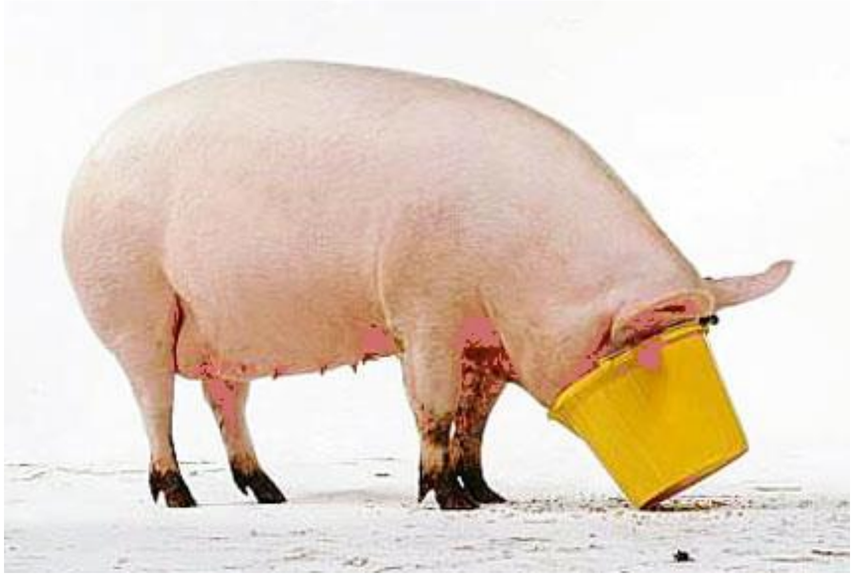
Recap

- Today we saw examples of how to come up with dynamic programming algorithms.
  - Longest Common Subsequence
  - Knapsack two ways
  - (If time) maximal independent set in trees.
- There is a **recipe** for dynamic programming algorithms.
- Sometimes coming up with the right substructure takes some creativity
  - Practice on homework! ☺
  - For even more practice check out additional examples/practice problems in CLRS or section!

Next time

- Greedy algorithms!



Before next time

- Pre-lecture exercise: Greed is good!