Lecture 8

Hashing

Announcements

- There will not be new HW posted this week, because it’s time to study for the…

Midterm!

- Wednesday, Feb 11, 6-9pm!
- Stay tuned for logistics email
- Covers up through last week (Lecture 7)

- Students with course conflicts:

  Email staff as soon as possible

  cs161-staff-win2526@cs.stanford.edu

How to study for the midterm?

- Go over lecture + homework + **section** + textbook
- DO PRACTICE PROBLEMS.
  - Algorithms Illuminated, CLRS have great problems!
  - Practice exam(s)
- Office Hours!
  - Most effective if you come with specific questions/topics
  - Note: some OH have moved earlier in the week
- Friday, here, 1:30-2:50: midterm review session

Outline

- **Hash tables** are another sort of data structure that allows fast INSERT/DELETE/SEARCH.
  - like self-balancing binary trees
  - The difference is we can get better performance in expectation by using randomness.
- **Hash families** are the magic behind hash tables.
- **Universal hash families** are even more magical.

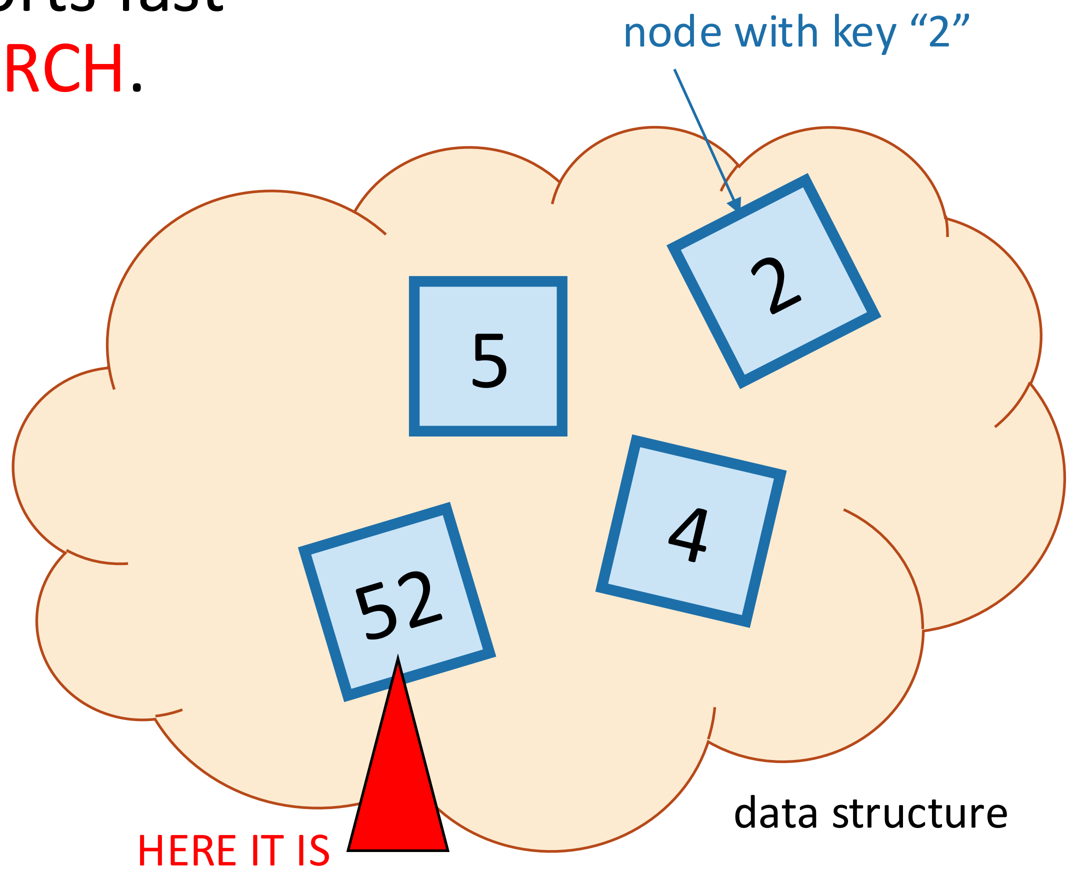
Goal

- We want to store nodes with keys in a data structure that supports fast INSERT/DELETE/SEARCH.

- INSERT 

- DELETE 

- SEARCH 



Last time

- Self balancing trees:
  - O(log(n)) deterministic INSERT/DELETE/SEARCH

Today:

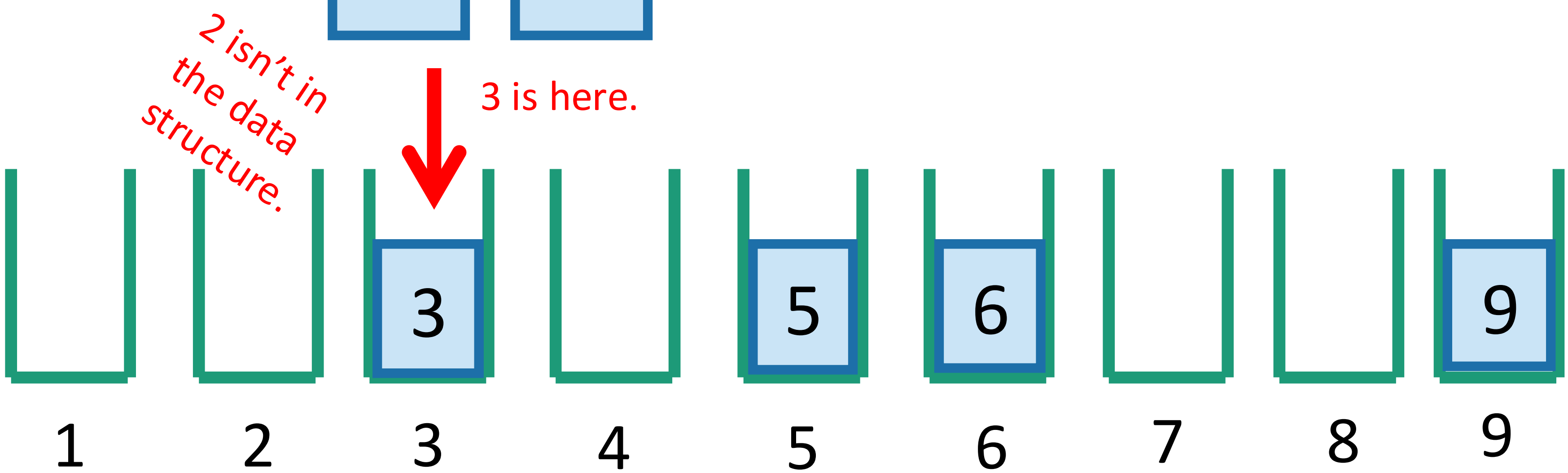
- Hash tables:
  - O(1) expected time INSERT/DELETE/SEARCH
- Worse worst-case performance, but often great in practice.

eg, Python's `dict`, Java's `HashSet/HashMap`, C++'s `unordered_map`
Hash tables are used for databases, caching, object representation, …

One way to get O(1) time

This is called “direct addressing”

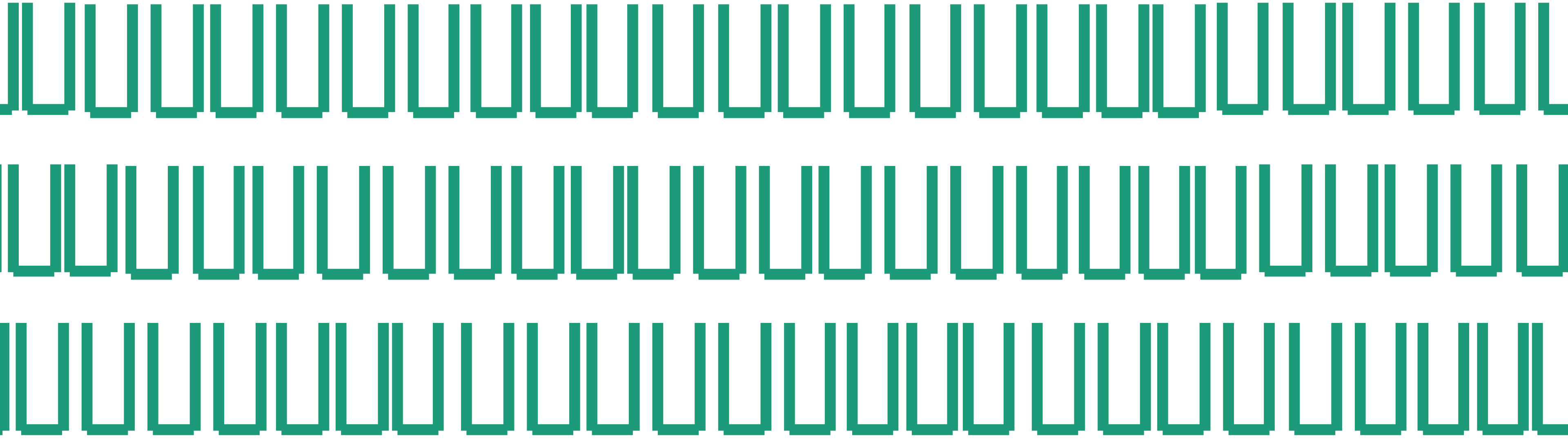
- Say all keys are in the set {1,2,3,4,5,6,7,8,9}.
- INSERT: 9 6 3 5
- DELETE: 6
- SEARCH: 3 2



Problem

The universe is really big!

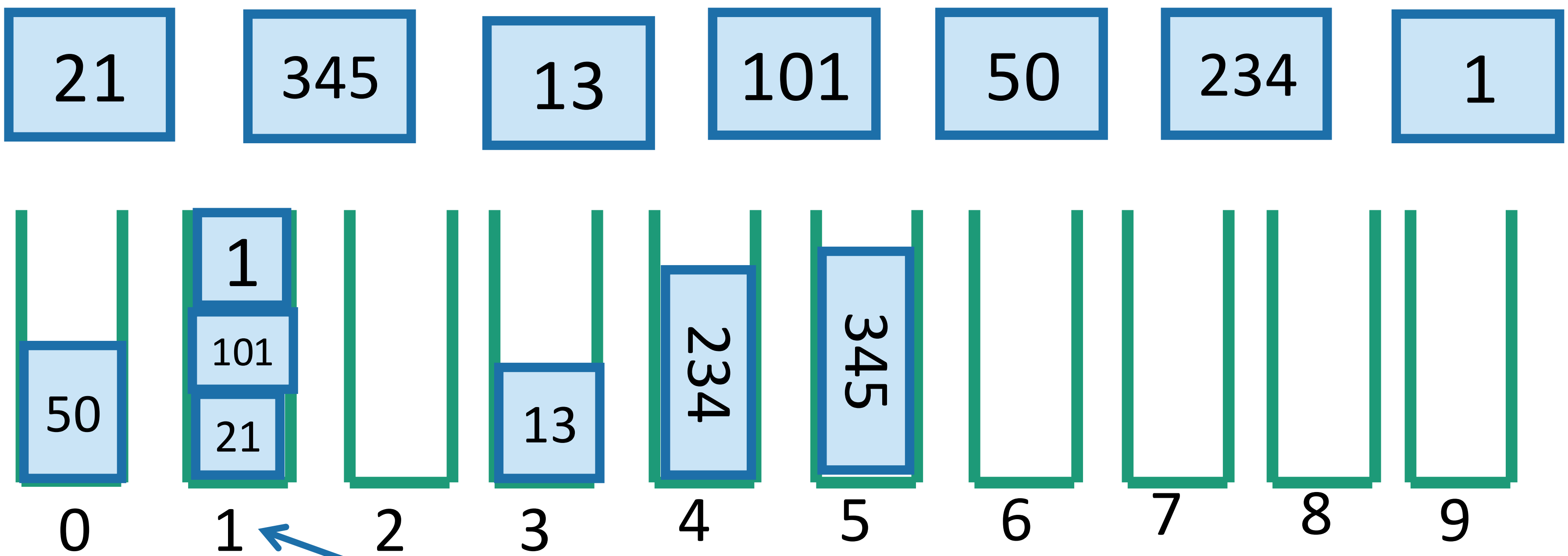
- If the keys may come from a "universe" $U = \{1,2, \ldots., 10000000000\}$, direct addressing takes a lot of space.



Solution?

Put things in buckets based on one digit

INSERT:

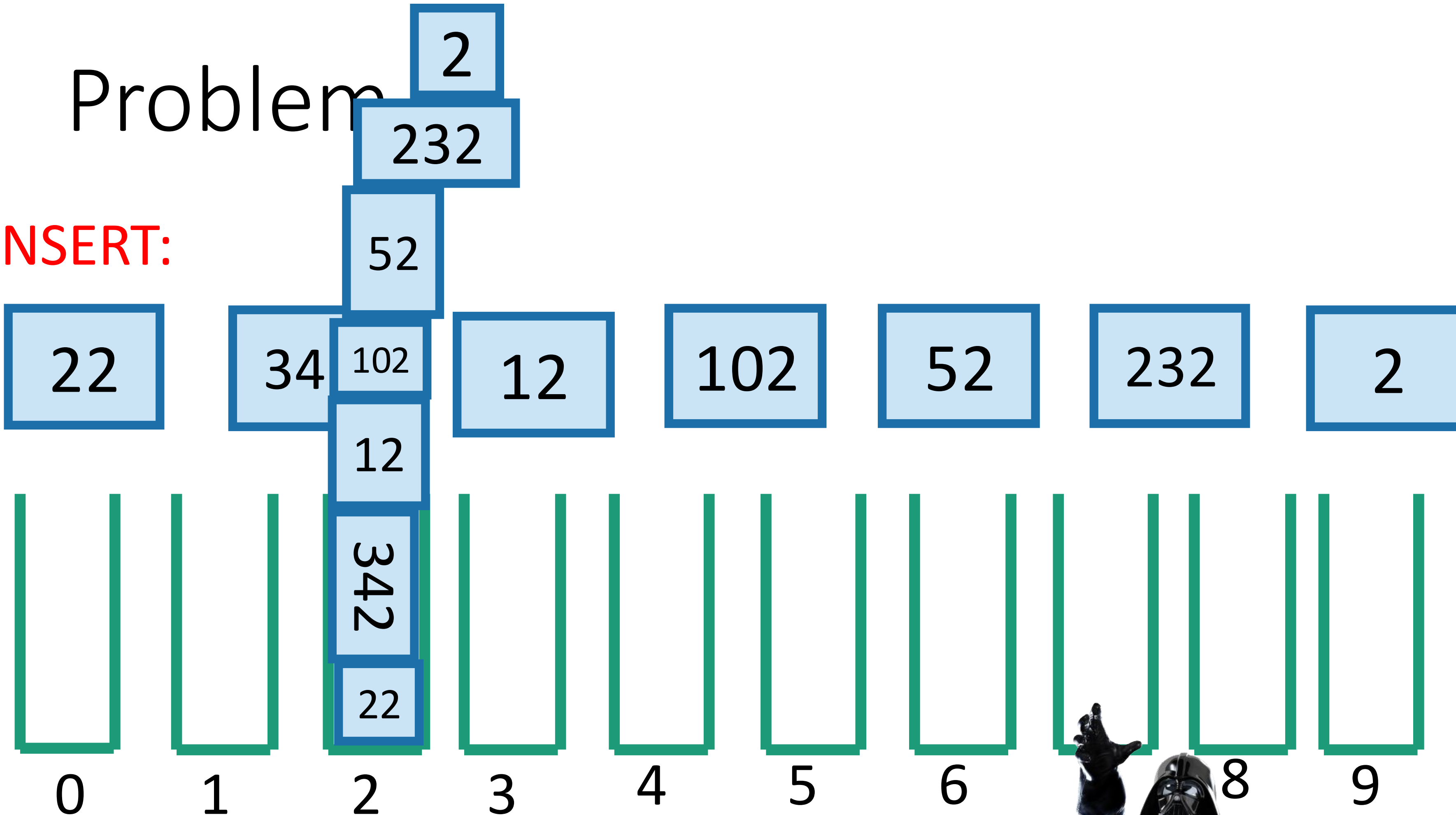


It's in this bucket somewhere…
go through until we find it.

Now SEARCH 21

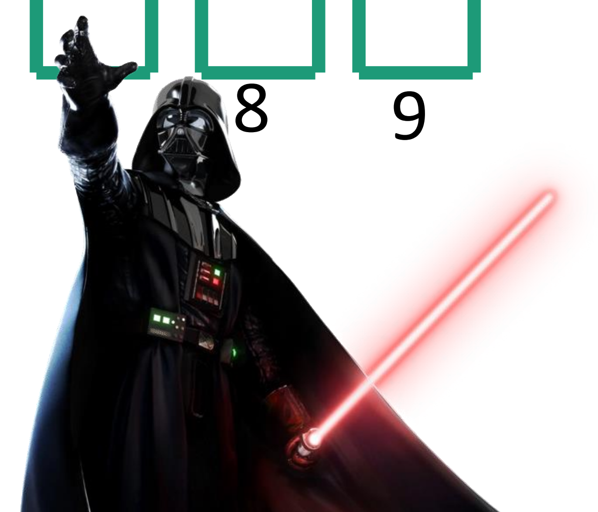
Problem

INSERT:



Now SEARCH 22

....this hasn't made our lives easier...



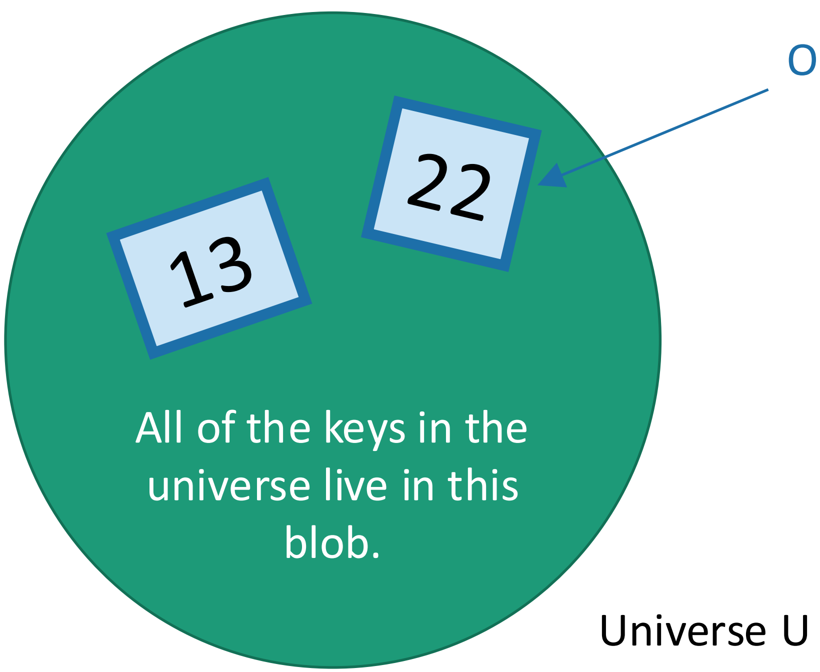
Hash tables

- That was an example of a hash table.
  - not a very good one, though.
- We will be **more clever** (and less deterministic) about our bucketing.
- This will result in fast (expected time) INSERT/DELETE/SEARCH.

But first! Terminology.

- U is a *universe* of size M.
  - M is really big.
- But only a few (at most n) elements of U are ever going to show up.
  - M is waaaayyyyyyy bigger than n.
- But we don't know which ones will show up in advance.

Only a n keys will ever show up.



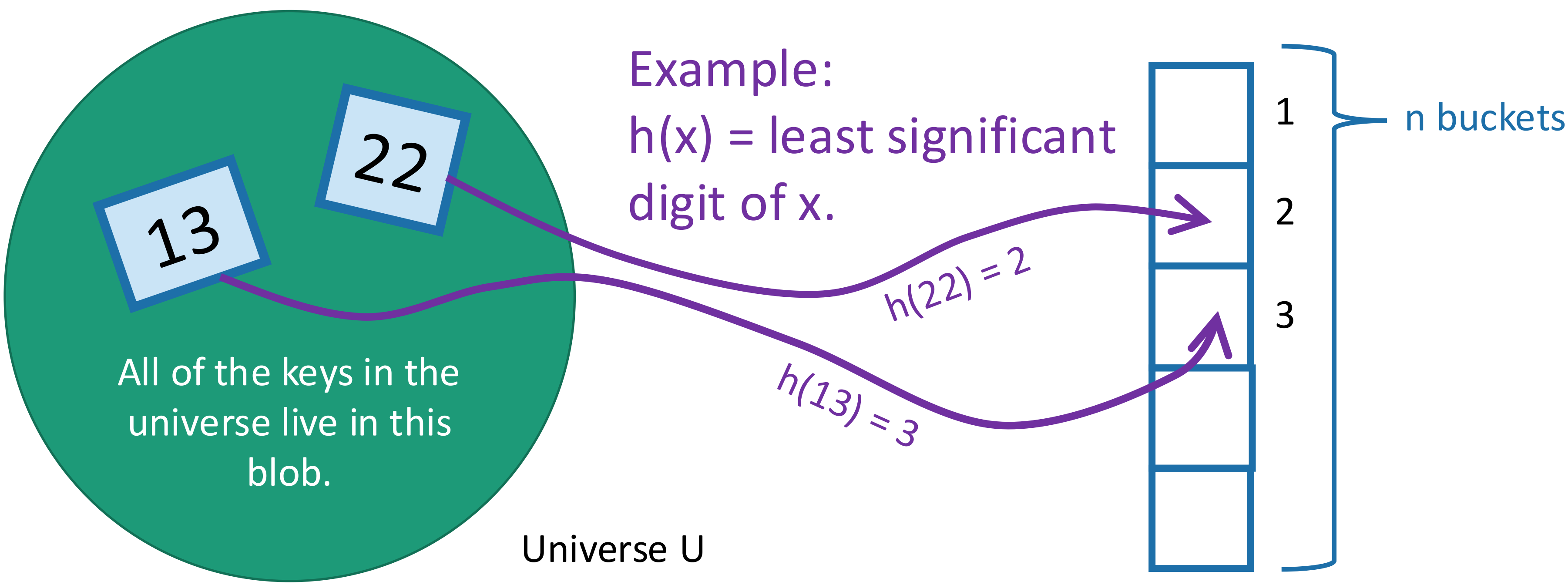
Example: U is the set of all strings of at most 280 ascii characters. (128^{280} of them).

The only ones which I care about are those which appear as trending hashtags on twitter. #hashinghashtags
There are way fewer than 128^{280} of these.

Hash Functions

- A *hash function* $h: U \to \{1, \ldots, n\}$ is a function that maps elements of U to buckets 1, ..., n.

- **Note!** For this lecture, n is **both** #buckets and #(things that might show up).
- That doesn't need to be the case, but in general we should think of those two things as being on the same order.

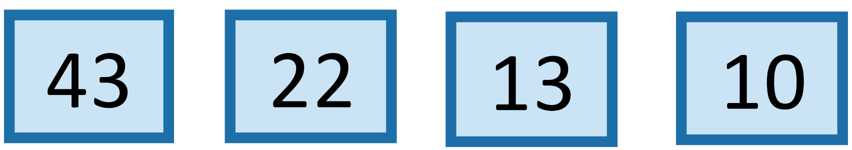


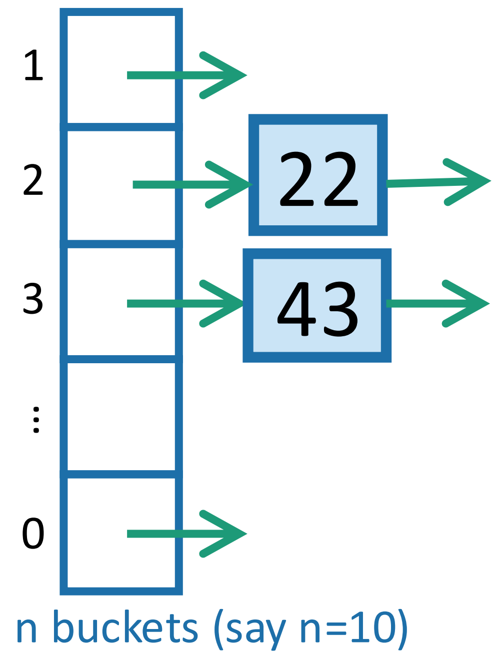
Hash Tables (with chaining)

A **hash table** consists of:

- An array of n buckets.
- Each bucket stores a linked list.
  - We can insert into a linked list in time O(1)
  - To find something in the linked list takes time O(length(list)).
- A hash function $h: U \rightarrow \{1, \ldots, n\}$.
  - For example, h(x) = least significant digit of x.

INSERT:





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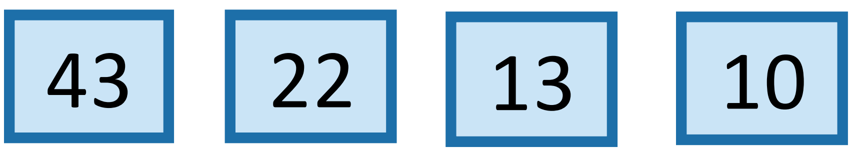
- A hash function $h: U \rightarrow \{1, \ldots, n\}$.
  - For example, h(x) = least significant digit of x.

For demonstration purposes only!
This is a terrible hash function! Don't use this!

INSERT:

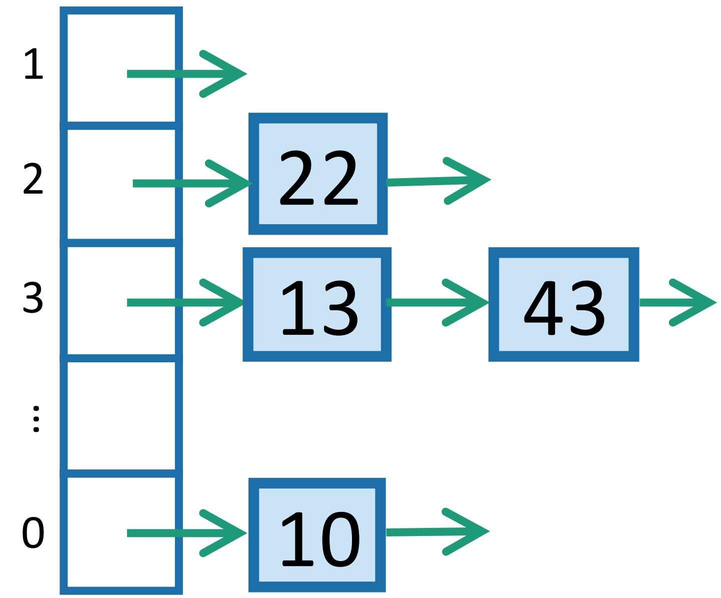


SEARCH 43:

Scan through all the elements in bucket h(43) = 3.

DELETE 43:

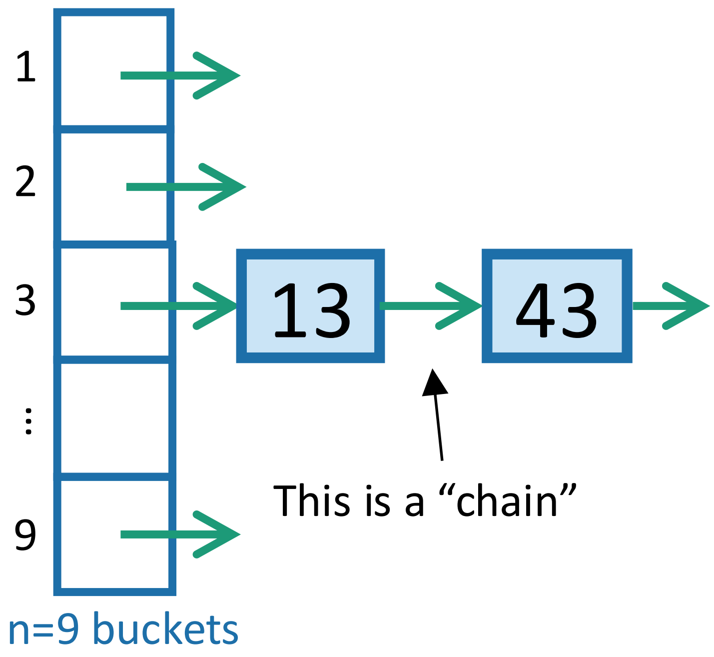
Search for 43 and remove it.

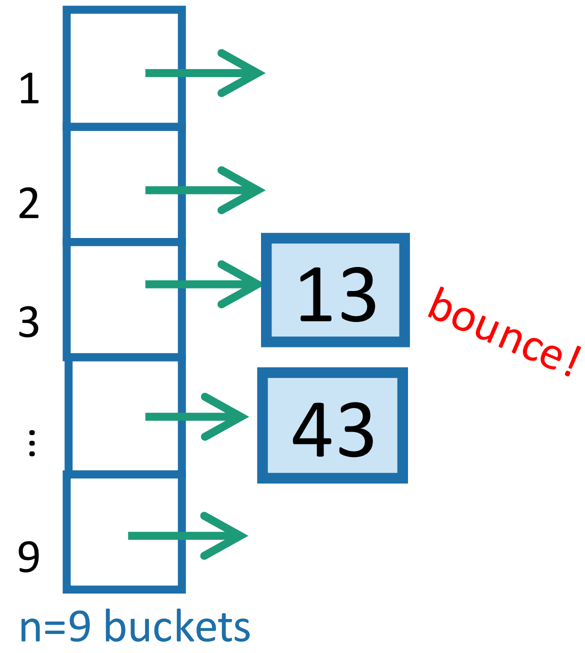


n buckets (say n=10)

Aside: Hash tables with open addressing

- The previous slide is about hash tables with chaining.
- There's also something called "open addressing"
- You don't need to know about it for this class.





\end{Aside}

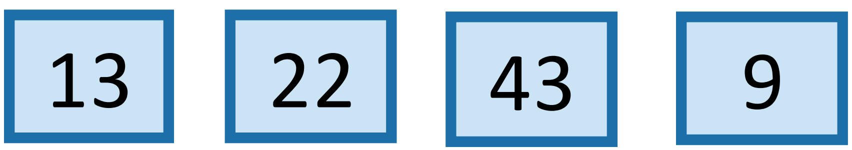
Hash Tables (with chaining)

A **hash table** consists of:

- Array of n buckets.
- Each bucket stores a linked list.
  - We can insert into a linked list in time O(1)
  - To find something in the linked list takes time O(length(list)).
- A hash function $h: U \rightarrow \{1, \ldots, n\}$.
  - For example, h(x) = least significant digit of x.

For demonstration purposes only!
This is a terrible hash function! Don't use this!

INSERT:



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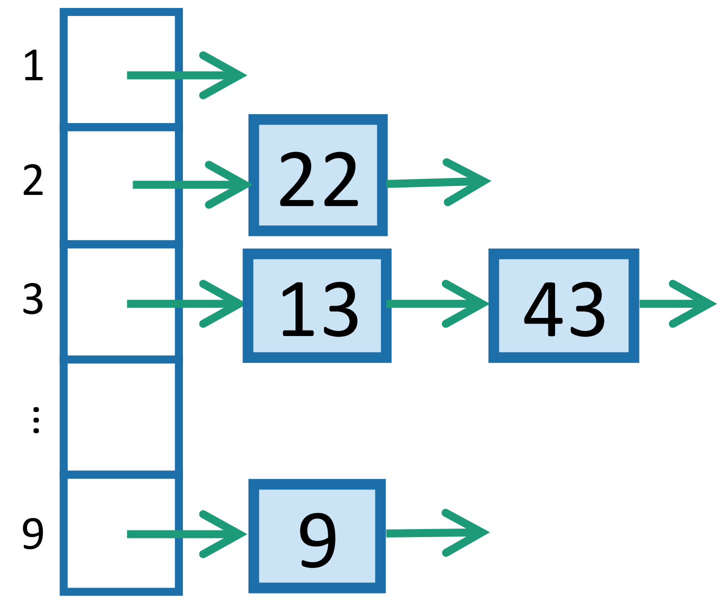


n buckets (say n=9)

Outline

- **Hash tables** are another sort of data structure that allows fast INSERT/DELETE/SEARCH.
  - (We still need to figure out how to do the bucketing)

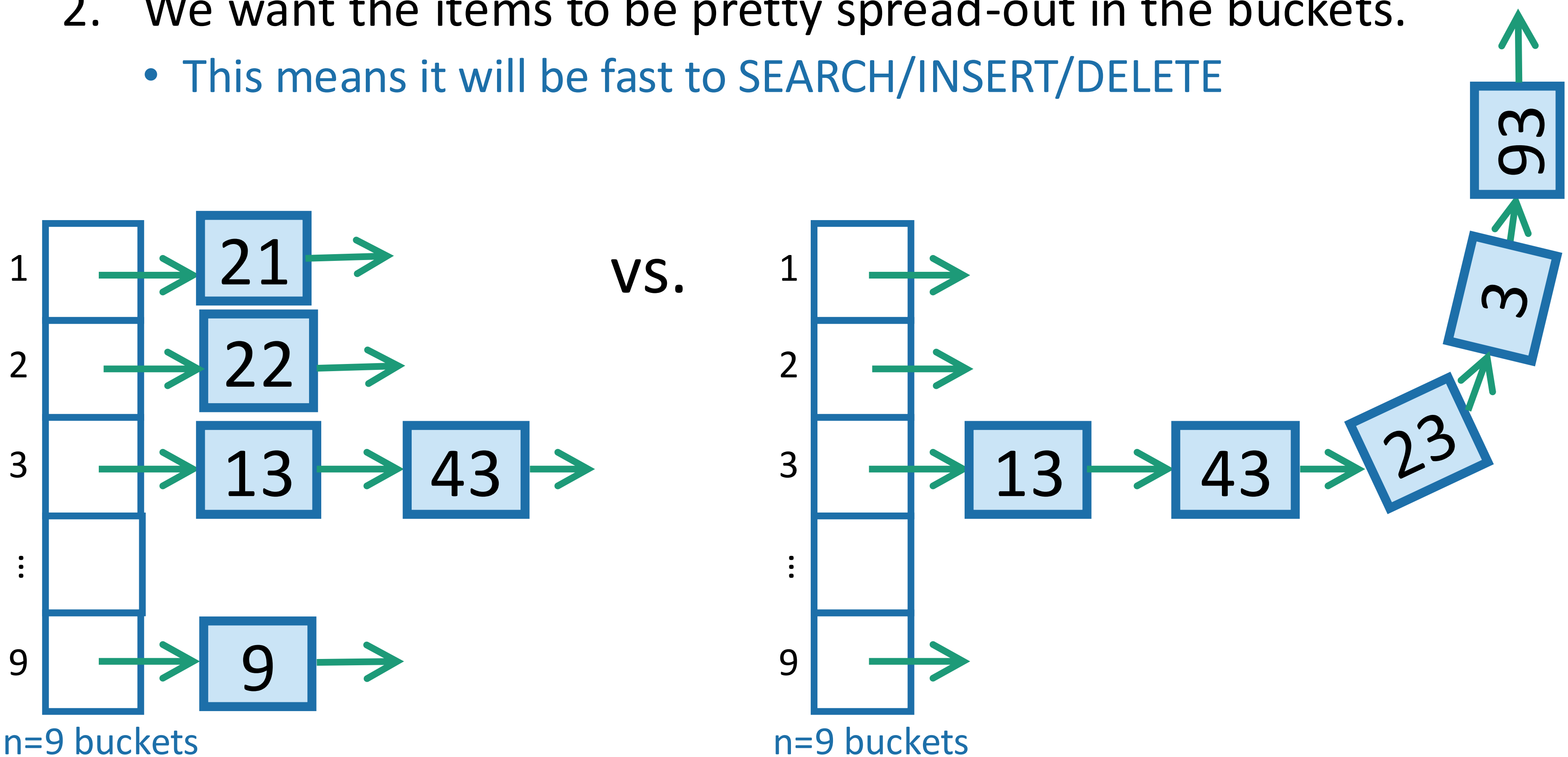
Interlude: motivation for hash families.

- **Hash families** are the magic behind hash tables.

- **Universal hash families** are even more magical.

What we want from a hash table

1. We want there to be not many buckets (say, n).
   - This means we don't use too much space
2. We want the items to be pretty spread-out in the buckets.
   - This means it will be fast to SEARCH/INSERT/DELETE



*Assuming it takes time O(1) to evaluate h(u)

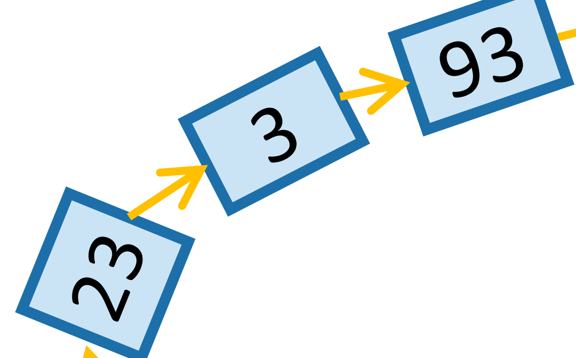
Worst-case analysis

- Goal: Design a function $h: U \to \{1, \ldots, n\}$ so that:
  - No matter what n items of U a bad guy chooses, the buckets will be balanced.
  - Here, balanced means O(1) entries per bucket.

- If we had this*, then we'd achieve our dream of O(1) INSERT/DELETE/SEARCH

So, does such a function exist?

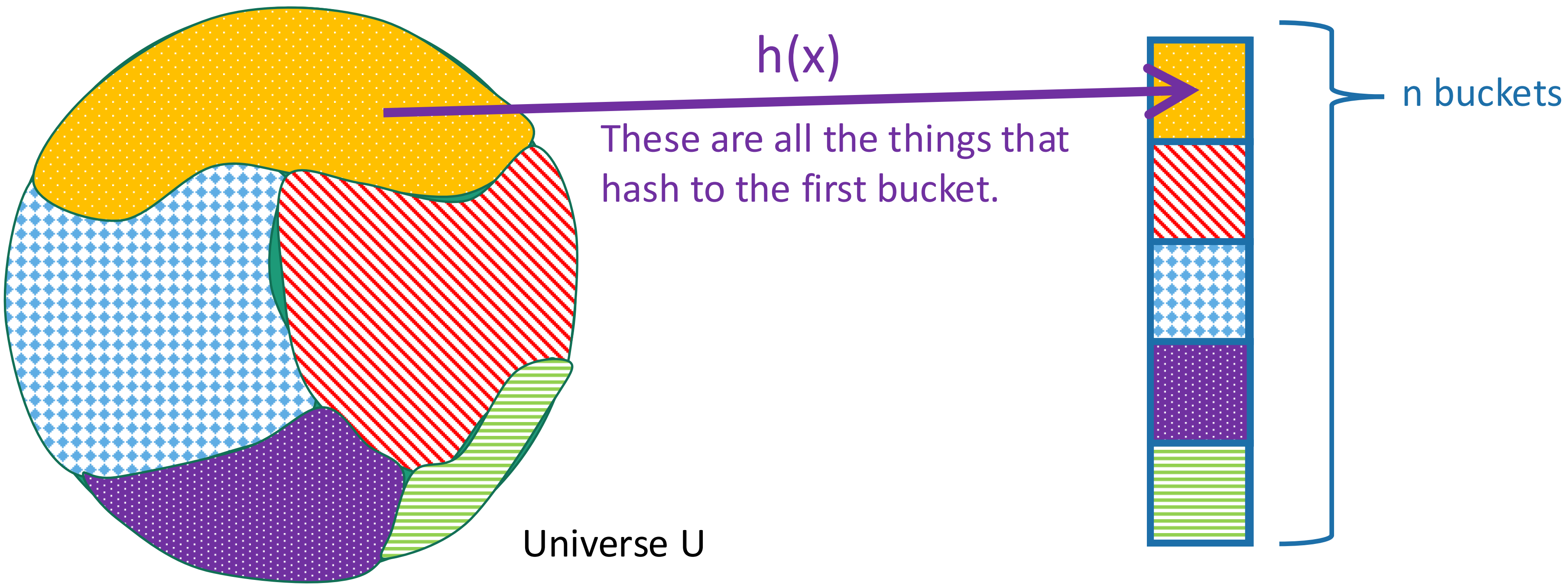
This is impossible!



No deterministic hash function can defeat worst-case input!

We really can’t beat the bad guy here.

- The universe U has M items
- They get hashed into n buckets
- At least one bucket has at least M/n items hashed to it.
- M is waayyyy bigger then n, so M/n is bigger than n.
- **Bad guy chooses n of the items that landed in this very full bucket.**



Solution:
Randomness

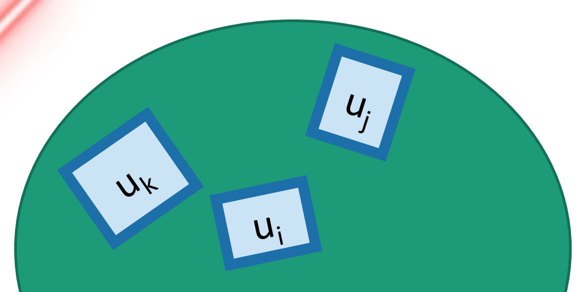
The game

What does **random** mean here? Uniformly random?

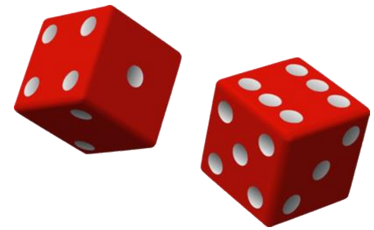
Plucky the pedantic penguin

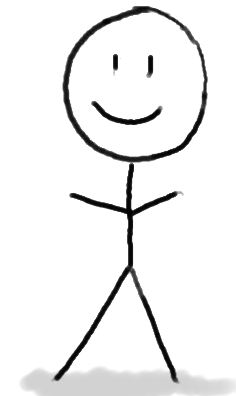
1. An adversary chooses any n items $u_1, u_2, \ldots, u_n \in U$, and any sequence of INSERT/DELETE/SEARCH operations on those items.

INSERT 13, INSERT 22, INSERT 43, INSERT 92, INSERT 7, SEARCH 43, DELETE 92, SEARCH 7, INSERT 92

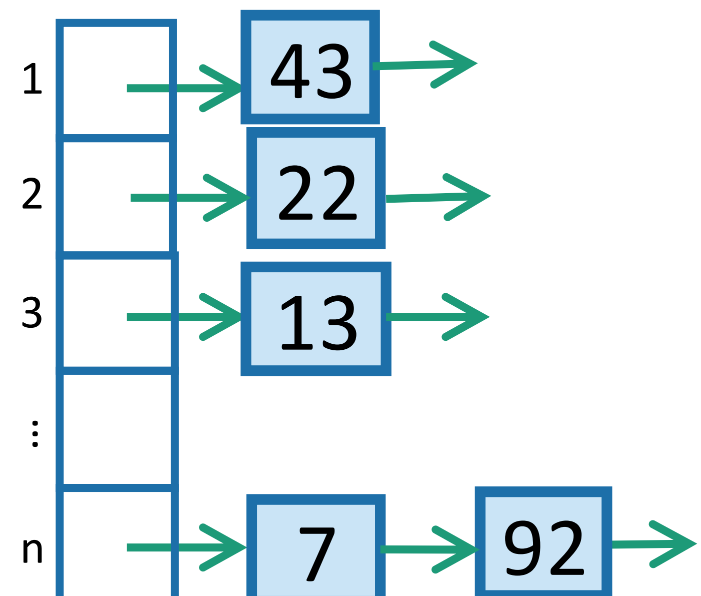


2. You, the algorithm, chooses a **random** hash function $h: U \to \{1, \ldots, n\}$.

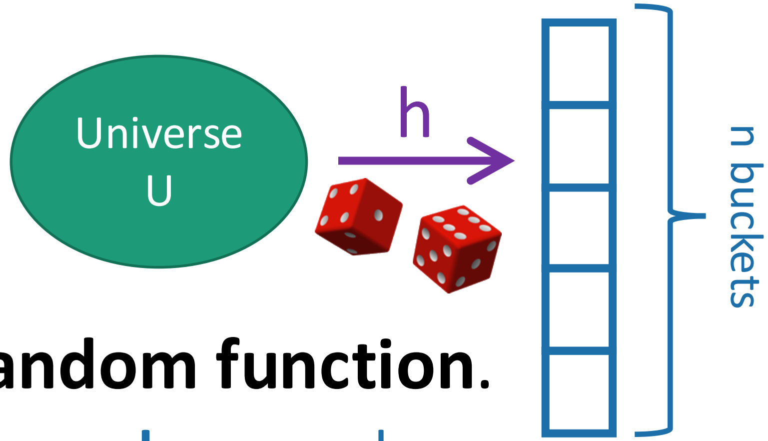




3. **HASH IT OUT** #hashpuns



Example of a random hash function



- $h: U \to \{1, \ldots, n\}$ is a **uniformly random function.**
  - That means that **h(1)** is a **uniformly random** number between 1 and n.
  - **h(2)** is also a **uniformly random** number between 1 and n, independent of h(1).
  - **h(3)** is also a **uniformly random** number between 1 and n, independent of h(1), h(2).
  - …
  - **h(M)** is also a **uniformly random** number between 1 and n, independent of h(1), h(2), …, h(M-1).

Randomness can help!

Intuitively: The bad guy can't foil a hash function that they don't yet know.

Lucky the Lackadaisical Lemur

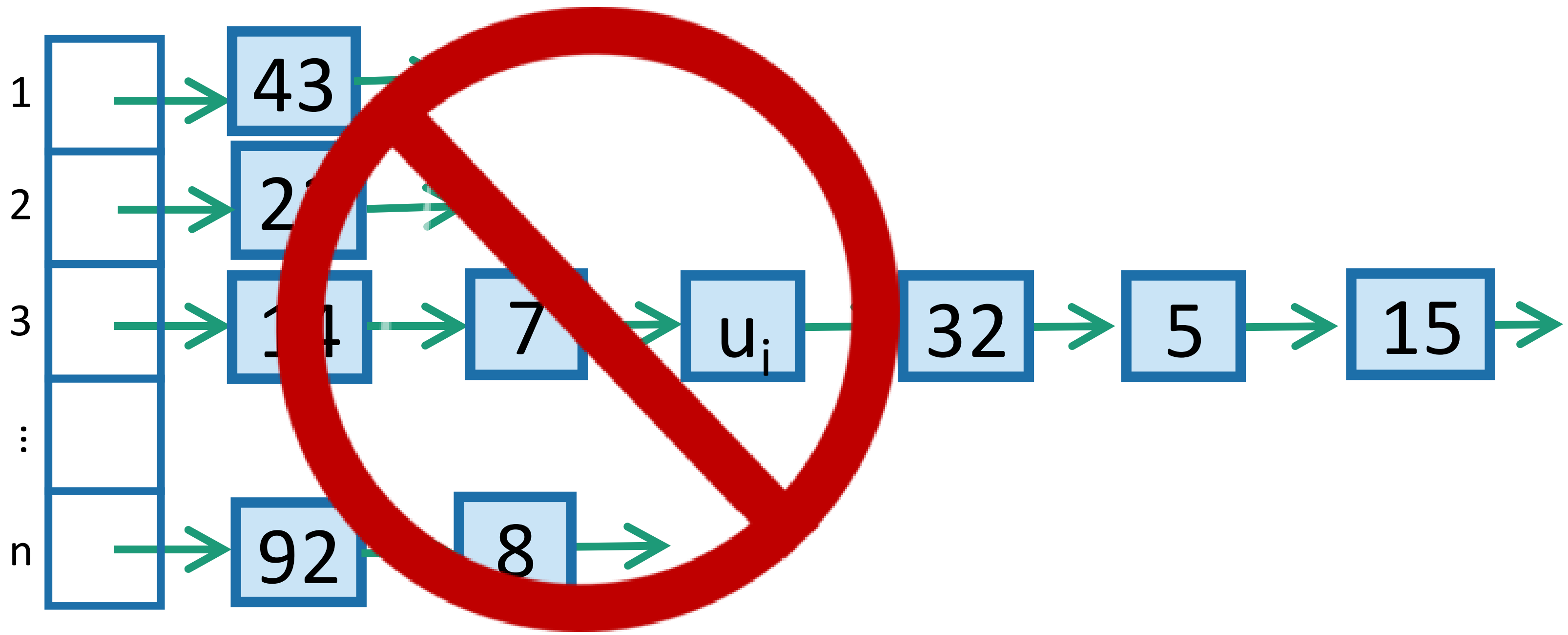
Plucky the Pedantic Penguin

Why not? What if there's some strategy that foils a random function with high probability?

We'll need to do some analysis…

Intuitive goal

It's **bad** if lots of items land in u_i's bucket.
So we want **not that**.



Formal goal

We could replace "2" here with any constant; it would still be good. But "2" will be convenient.

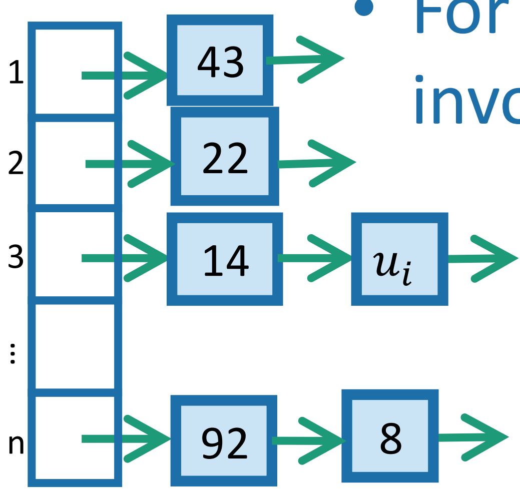
- Let h be a random hash function.
- Want: For all ways a bad guy could choose $u_1, u_2, \ldots, u_n$ to put into the hash table, and for all $i \in \{1, \ldots, n\}$,

  $$E[\text{ number of items in } u_i\text{'s bucket }] \leq 2.$$

- If that were the case*:
  - For each INSERT/DELETE/SEARCH operation involving u_i,

    $$E[\text{ time of operation }] = O(1)$$

This is what we wanted at the beginning of lecture!



*Assuming h(u) takes O(1) time to compute

Goal:

- Come up with a distribution on hash functions so that:
- For all i=1, …, n,

$$E[\text{ number of items in } u_i\text{'s bucket }] \leq 2.$$

Aside

- For all i=1, …, n,

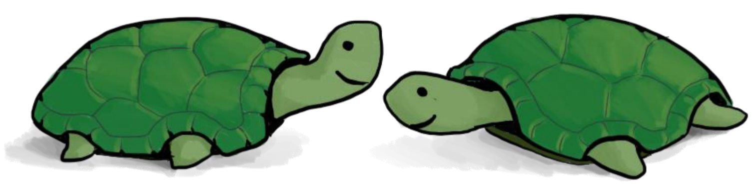
$$E[\text{ number of items in } u_i \text{ 's bucket }] \leq 2.$$

vs

- For all i=1,…,n:

$$E[\text{ number of items in bucket } i\ ] \leq 2$$

Are these the same?



Think-Pair-Share Terrapins

No! (This was your pre-lecture exercise!)

Aside

- For all i=1, …, n,

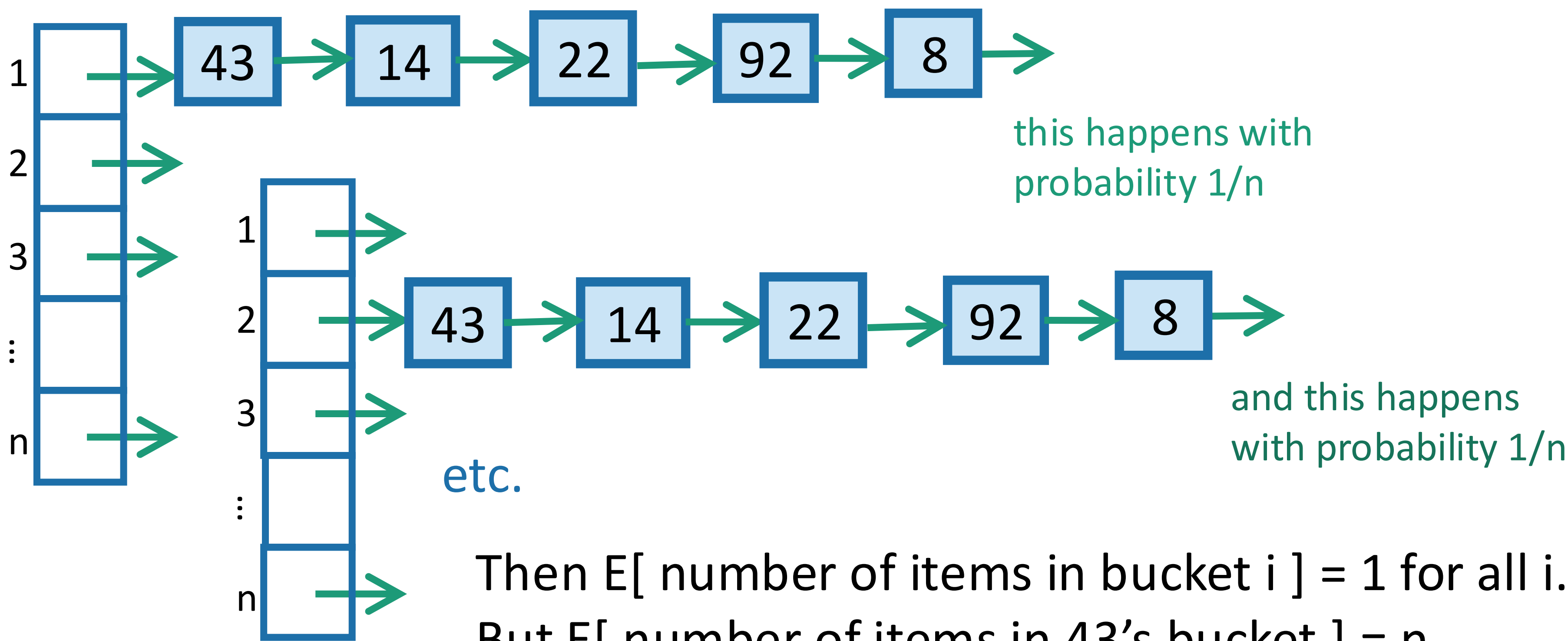
$$E[\text{ number of items in } u_i \text{ 's bucket }] \leq 2.$$

vs

- For all i=1,…,n:

$$E[\text{ number of items in bucket } i\ ] \leq 2$$

Suppose that:



Then E[number of items in bucket i] = 1 for all i.

But E[number of items in 43's bucket] = n

Goal:

- Come up with a distribution on hash functions so that:
- For all $i = 1, \dots, n$,

  E[number of items in u_i's bucket] $\leq$ 2.

Claim:

- The goal is achieved by a uniformly random hash function.

Proof of Claim

- Let h be a uniformly random hash function.
- Then for all $i = 1, \ldots, n$,
  E[number of items in u_i's bucket] ≤ 2.

- $E[$ # items in u_i's bucket $] =$
- $= E\left[\sum_{j=1}^{n} \mathbf{1}\{h(u_i) = h(u_j)\}\right]$
- $= \sum_{j=1}^{n} P\{ h(u_i) = h(u_j)\}$
- $= 1 + \sum_{j \neq i} P\{ h(u_i) = h(u_j)\}$
- $= 1 + \sum_{j \neq i} 1/n$
- $= 1 + \frac{n-1}{n} \leq 2.$

Exercise: show this formally!
Intuitively, there are n possibilities where u_j can land, and only one of them is $h(u_i)$.

A uniformly random hash function leads to balanced buckets

- We just showed:
  - For all ways a bad guy could choose $u_1, u_2, \ldots, u_n$, to put into the hash table, and for all $i \in \{1, \ldots, n\}$,

    E[number of items in u_i 's bucket] $\leq$ 2.
- Which implies*:
  - No matter what sequence of operations and items the bad guy chooses,

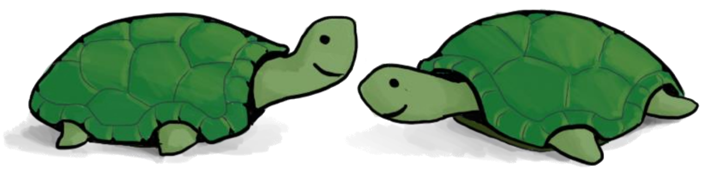
    E[time of INSERT/DELETE/SEARCH] = O(1)
- So our solution is:

Pick a uniformly random hash function?

*Assuming h(u) takes O(1) time to compute

What’s wrong with this plan?

- Hint: How would you implement (and store) and uniformly random function $h: U \rightarrow \{1, \ldots, n\}$?



Think-Pair-Share Terrapins
1 minute think
1 minute pair and share

- If h is a uniformly random function:
  - That means that **h(1)** is a **uniformly random** number between 1 and n.
  - **h(2)** is also a **uniformly random** number between 1 and n, independent of h(1).
  - **h(3)** is also a **uniformly random** number between 1 and n, independent of h(1), h(2).
  - …
  - **h(M)** is also a **uniformly random** number between 1 and n, independent of h(1), h(2), …, h(M-1).

A uniformly random hash function is not a good idea.

- In order to store/evaluate a uniformly random hash function, we'd use a lookup table:

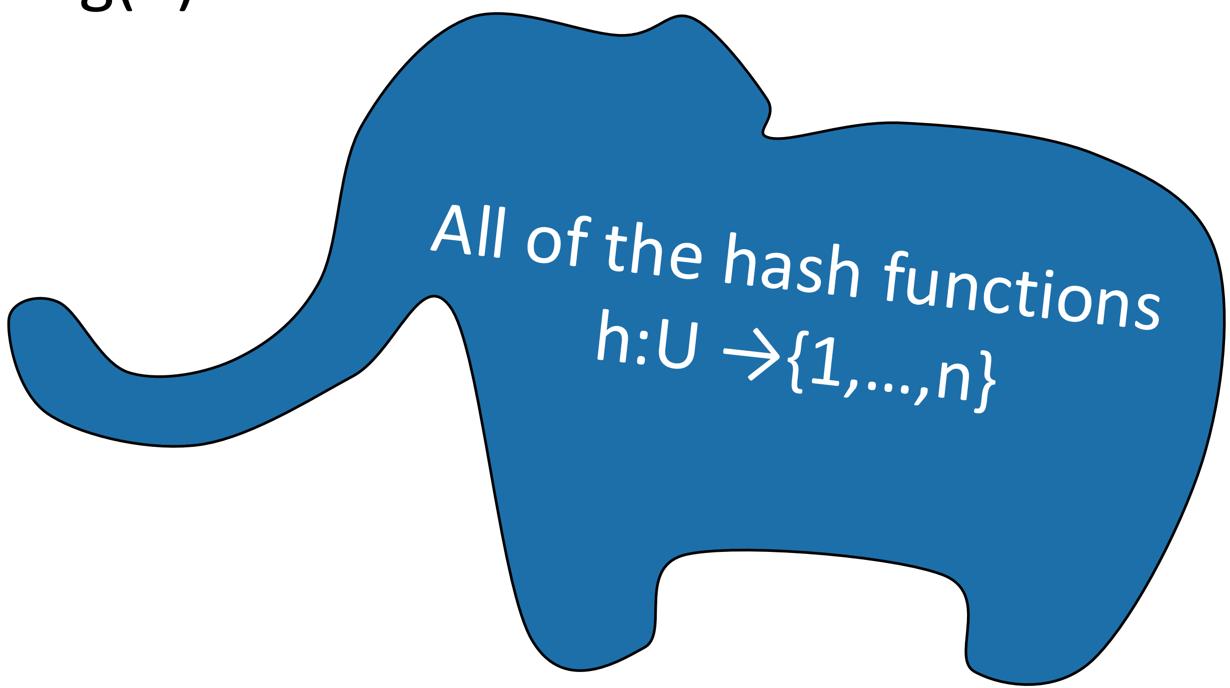
All of the M things in the universe

x	h(x)
AAAAAA	1
AAAAAB	5
AAAAAC	3
AAAAAD	3
…	
ZZZZZY	7
ZZZZZZ	3

- Each value of h(x) takes log(n) bits to store.
- Storing M such values requires Mlog(n) bits.
- In contrast, direct addressing (initializing a bucket for every item in the universe) requires only M bits.

Another way to say this

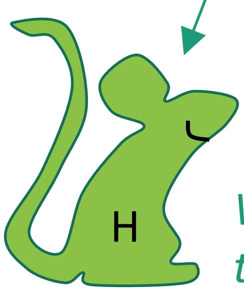
- There are lots of hash functions.
- There are n^M of them.
- Writing down a random one of them takes $\log(n^M)$ bits, which is $M \log(n)$.



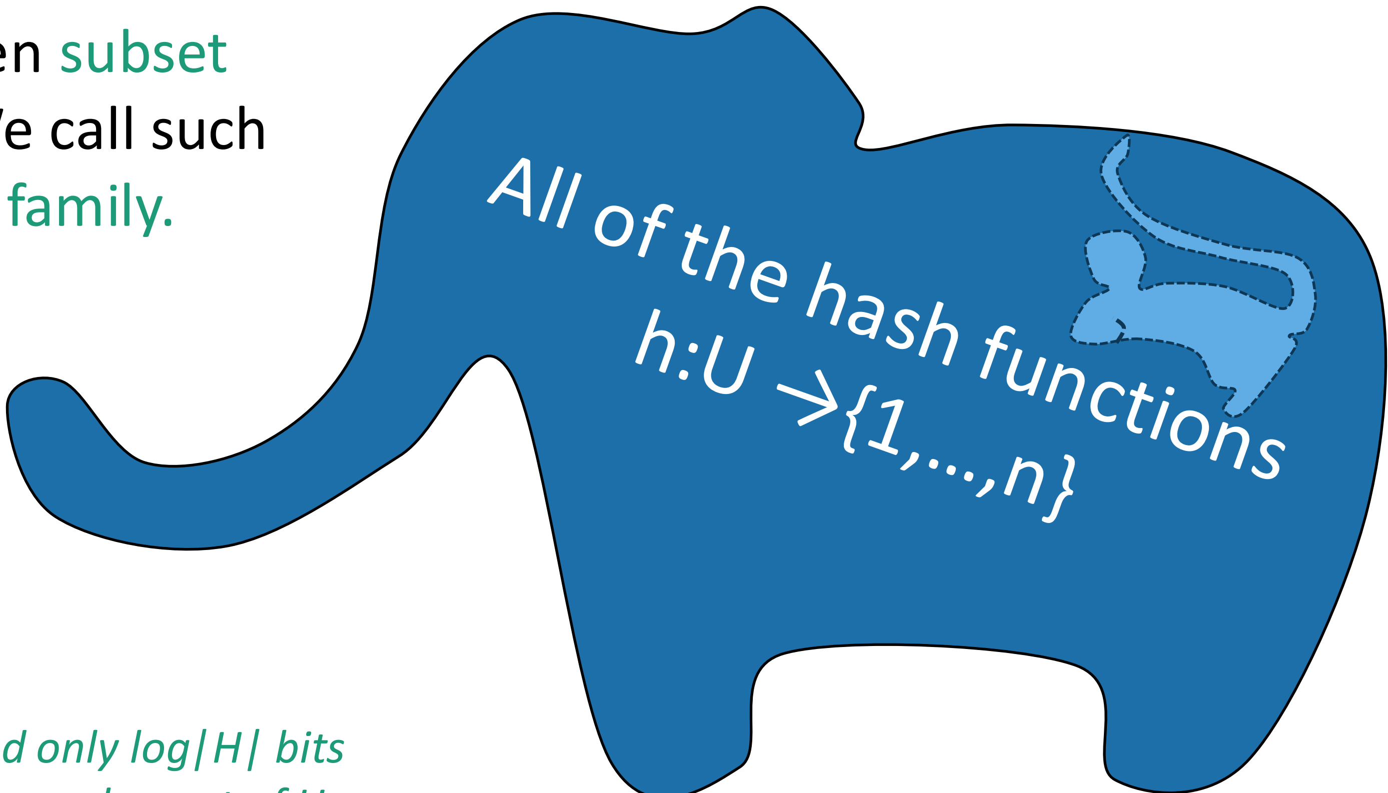
Solution

- Pick from a smaller set of functions.

A cleverly chosen subset of functions. We call such a subset a hash family.



We need only log|H| bits to store an element of H.

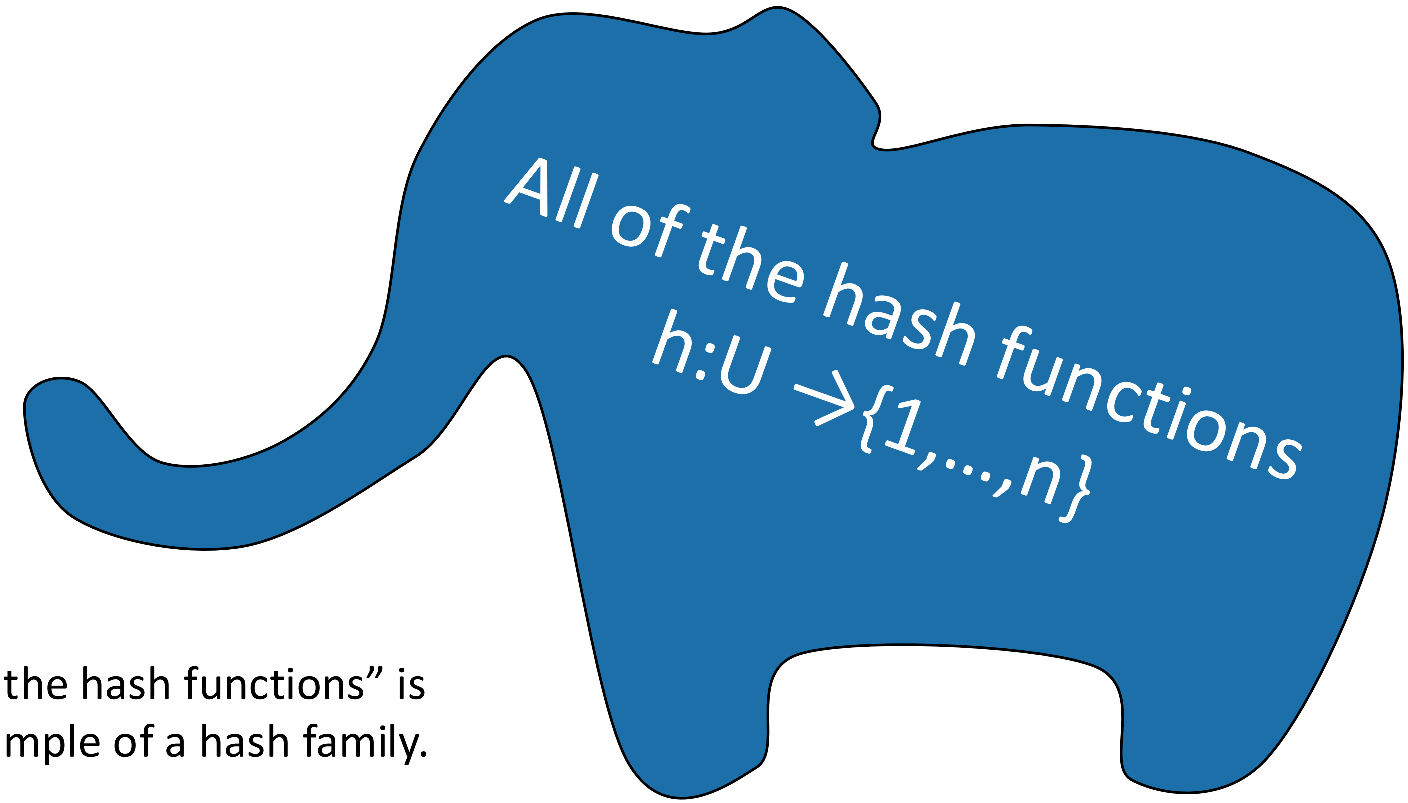


Outline

- **Hash tables** are another sort of data structure that allows fast INSERT/DELETE/SEARCH.
  - like self-balancing binary trees
  - The difference is we can get better performance in expectation by using randomness.
- **Hash families** are the magic behind hash tables.
- **Universal hash families** are even more magic.

Hash families

- A hash family is a collection of hash functions.

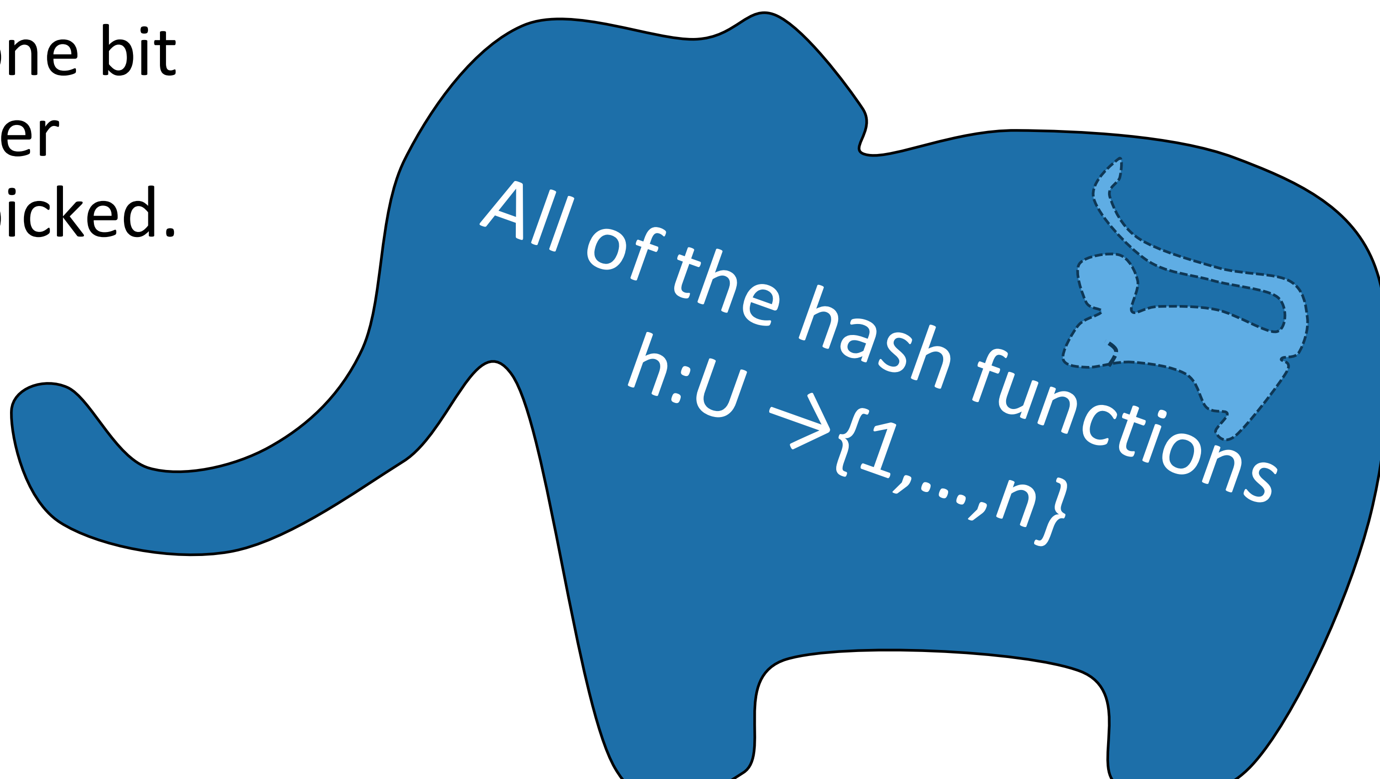


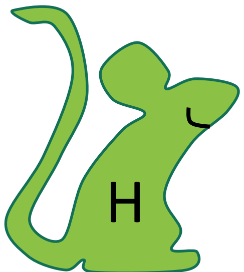
"All of the hash functions" is an example of a hash family.

Example:
a smaller hash family

This is still a terrible idea!
Don't use this example!
For pedagogical purposes only!

- H = { function which returns the least sig. digit,
  function which returns the most sig. digit }
- Pick h in H at random.
- Store just one bit to remember which we picked.



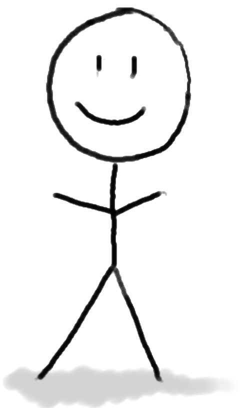


The game

h_0 = Most_significant_digit
h_1 = Least_significant_digit
H = $\{h_0, h_1\}$

2. You, the algorithm, chooses a **random** hash function $h: U \to \{0, \ldots, 9\}$. Choose it randomly from H.

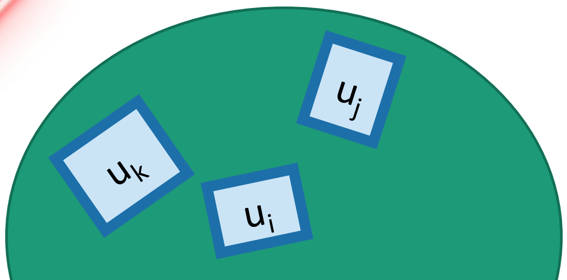
I picked h_1



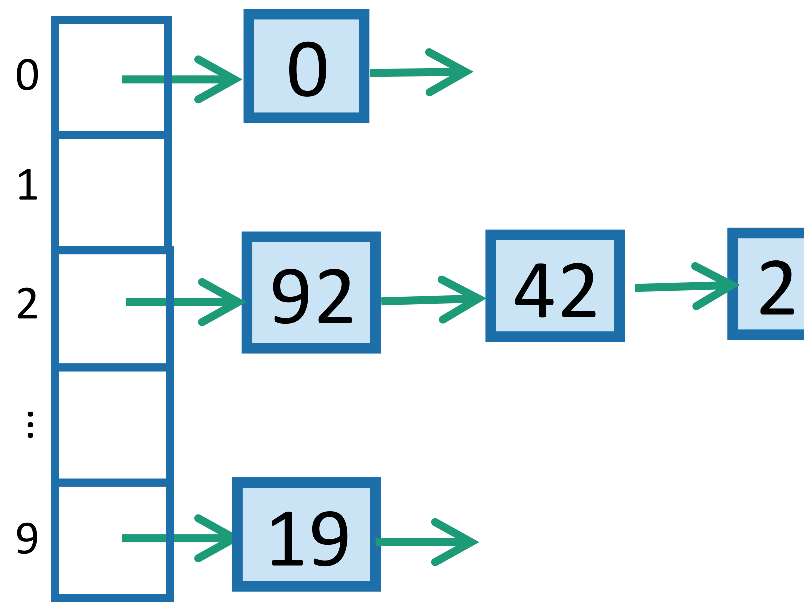
1. An adversary (who knows H) chooses any n items $u_1, u_2, \ldots, u_n \in U$, and any sequence of INSERT/DELETE/SEARCH operations on those items.

19	22	42	92	0

INSERT 19, INSERT 22, INSERT 42, INSERT 92, INSERT 0, SEARCH 42, DELETE 92, SEARCH 0, INSERT 92



3. HASH IT OUT #hashpuns



This is not a very good hash family

- H = { function which returns least sig. digit,

  function which returns most sig. digit }
- On the previous slide, the adversary could have been a lot more adversarial…

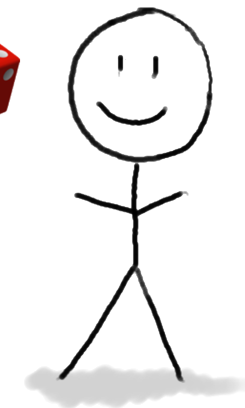
The game

h_0 = Most_significant_digit

h_1 = Least_significant_digit

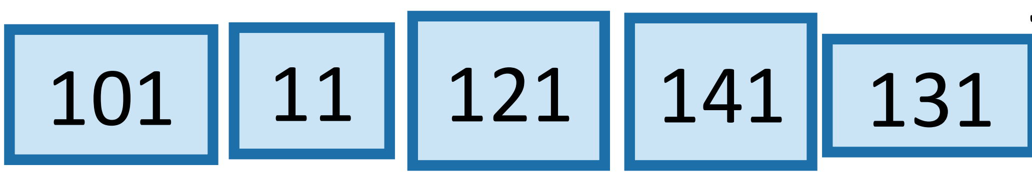
$H = \{h_0, h_1\}$

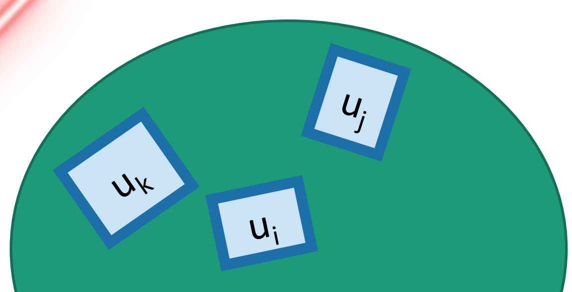
2. You, the algorithm, chooses a **random** hash function $h: U \rightarrow \{0, \ldots, 9\}$. Choose it randomly from H.



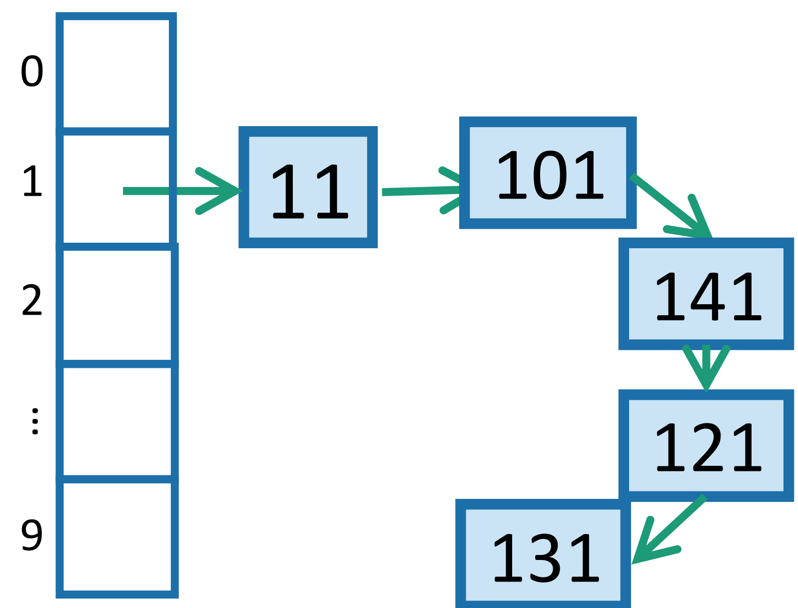
I picked h_0

1. An adversary (who knows H) chooses any n items $u_1, u_2, \ldots, u_n \in U$, and any sequence of INSERT/DELETE/SEARCH operations on those items.





3. HASH IT OUT #hashpuns



Outline

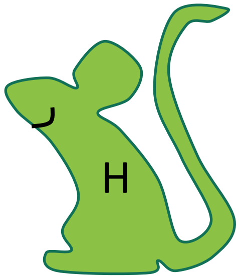
- **Hash tables** are another sort of data structure that allows fast INSERT/DELETE/SEARCH.
  - like self-balancing binary trees
  - The difference is we can get better performance in expectation by using randomness.

- **Hash families** are the magic behind hash tables.

- **Universal hash families** are even more magic.

How to pick the hash family?

- Definitely not like in that example.
- Let’s go back to that computation from earlier….



Proof of Claim

- Let h be a uniformly random hash function.
- Then for all $i = 1, \dots, n$,
  E[number of items in u_i's bucket] ≤ 2.

- $E[$ # items in u_i's bucket $] =$
- $= E\left[\sum_{j=1}^{n} \mathbf{1}\{h(u_i) = h(u_j)\}\right]$
- $= \sum_{j=1}^{n} P\{ h(u_i) = h(u_j)\}$
- $= 1 + \sum_{j \neq i} P\{ h(u_i) = h(u_j)\}$
- $= 1 + \sum_{j \neq i} 1/n$
- $= 1 + \frac{n-1}{n} \leq 2.$

All that we needed was that this is 1/n

*assuming h(u) takes O(1) time to compute.

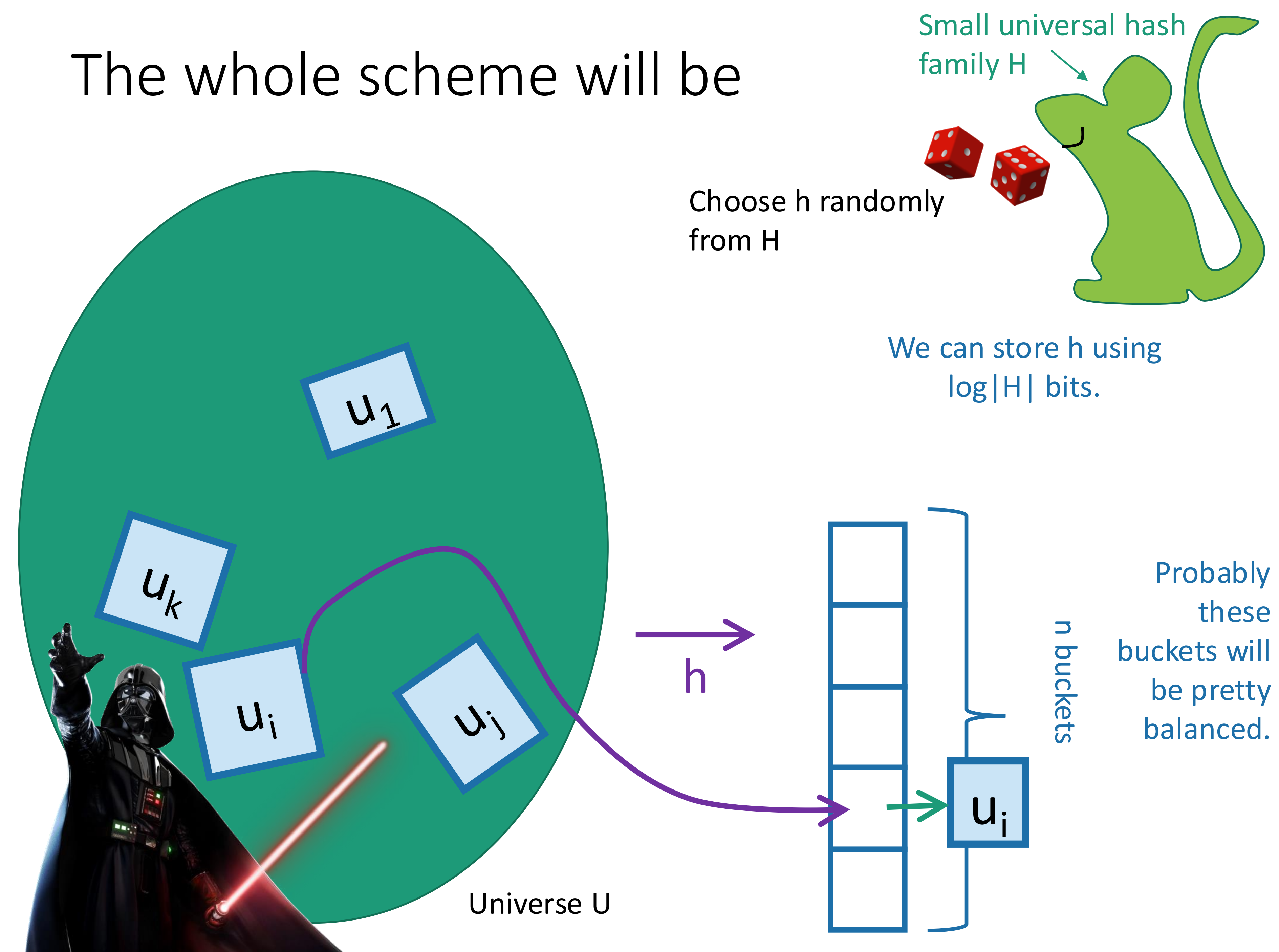
Universal hash families

- H is a ***universal hash family*** if, when h is chosen uniformly at random from H,

$$\text{for all } u_i, u_j \in U \quad \text{with } u_i \neq u_j,$$
$$P_{h \in H}\{\, h(u_i) = h(u_j)\} \leq \frac{1}{n}$$

- Earlier analysis shows: if we draw h uniformly at random from a universal hash family H, we will have expected time* O(1) INSERT/DELETE/SEARCH!
- And if H is small, we can store a random $h \in H$ efficiently!

The whole scheme will be



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Example

- H = the set of all functions $h: U \to \{1, \ldots, n\}$
  - We saw this earlier – it corresponds to picking a uniformly random hash function.
  - Unfortunately this H is really really large.

- **Universal hash family:** if you choose h randomly from H,

$$\text{for all } u_i, u_j \in U \quad \text{with } u_i \neq u_j,$$
$$P_{h \in H}\{ h(u_i) = h(u_j)\} \leq \frac{1}{n}$$

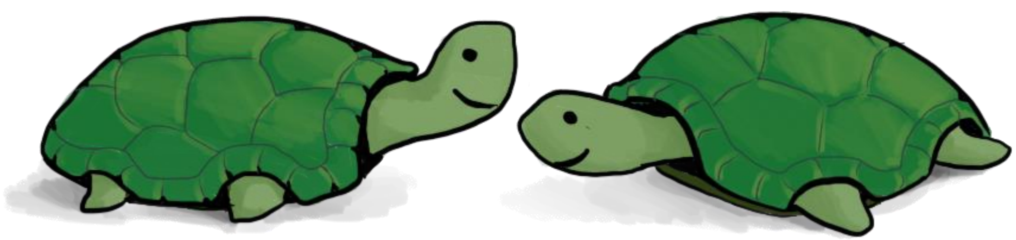
Non-example

- h_0 = Most_significant_digit
- h_1 = Least_significant_digit
- H = $\{h_0, h_1\}$

Prove that this choice of H is NOT a universal hash family!

2 minutes think

1 minute pair and share



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Non-example

- h_0 = Most_significant_digit
- h_1 = Least_significant_digit
- H = $\{h_0, h_1\}$

NOT a universal hash family:

$$P_{h \in H}\{h(101) = h(111)\} = 1 > \frac{1}{10}$$

A small universal hash family??

- Here's one:
  - Pick a prime $p \geq M$. (And not much bigger than M)
  - Define

$$f_{a,b}(x) = ax + b \quad mod\ p$$

$$h_{a,b}(x) = f_{a,b}(x) \quad mod\ n$$

  - Define:

$$H = \{\ h_{a,b}(x)\ :\ a \in \{1, \ldots, p-1\}, b \in \{0, \ldots, p-1\}\ \}$$

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$$H = \{ h_{a,b}(x) \ : \ a \in \{1, \ldots, p-1\}, b \in \{0, \ldots, p-1\} \}$$

See CLRS (Thm 11.5) if you are curious, but you don't need to know why this is true for this class.

- Claims:

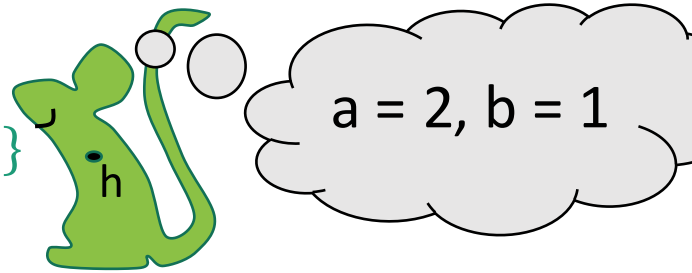
H is a universal hash family.

A random $h \in H$ takes $O(\log M)$ bits to store.

A random $h \in H$ takes $O(\log M)$ bits to store (And more!)

$H = \{ h_{a,b}(x) \ : \ a \in \{1, \ldots, p-1\}, b \in \{0, \ldots, p-1\} \}$

$|H| = p \cdot (p-1) = O(M^2)$



- Just need to store two numbers:
  - a is in $\{1, \ldots, p-1\}$
  - b is in $\{0, \ldots, p-1\}$
  - Store a and b with $2\log(p)$ bits
  - By our choice of p (close to M), that's $O(\log(M))$ bits.
- Also, given a and b, h is fast to evaluate!
  - It takes time $O(1)$ to compute $h(x)$.
- Compare: direct addressing was M bits!
  - Example: If M = $\mathbf{128^{280}}$, log(M) = **1960.**

A small universal hash family??

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  - Pick a prime $p \geq M$. (And not much bigger than M)
  - Define

$$f_{a,b}(x) = ax + b \quad mod\ p$$

$$h_{a,b}(x) = f_{a,b}(x) \quad mod\ n$$

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$$H = \{ h_{a,b}(x) \ : \ a \in \{1, \ldots, p-1\}, b \in \{0, \ldots, p-1\} \}$$

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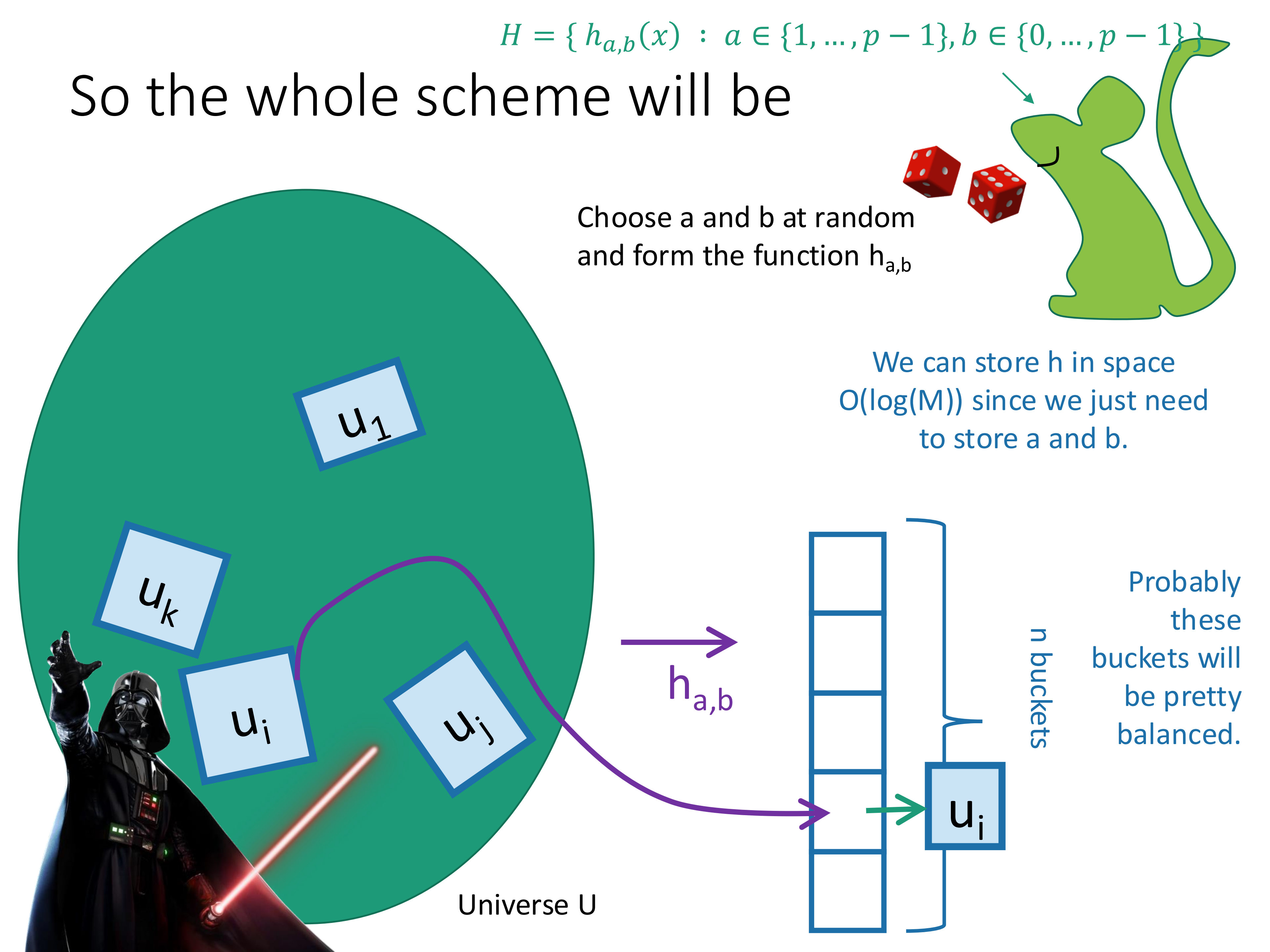
$H = \{\, h_{a,b}(x) \;:\; a \in \{1, \dots, p-1\}, b \in \{0, \dots, p-1\} \,\}$
So the whole scheme will be
Choose a and b at random and form the function $h_{a,b}$
We can store h in space O(log(M)) since we just need to store a and b.
u_1
u_k
u_i
u_j
$h_{a,b}$
n buckets
u_i
Probably these buckets will be pretty balanced.
Universe U

Outline

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- **Hash families** are the magic behind hash tables.

- **Universal hash families** are even more magic.

Recap

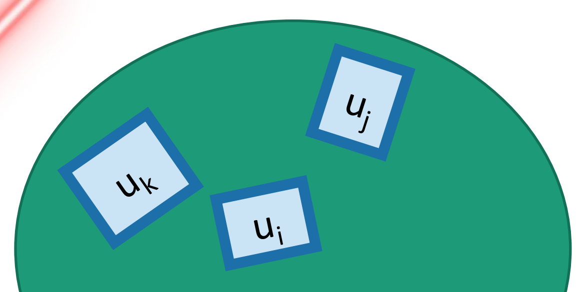
Want O(1)

INSERT/DELETE/SEARCH

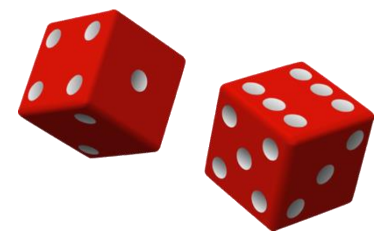
We studied this game

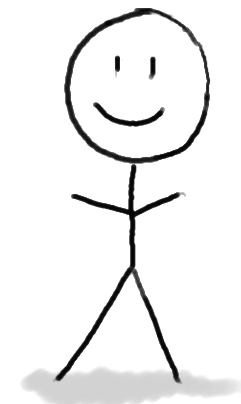
1. An adversary chooses any n items $u_1, u_2, \ldots, u_n \in U$, and any sequence of L INSERT/DELETE/SEARCH operations on those items.

INSERT 13, INSERT 22, INSERT 43, INSERT 92, INSERT 7, SEARCH 43, DELETE 92, SEARCH 7, INSERT 92

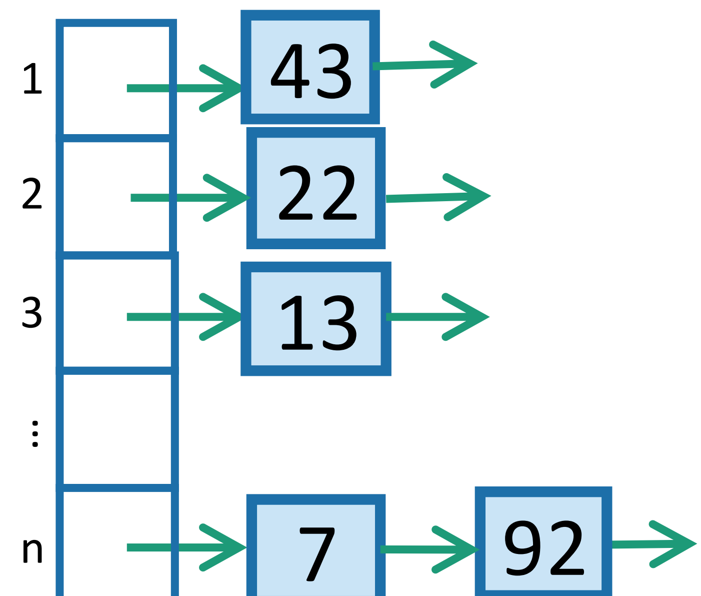


2. You, the algorithm, chooses a **random** hash function $h: U \to \{1, \ldots, n\}$.





3. **HASH IT OUT**



Uniformly random h was good

- If we choose h uniformly at random,

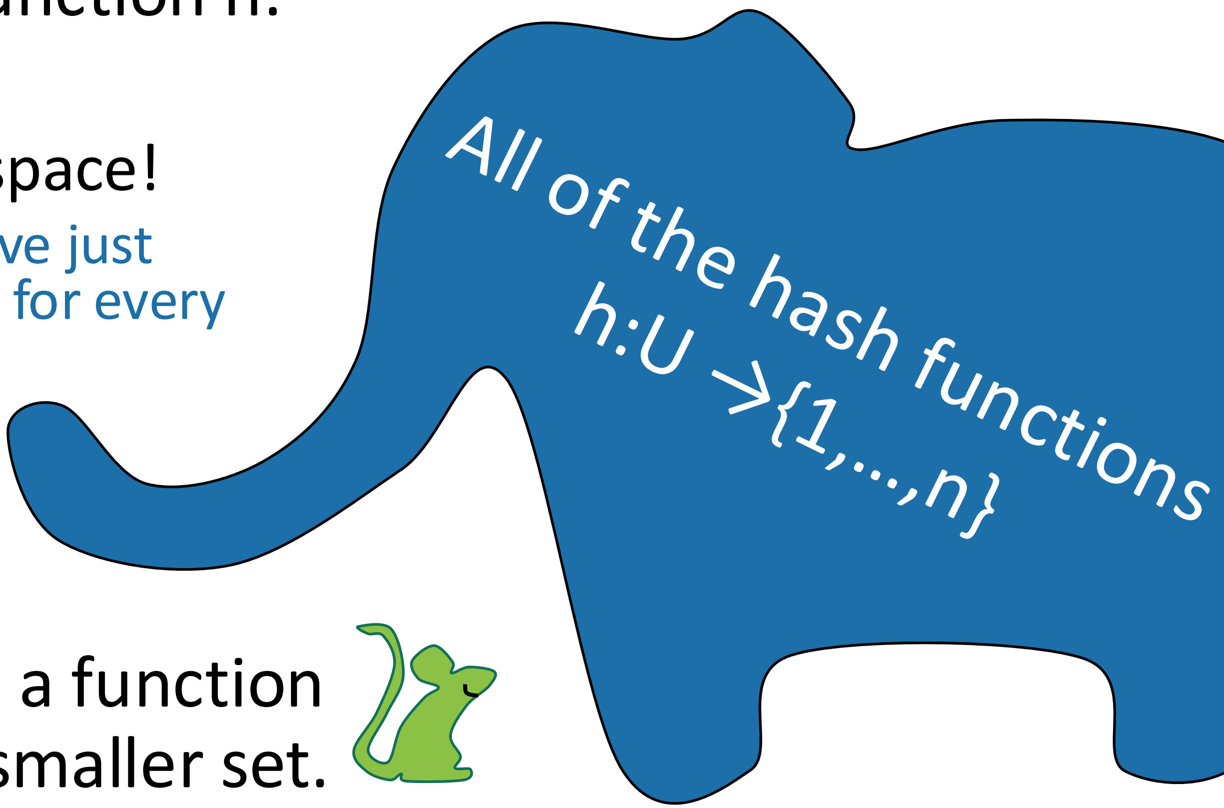
$$\text{for all } u_i, u_j \in U \quad \text{with } u_i \neq u_j,$$
$$P_{h \in H}\{\, h(u_i) = h(u_j)\} \leq \frac{1}{n}$$

- That was enough to ensure that all INSERT/DELETE/SEARCH operations took O(1) time in expectation, even on adversarial inputs.

Uniformly random h was bad

- If we actually want to implement this, we have to store the hash function h.

- That takes a lot of space!
  - We may as well have just initialized a bucket for every single item in U.



- Instead, we chose a function randomly from a smaller set.

Universal Hash Families

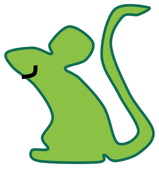
H is a universal hash family if:

- If we choose h uniformly at random in H,

$$\text{for all } u_i, u_j \in U \quad \text{with } u_i \neq u_j,$$
$$P_{h \in H}\{\, h(u_i) = h(u_j) \} \leq \frac{1}{n}$$

This was all we needed to make sure that the buckets were balanced in expectation!

- We gave an example of a really small universal hash family, of size $O(M^2)$
- That means we need only O(log M) bits to store it.

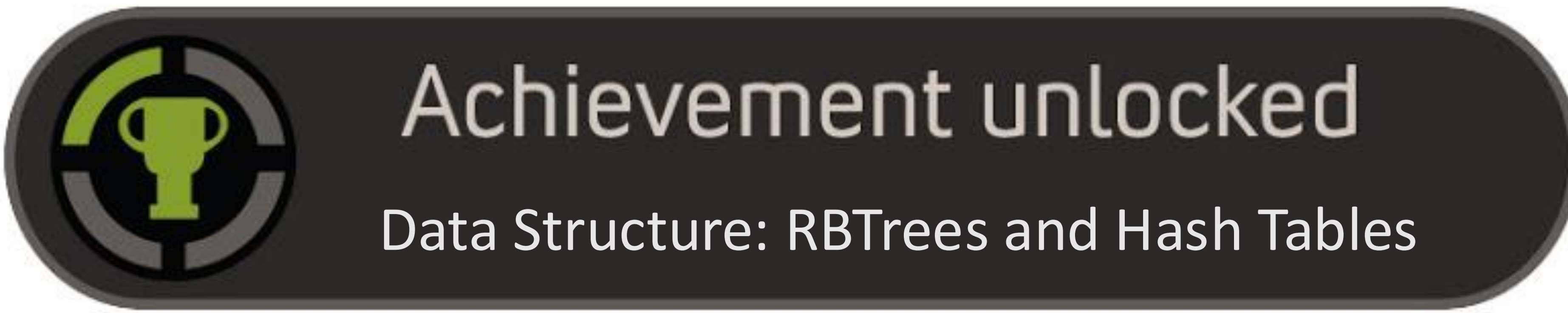


Hashing a universe of size M into n buckets, where at most n of the items in M ever show up.

Conclusion:

- We can build a hash table that supports INSERT/DELETE/SEARCH in O(1) expected time
- Requires O(n log(M)) bits of space.
  - O(n) buckets
  - O(n) items with log(M) bits per item
  - O(log(M)) to store the hash function

That's it for data structures (for now)



Now we can use these going forward!