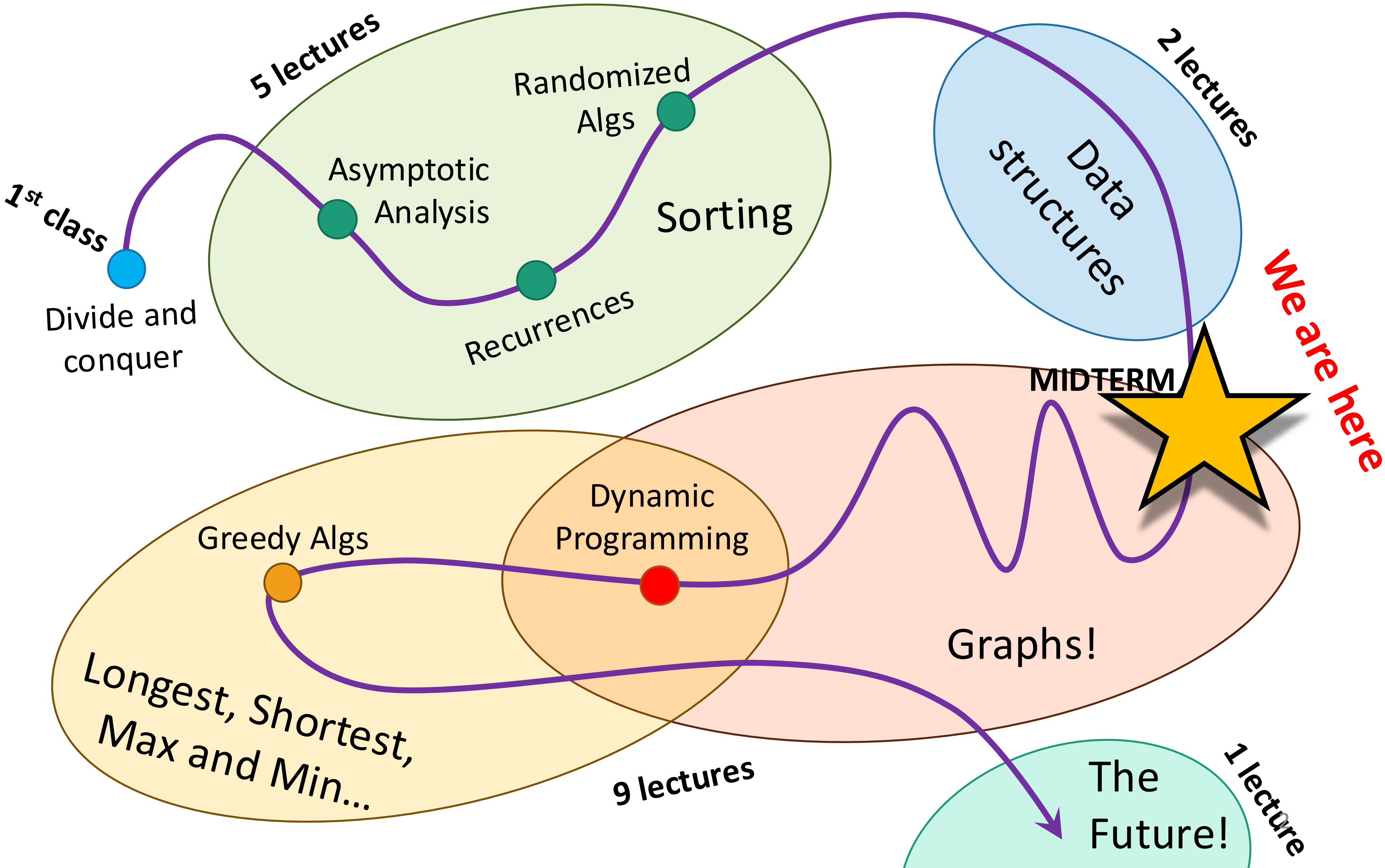
Lecture 9

Graphs, BFS and DFS

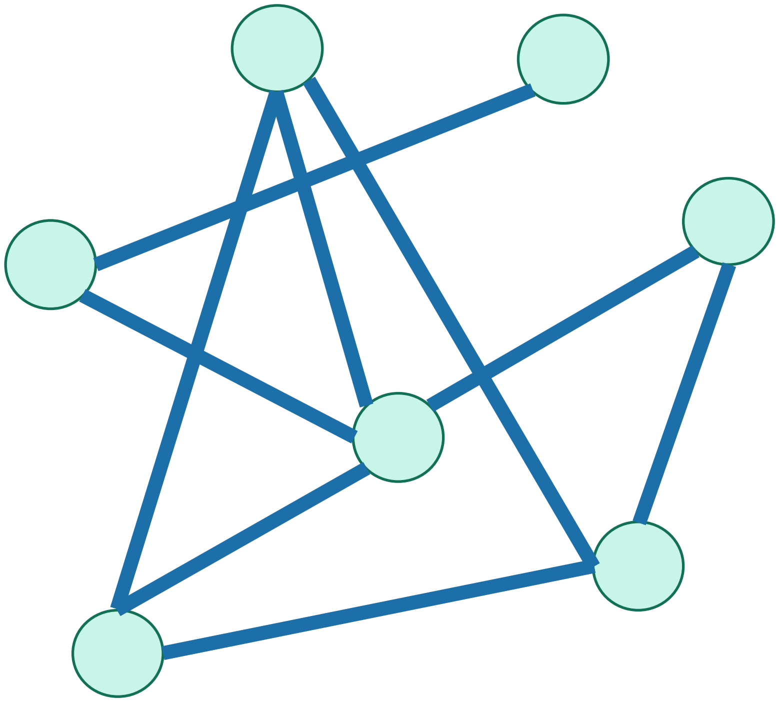
More detailed schedule on the website!

Roadmap



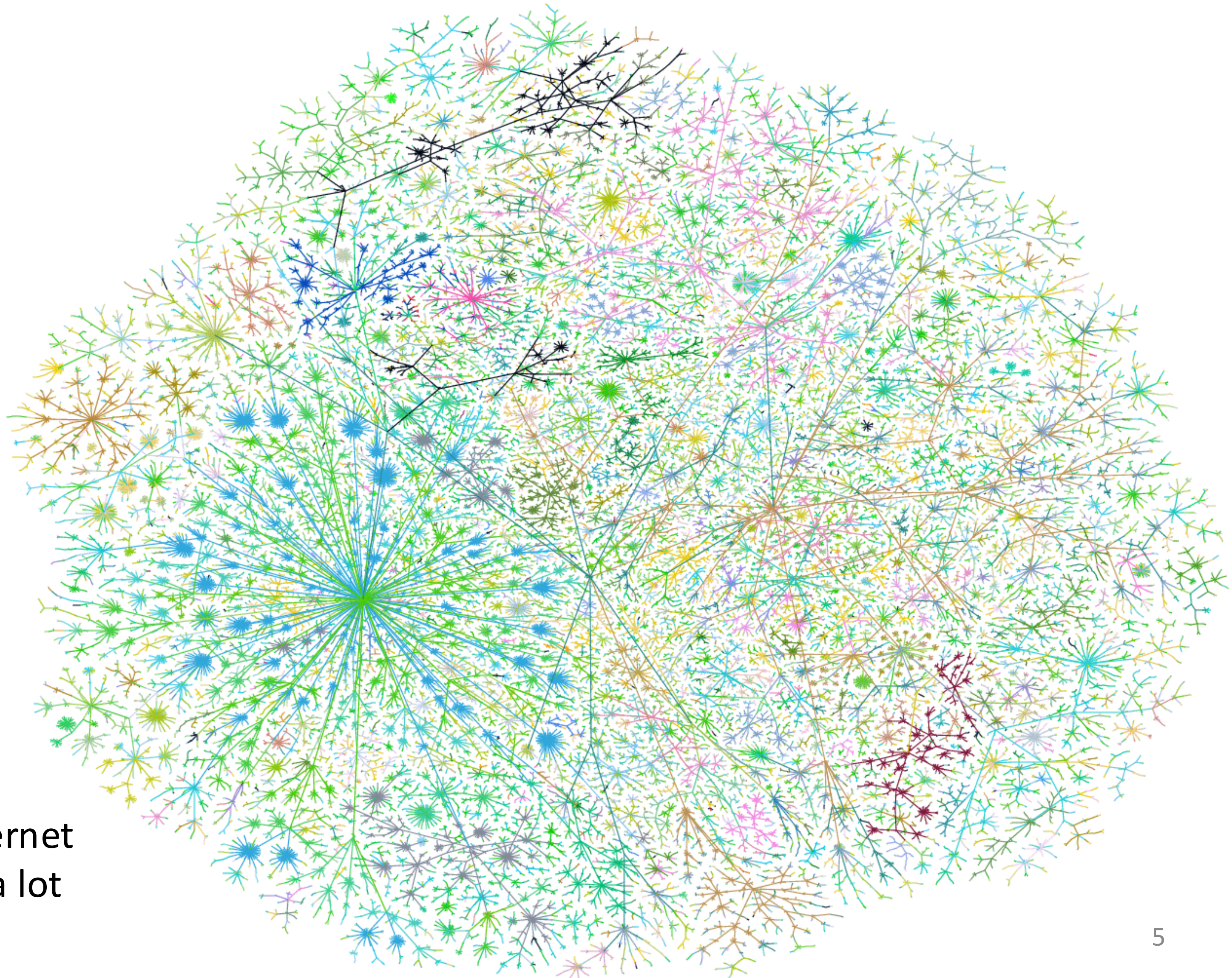
Outline

- Part 0: Graphs and terminology
- Part 1: Depth-first search
  - Application: topological sorting
  - Application: in-order traversal of BSTs
- Part 2: Breadth-first search
  - Application: shortest paths
  - Application (if time): is a graph bipartite?



Part 0: Graphs

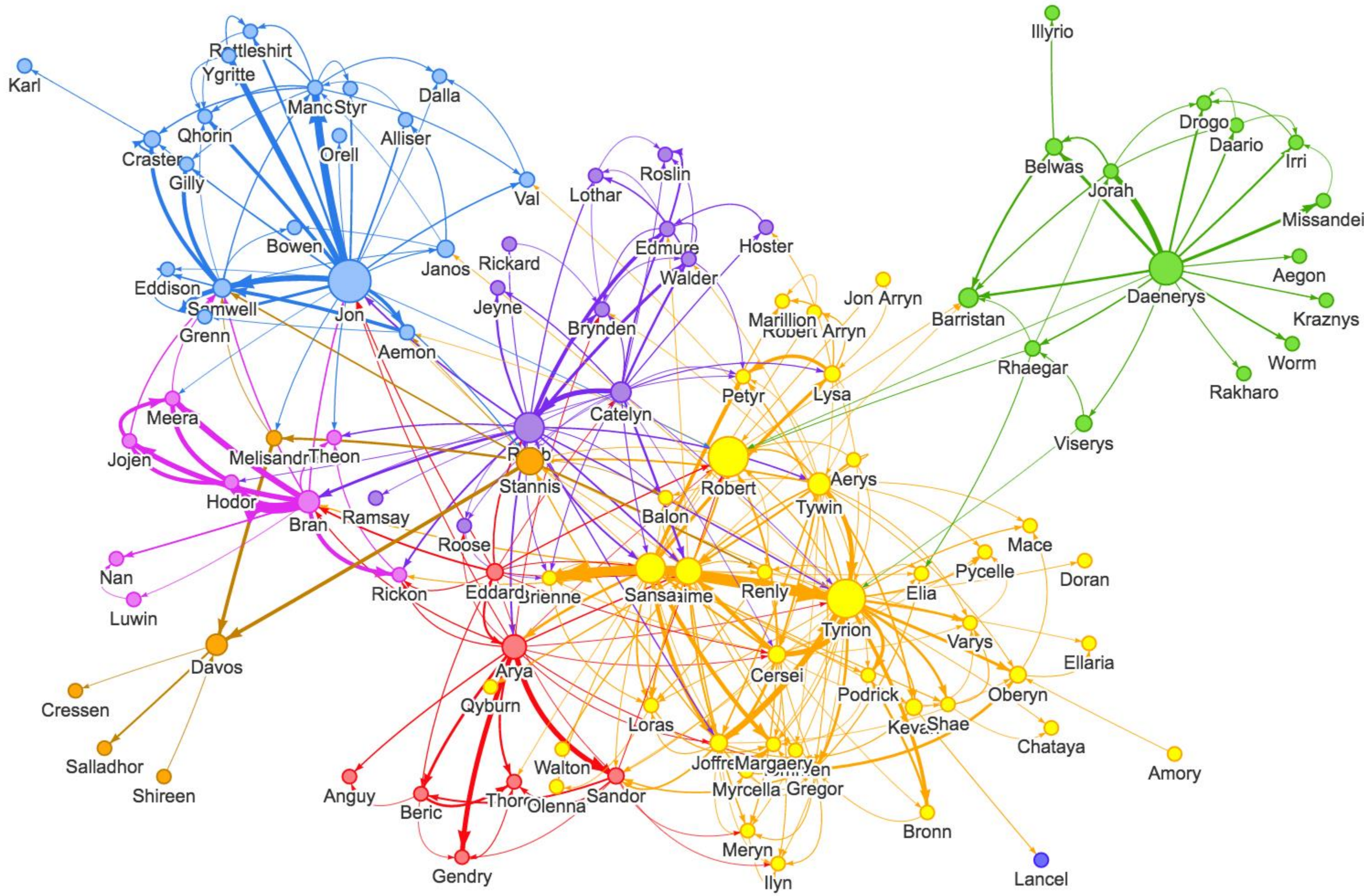
Graphs



Graph of the internet
(circa 1999…it's a lot
bigger now…)

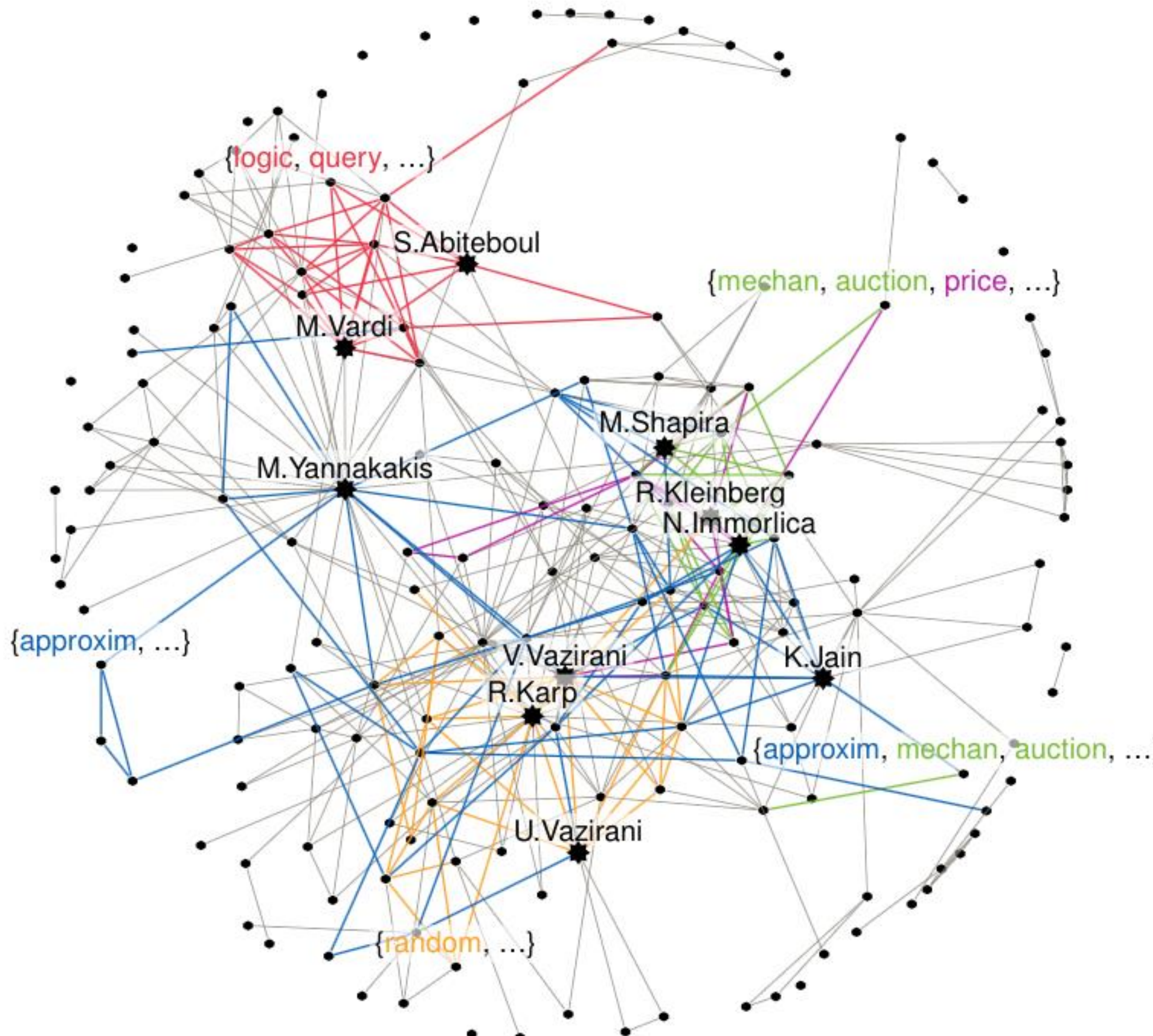
Graphs

Game of Thrones Character Interaction Network



Graphs

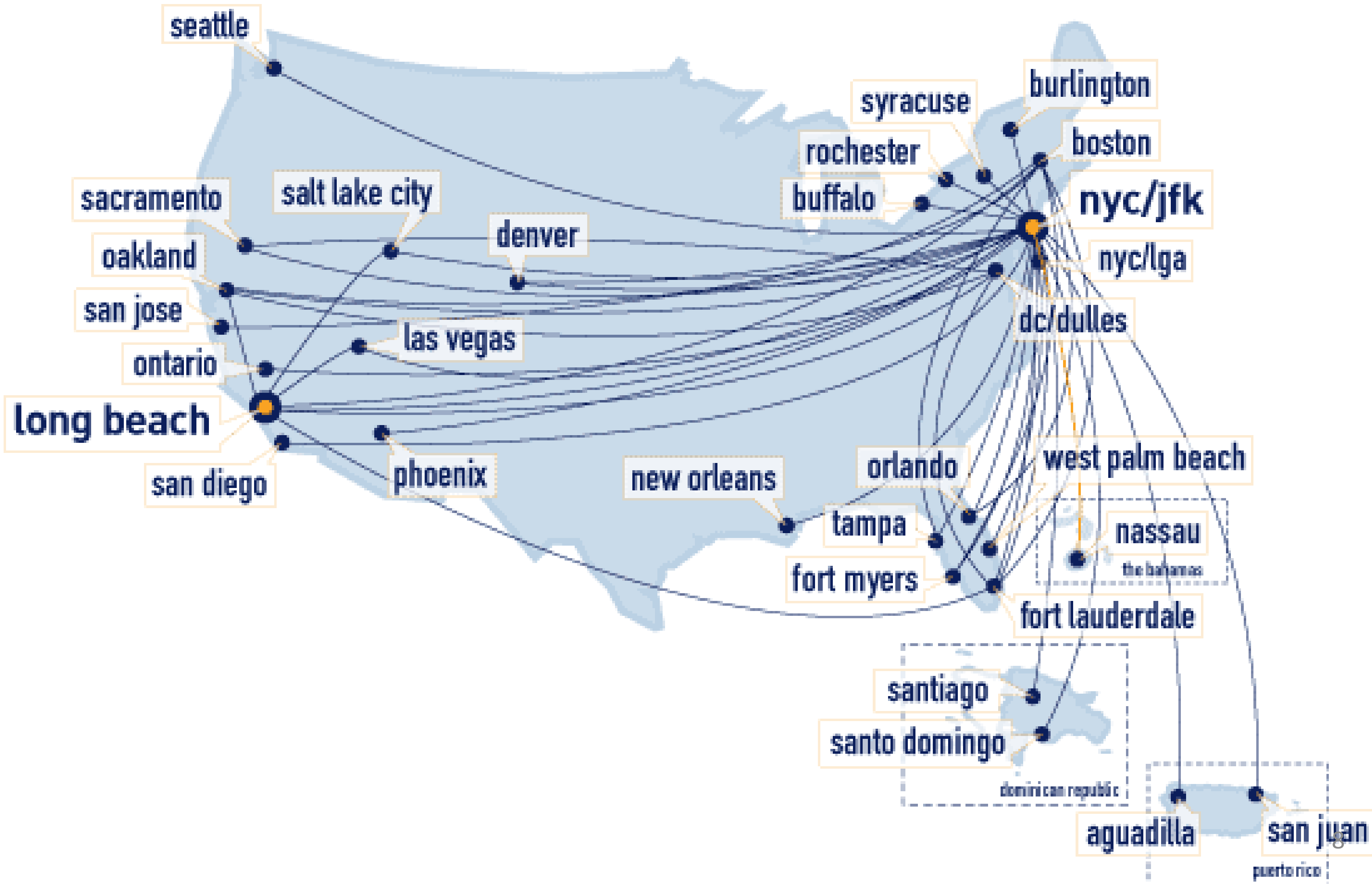
Theoretical Computer Science academic communities



Example from DBLP:
Communities within the co-authors of Christos H. Papadimitriou

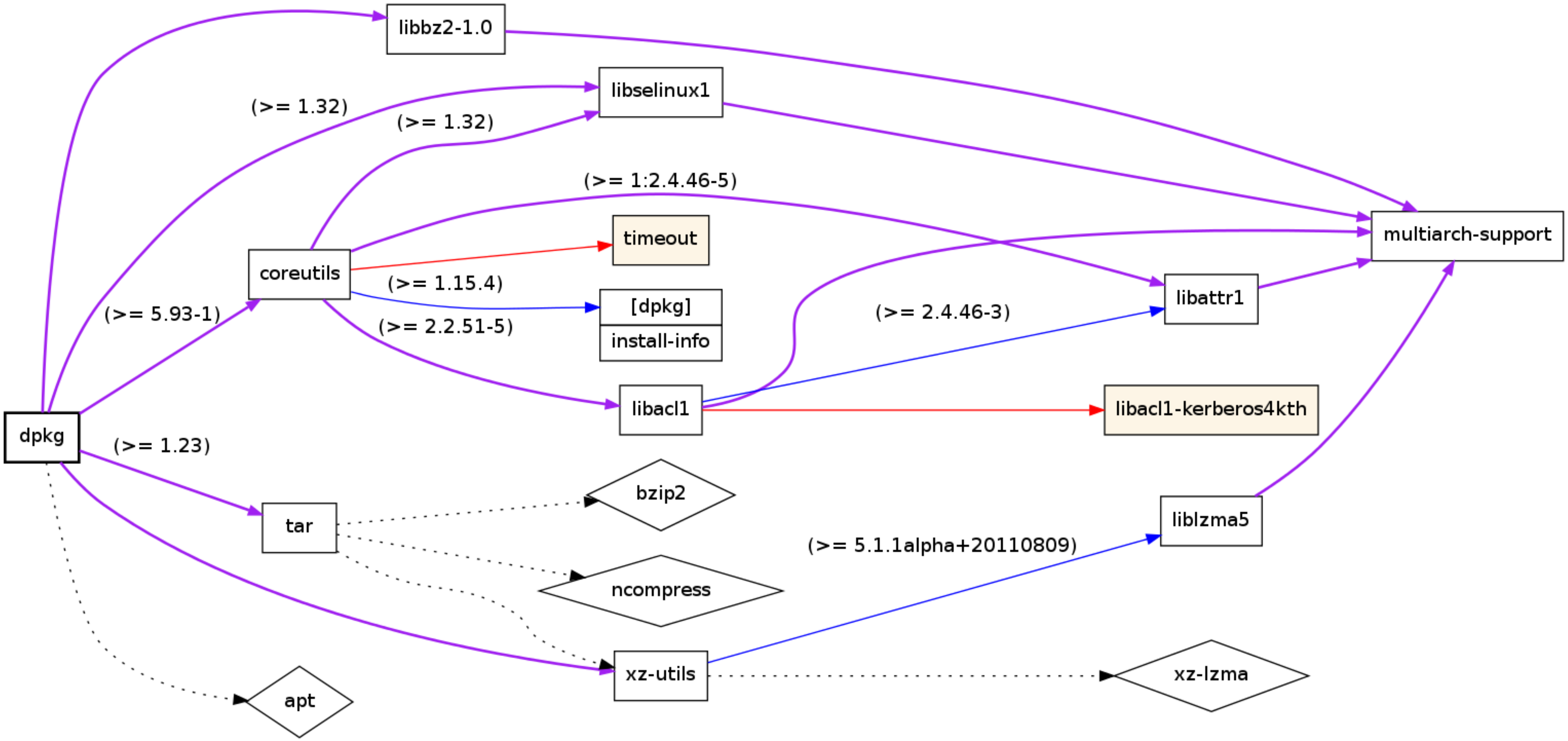
Graphs

jetblue flights

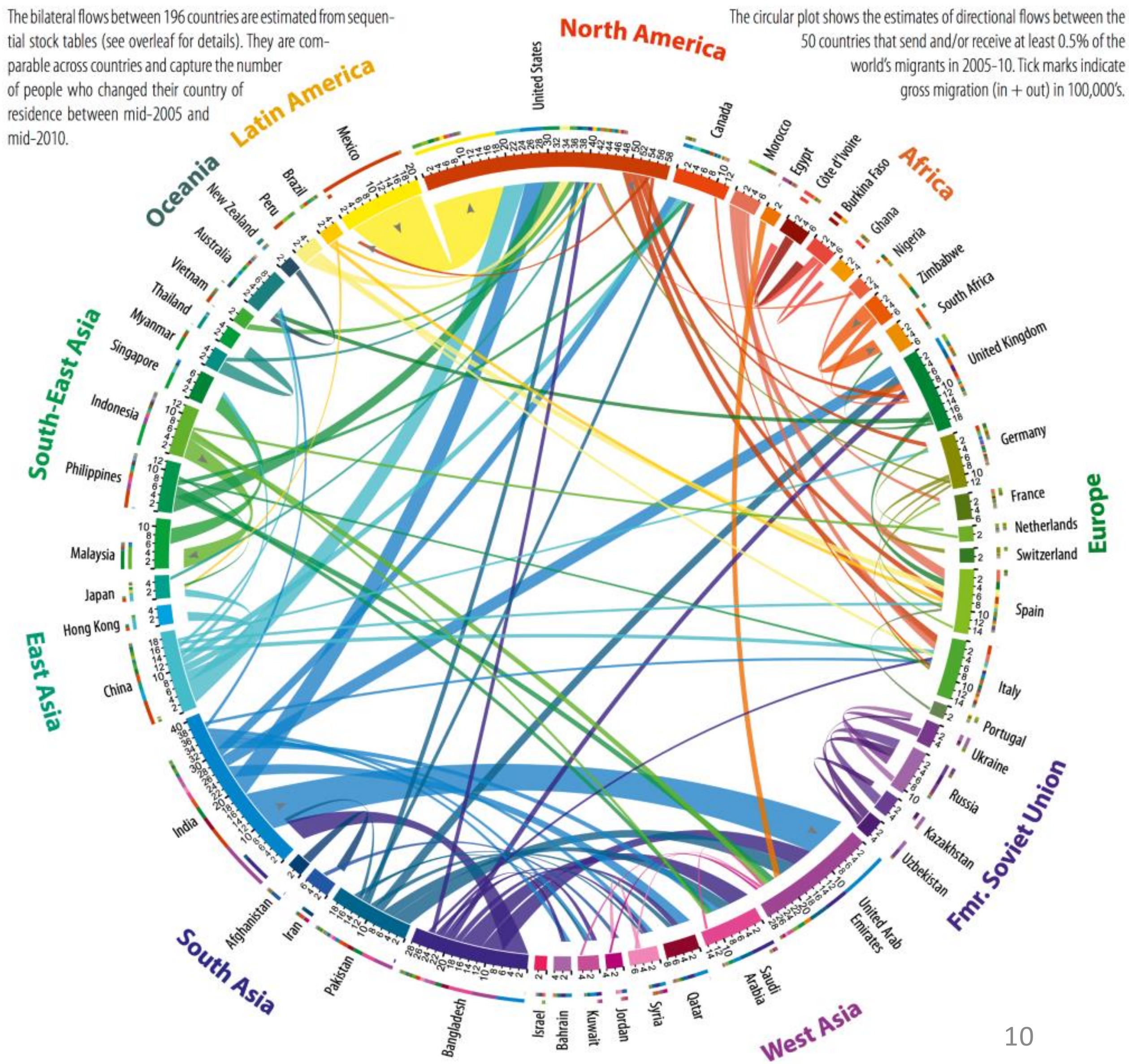


Graphs

debian dependency (sub)graph



Graphs

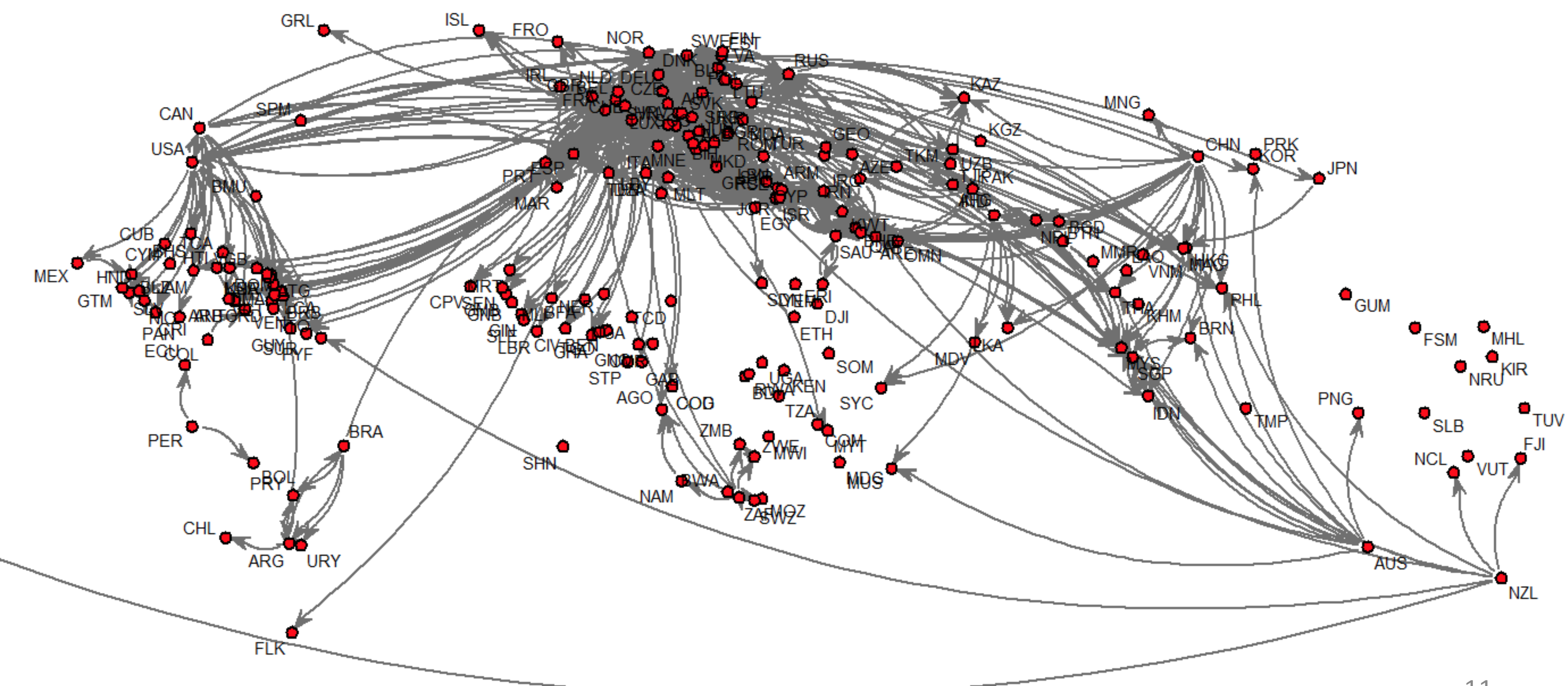


Immigration flows

Graphs

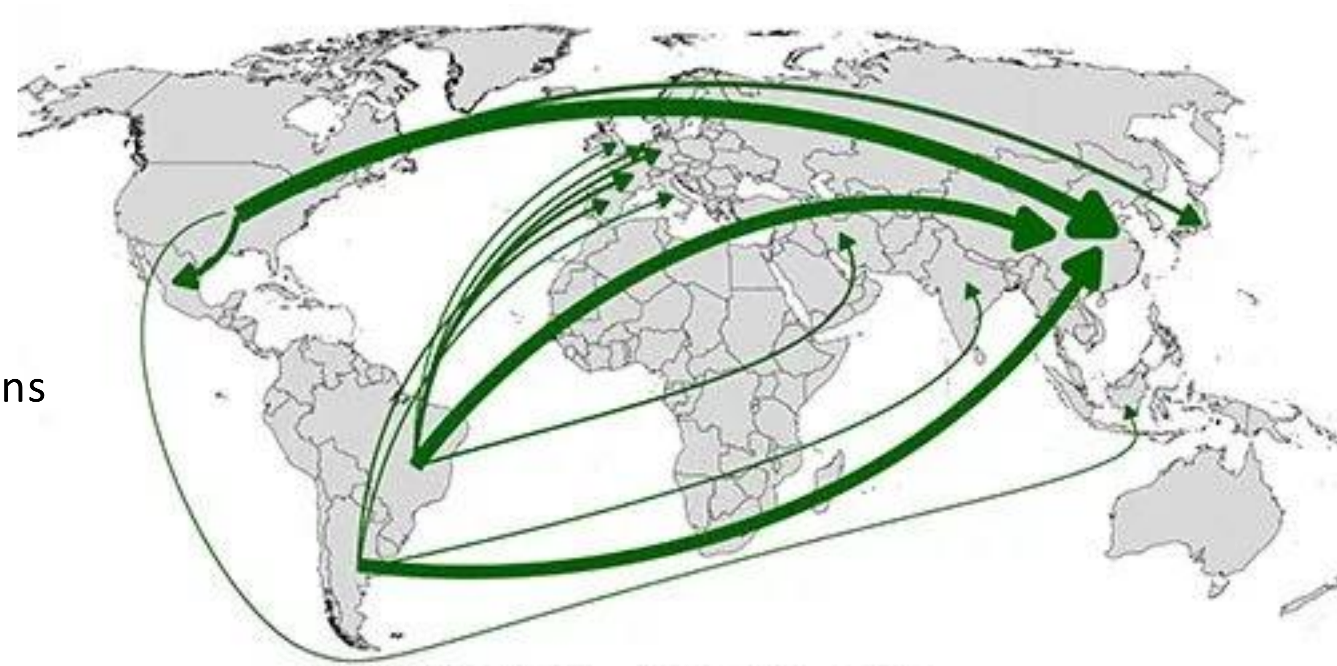
Potato trade

World trade in fresh potatoes, flows over 0.1 m US$ average 2005-2009

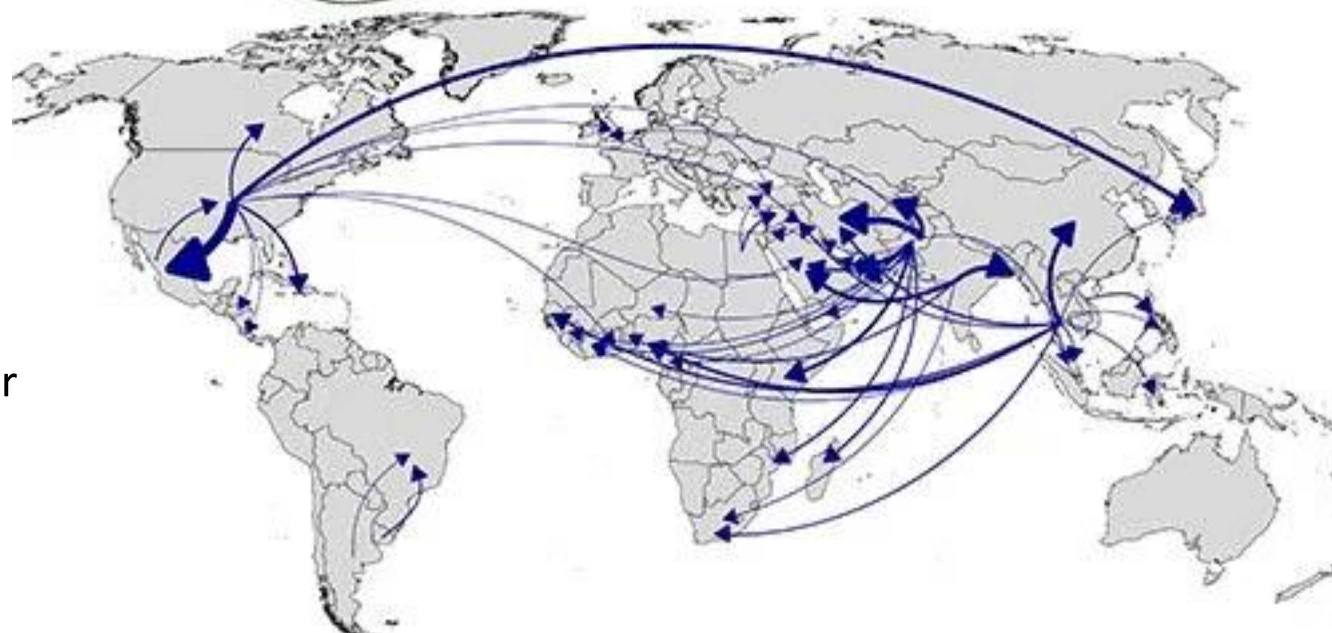


Graphs

Soybeans

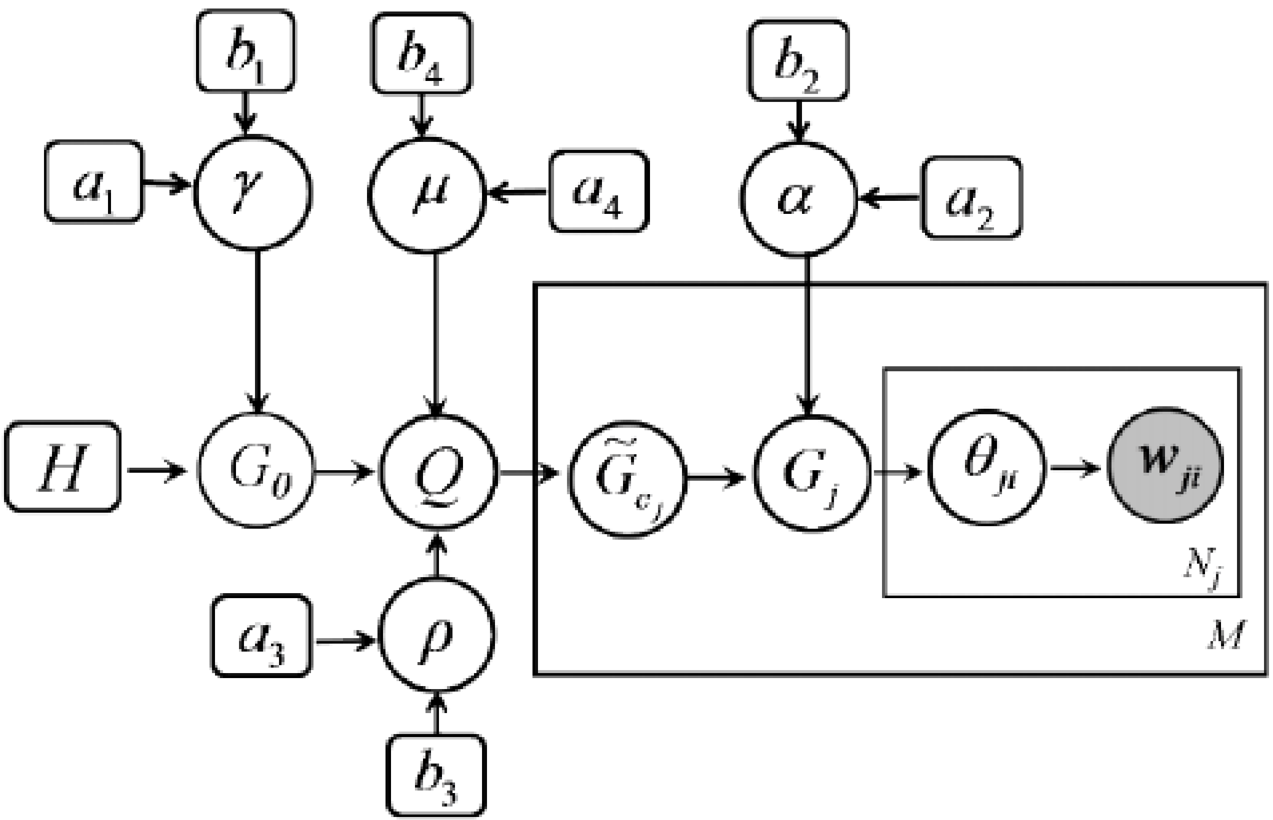


Water



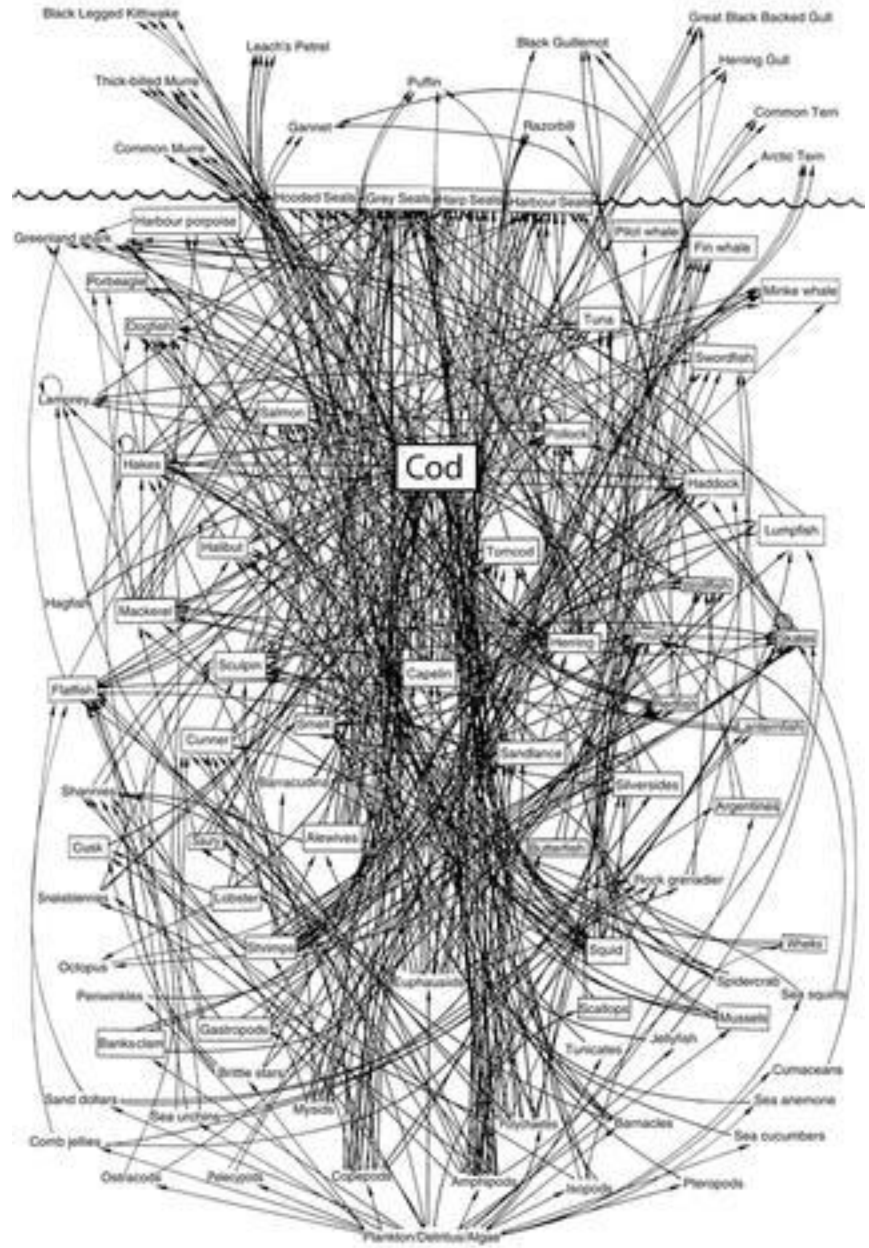
Graphs

Graphical models



Graphs

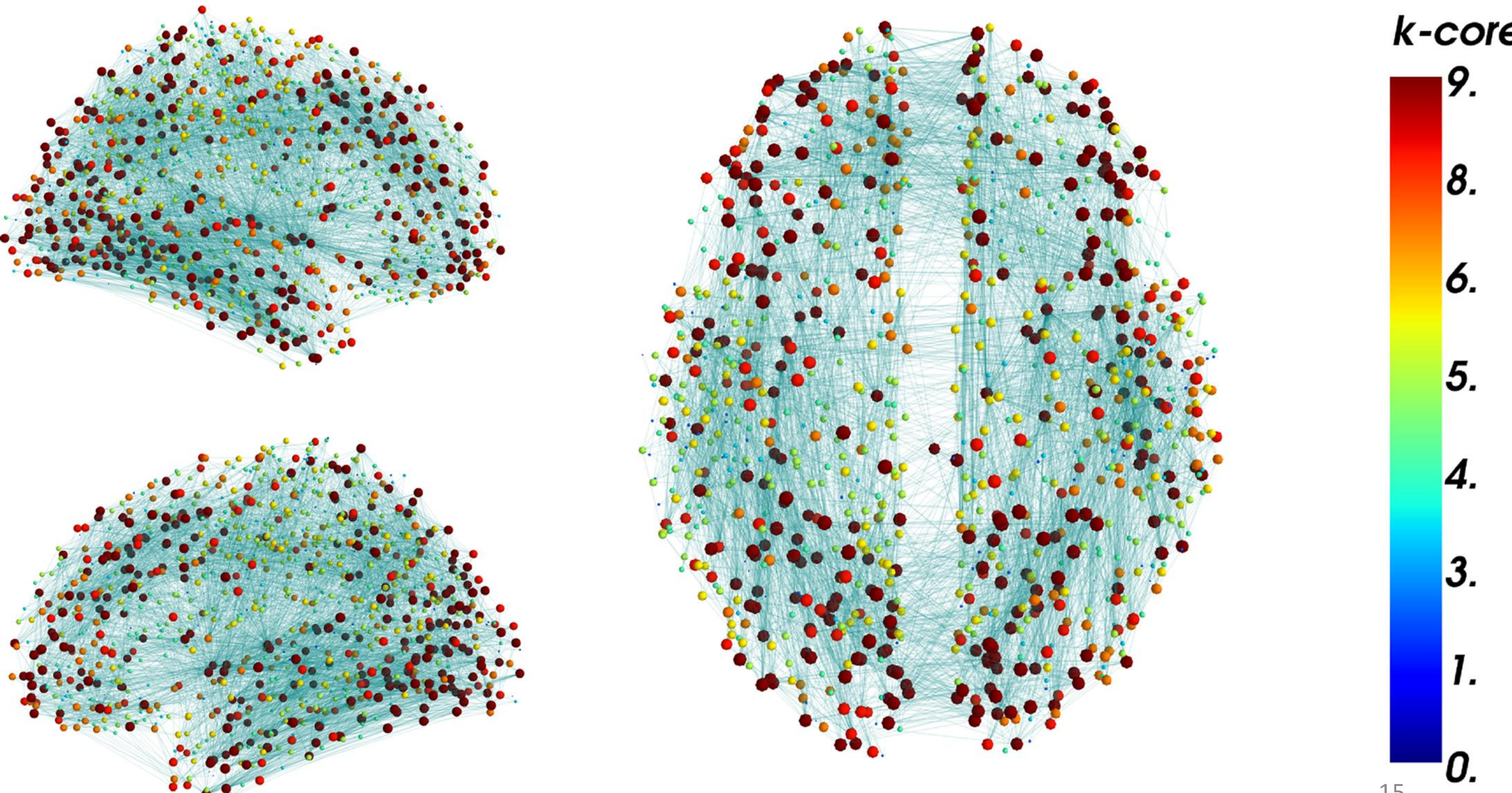
What eats what in the Atlantic ocean?



A simplified food web for the Northwest Atlantic. © IMMA

Graphs

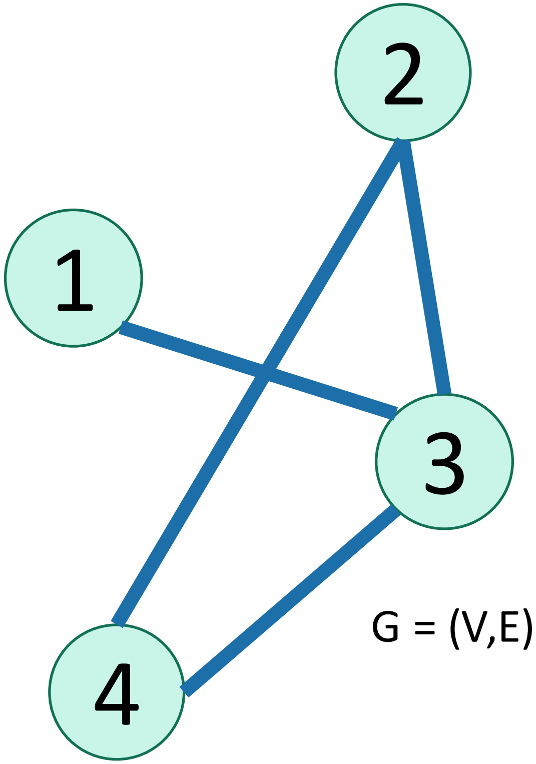
Neural connections in the brain



Graphs

- **There are a lot of graphs.**
- We want to answer questions about them.
  - Efficient routing?
  - Community detection/clustering?
  - From pre-lecture exercise:
    - Computing Bacon numbers
    - Signing up for classes without violating pre-req constraints
    - How to distribute fish in tanks so that none of them will fight.
- This is what we'll do for the next several lectures.

Undirected Graphs

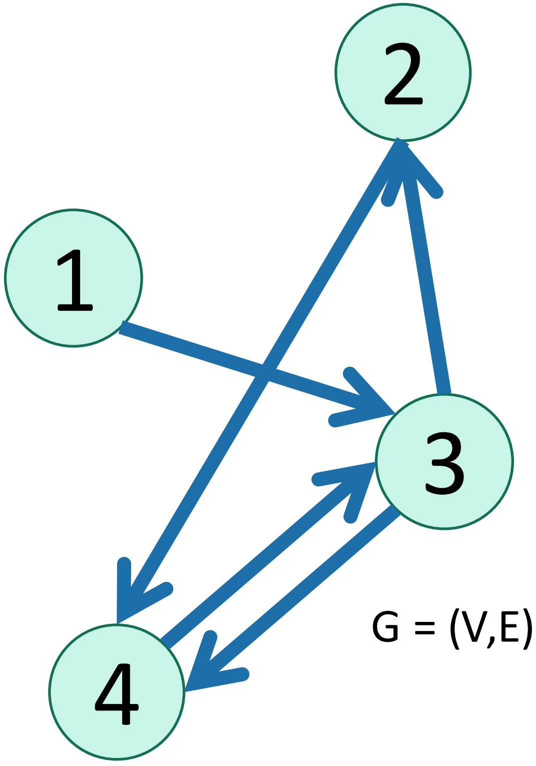


- An **undirected** graph G has:
  - A set V of vertices
  - A set E of edges
  - Formally, G = (V,E)
- The **degree** of vertex is the number of edges coming out.
- The connected vertices are called **neighbors**.
- Example
  - V = {1,2,3,4}
  - E = { {1,3}, {2,4}, {3,4}, {2,3} }

- The **degree** of vertex 4 is 2.
  - There are 2 edges coming out.
- Vertex 4's **neighbors** are 2 and 3

Directed Graphs

- A **directed** graph G has:
  - A set V of vertices
  - A set E of **DIRECTED** edges
  - Formally, G = (V,E)
- The **in-degree** of vertex is the number of edges coming in.
- The **out-degree** of vertex is the number of edges going out.
- Example
  - V = {1,2,3,4}
  - E = { (1,3), (2,4), (3,4), (4,3), (3,2) }

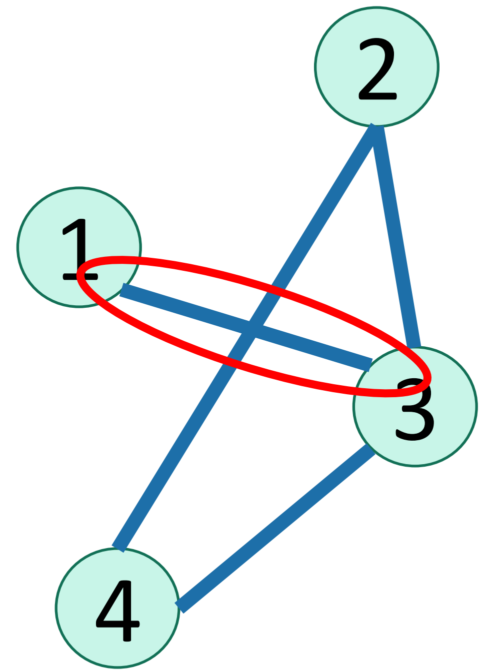


- The **in-degree** of vertex 4 is 2.
- The **out-degree** of vertex 4 is 1.
- Vertex 4's **incoming neighbors** are 2,3
- Vertex 4's **outgoing neighbor** is 3.

How do we represent graphs?

- Option 1: adjacency matrix

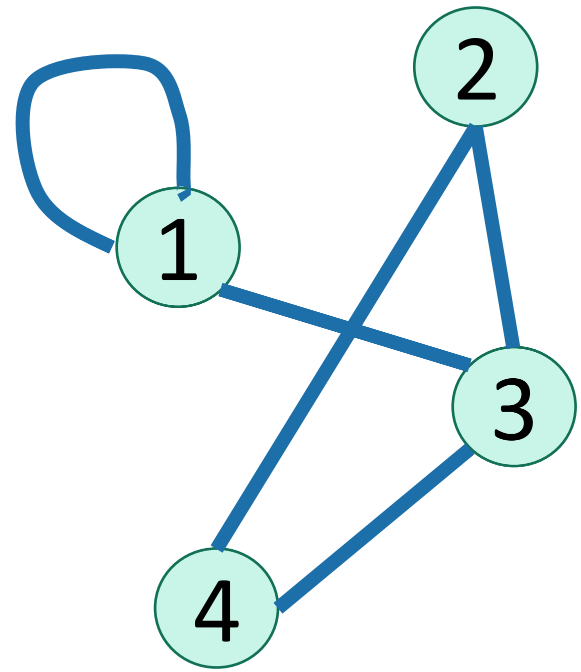
$$
\begin{array}{c c}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} &
\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}
\end{array}
$$



How do we represent graphs?

- Option 1: adjacency matrix

$$
\begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
1 & 1 & 0 & 1 & 0 \\
2 & 0 & 0 & 1 & 1 \\
3 & 1 & 1 & 0 & 1 \\
4 & 0 & 1 & 1 & 0
\end{array}
\qquad
\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}
$$



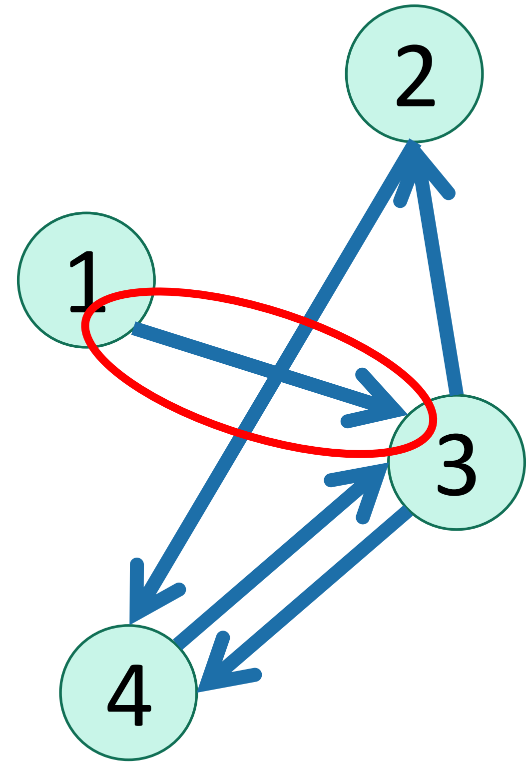
How do we represent graphs?

- Option 1: adjacency matrix

Destination

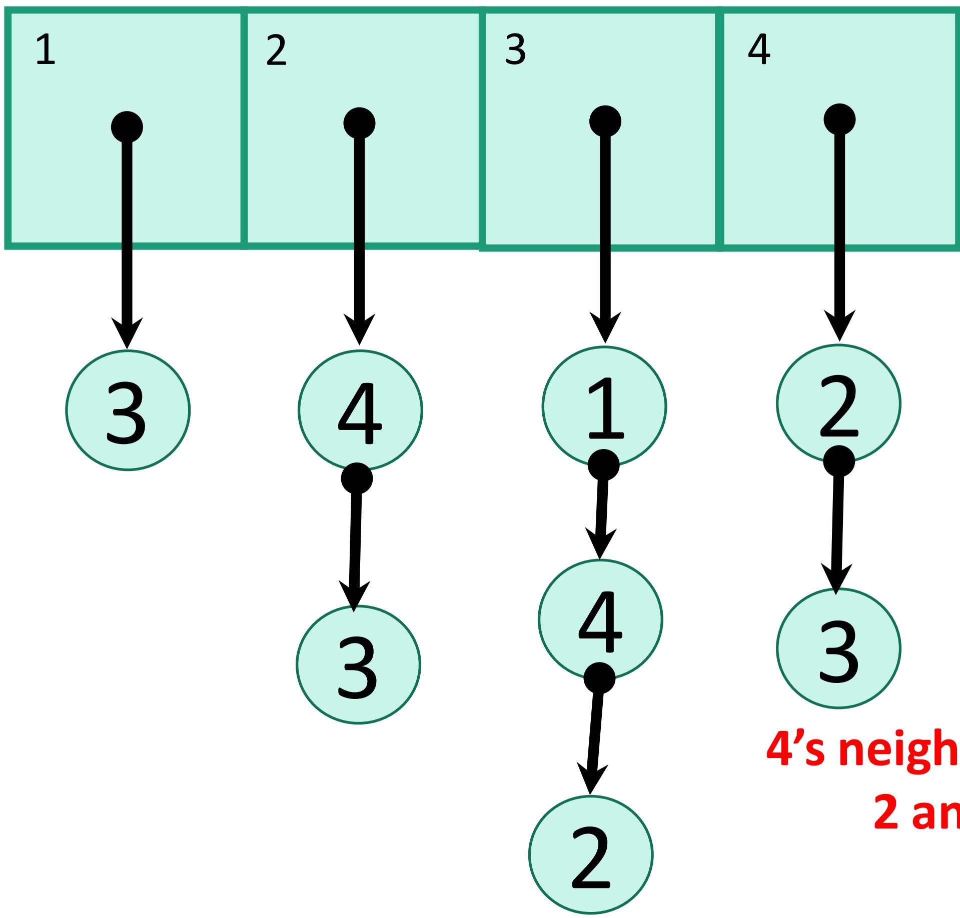
Source

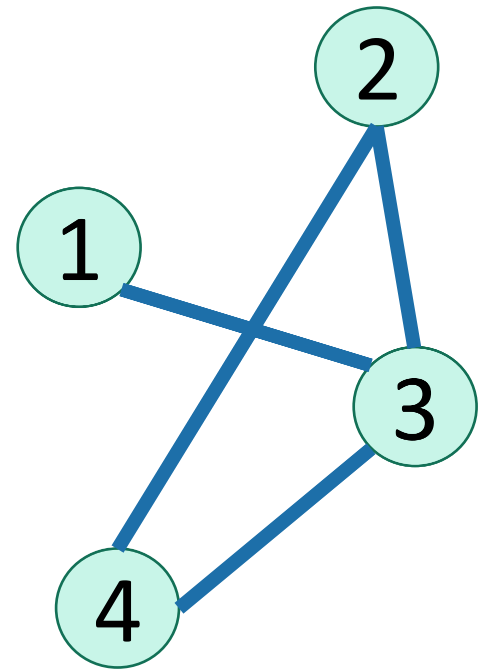
$$
\begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
1 & 0 & 0 & 1 & 0 \\
2 & 0 & 0 & 0 & 1 \\
3 & 0 & 1 & 0 & 1 \\
4 & 0 & 0 & 1 & 0
\end{array}
$$



How do we represent graphs?

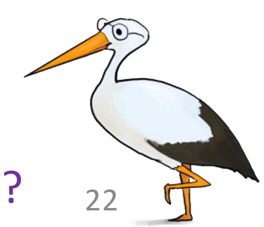
- Option 2: adjacency lists.





4's neighbors are 2 and 3

How would you modify this for directed graphs?

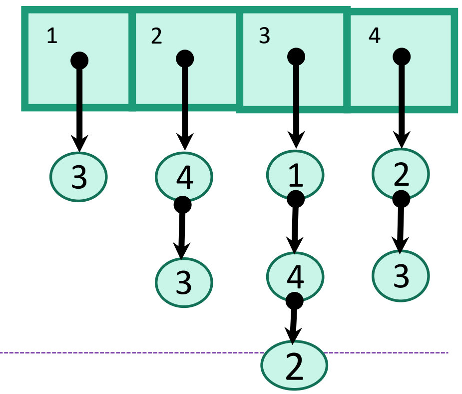


In either case

- Vertices can store other information
  - Attributes (name, IP address, …)
  - helper info for algorithms that we will perform on the graph
- Basic operations:
  - **Edge Membership**: Is edge e in E?
  - **Neighbor Query**: What are the neighbors of vertex v?

Trade-offs

Generally better for **sparse** graphs (where $m \ll n^2$)

Say there are n vertices and m edges.	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$	
Edge membership Is e = {v,w} in E?	O(1)	O(deg(v)) or O(deg(w))
Neighbor query Give me v's neighbors.	O(n)	O(deg(v))
Space requirements	$O(n^2)$	O(n + m)

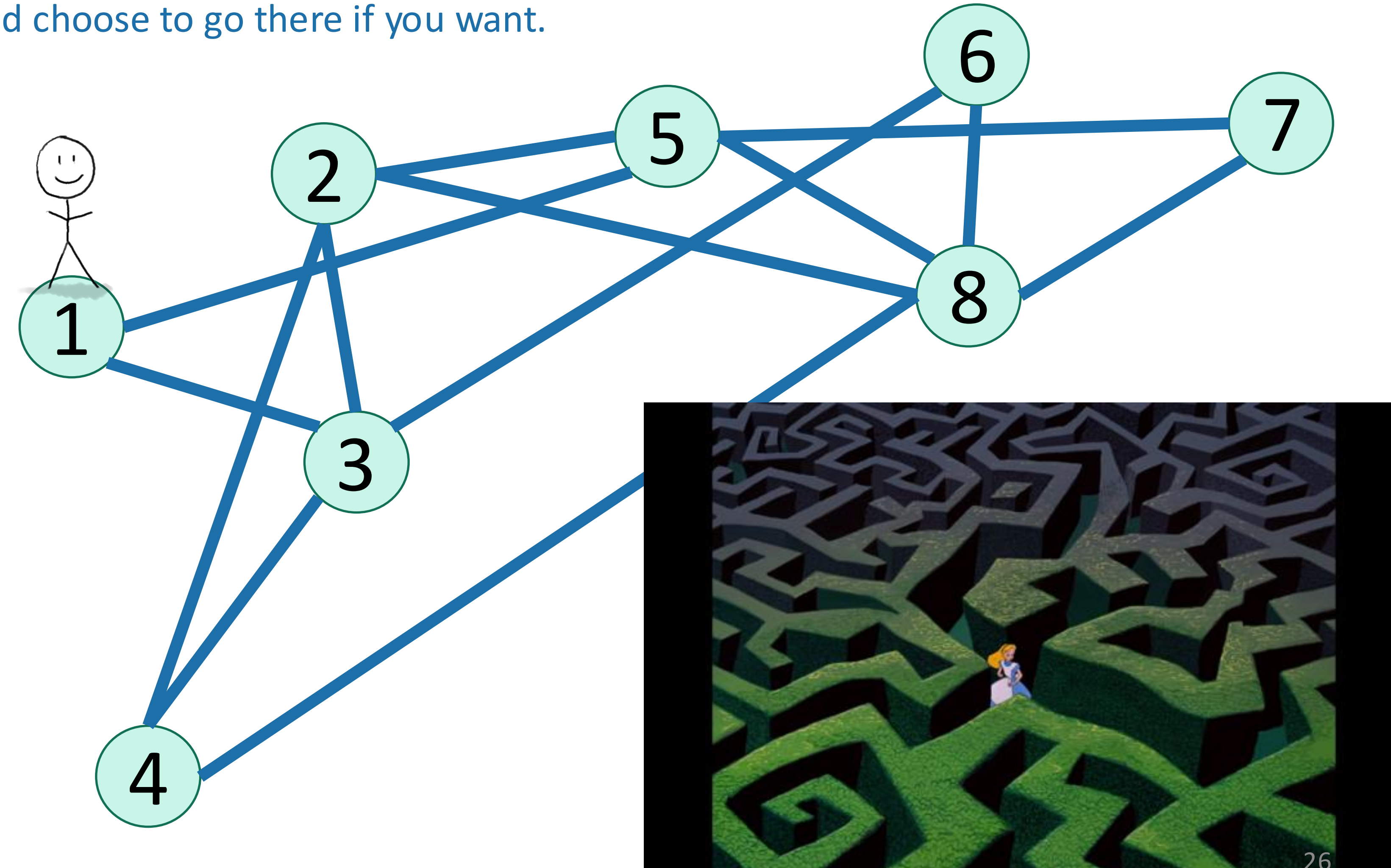
See Lecture 9 IPython notebook for an actual implementation!

We'll assume this representation for the rest of the class

Part 1: Depth-first search

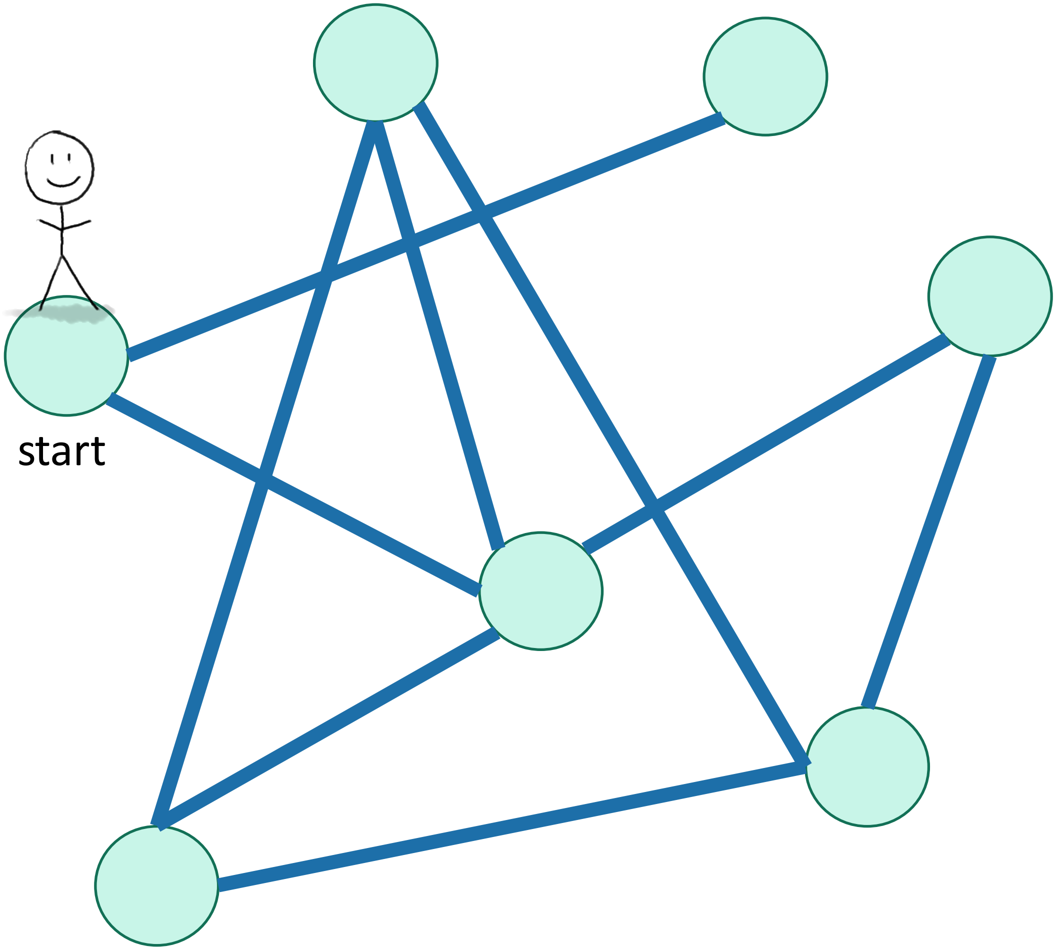
How do we explore a ~~graph~~ labyrinth?

At each node, you can get a list of neighbors, and choose to go there if you want.



Depth First Search

Exploring a labyrinth with chalk and a piece of string



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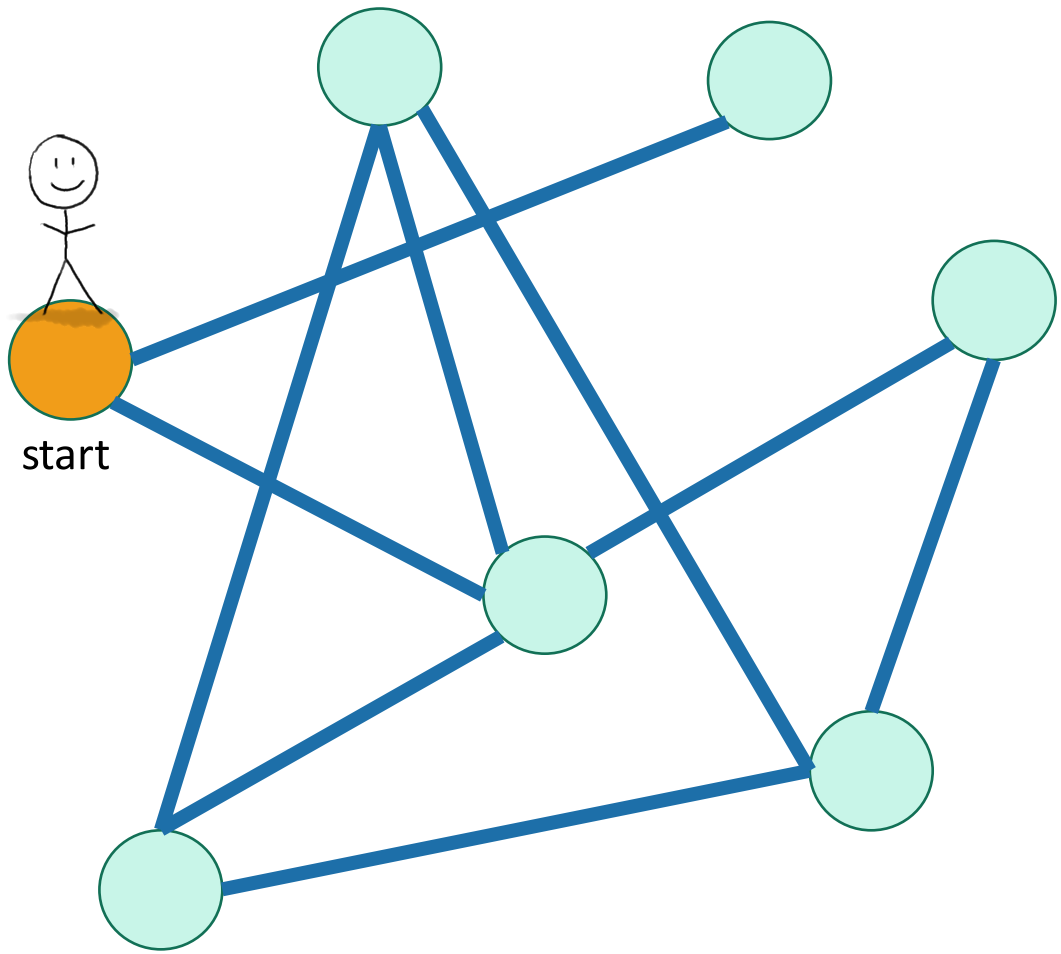
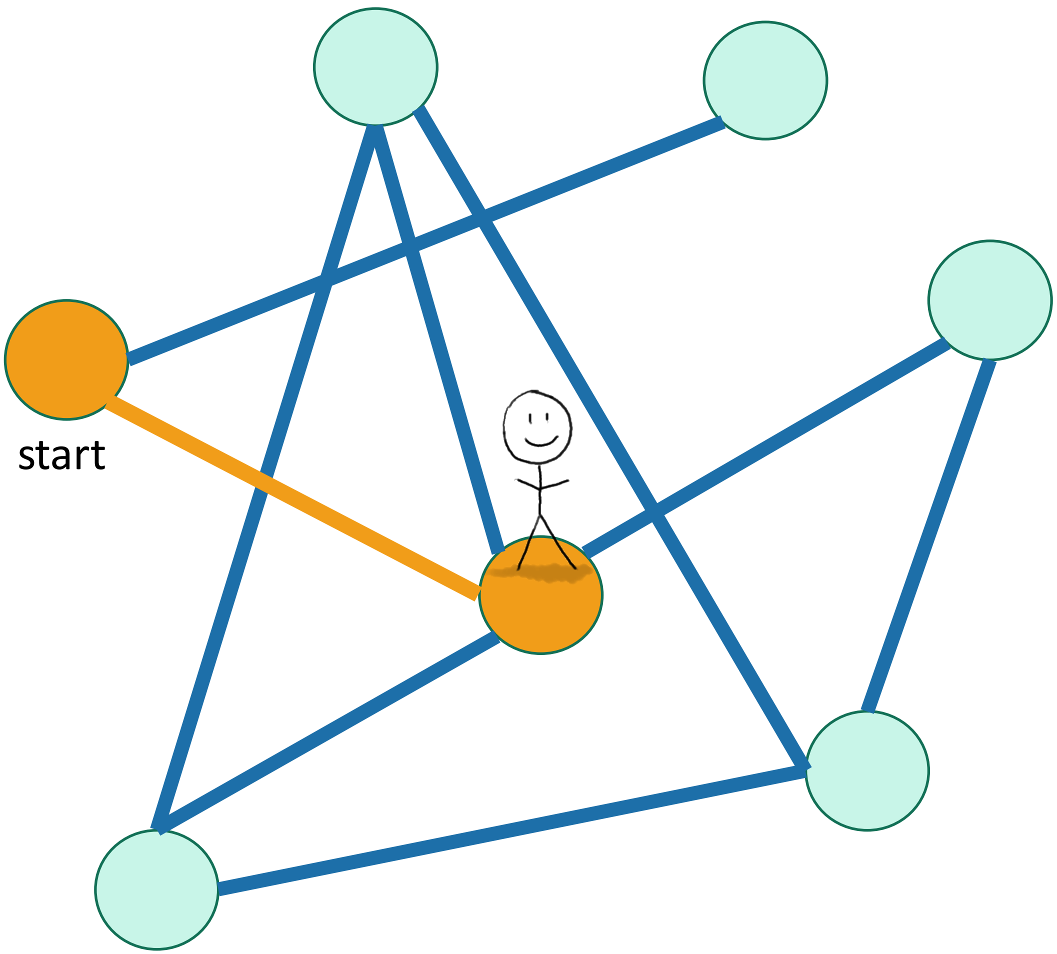


- Not been there yet
- Been there, haven't explored all the paths out.
- Been there, have explored all the paths out.

Depth First Search

Exploring a labyrinth with chalk and a piece of string



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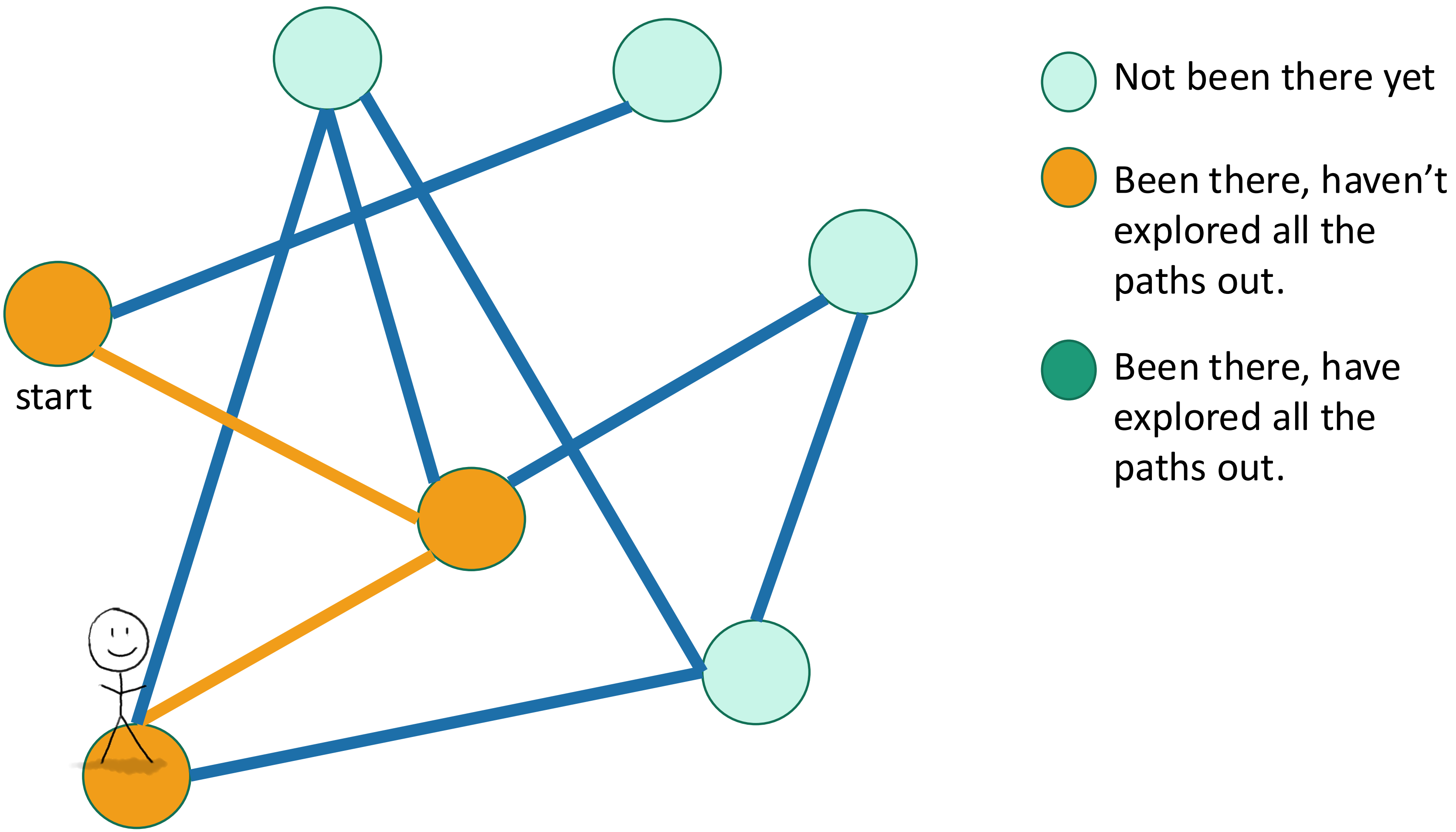

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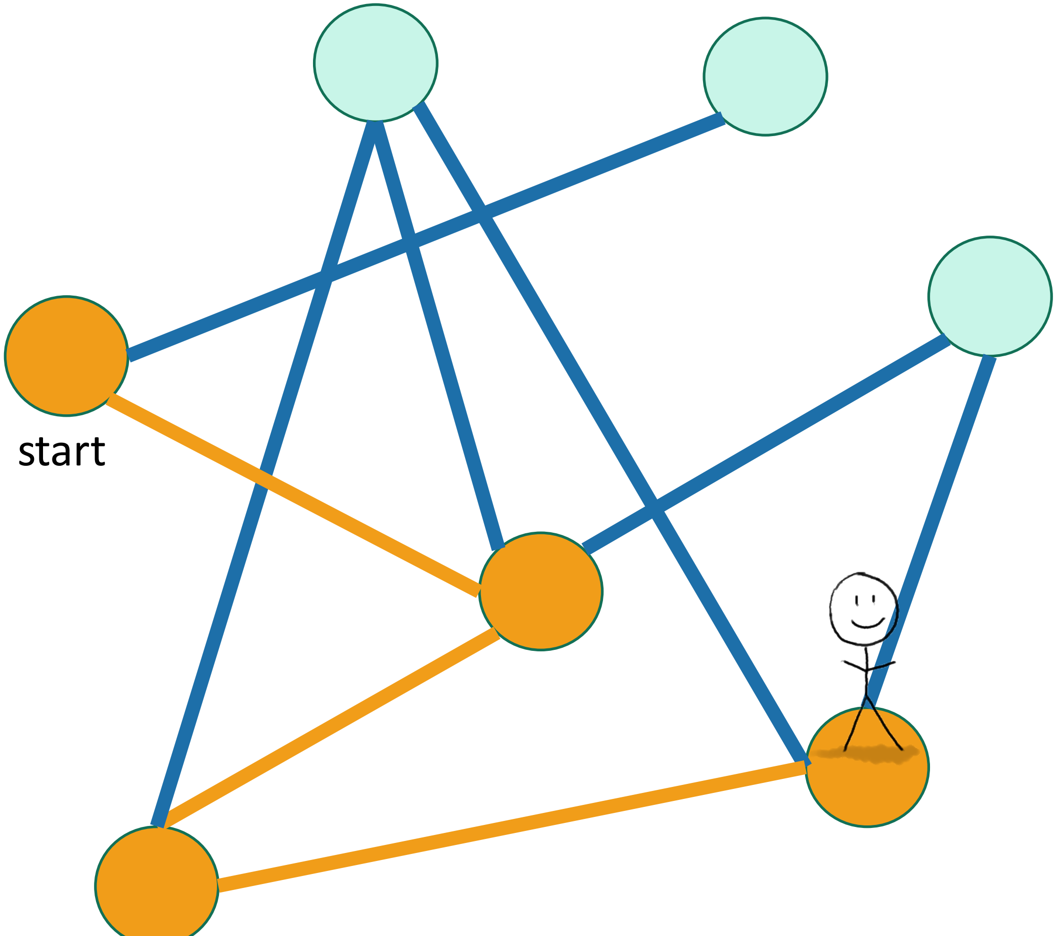

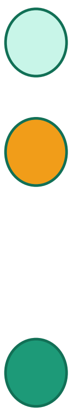

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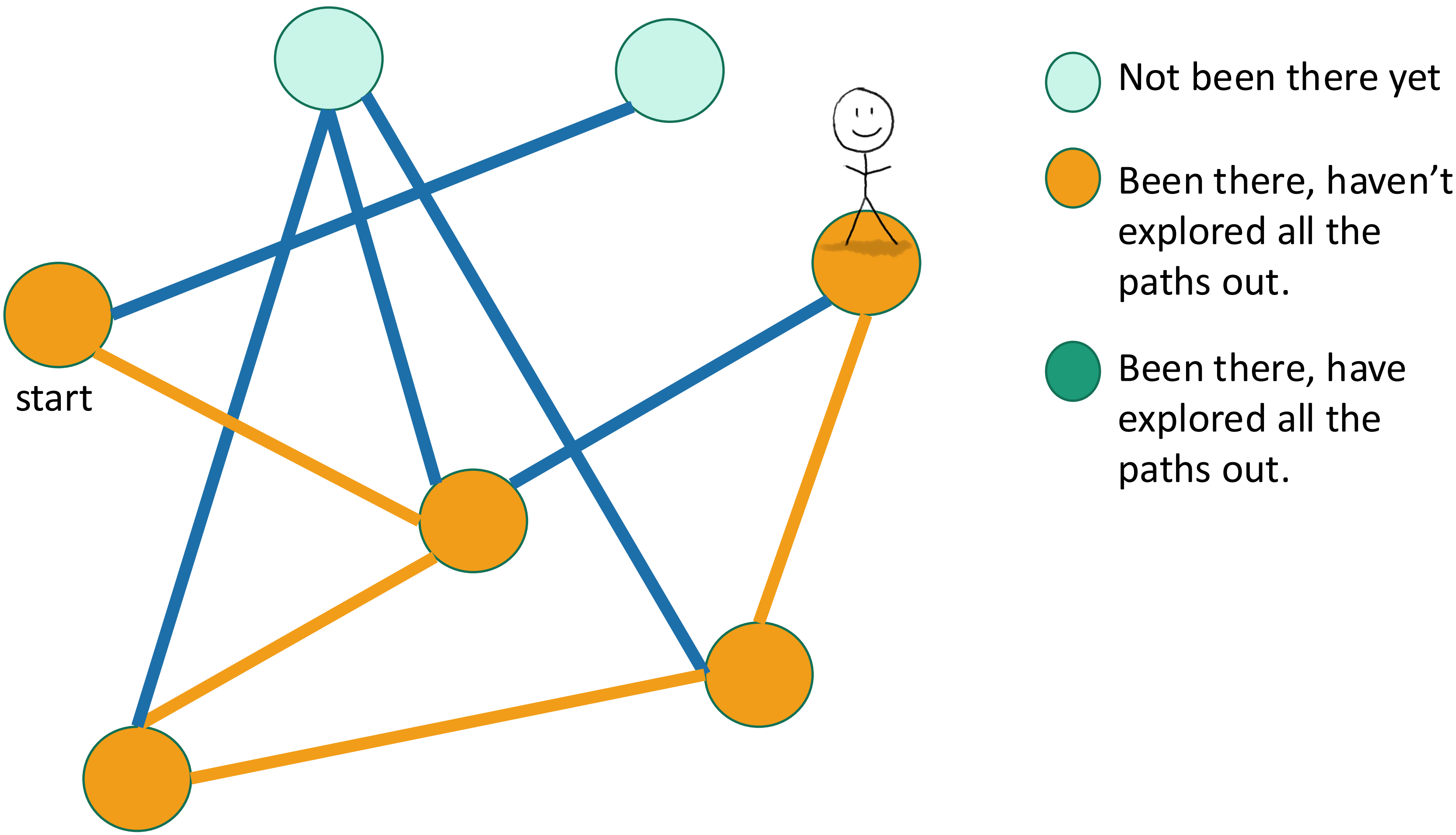

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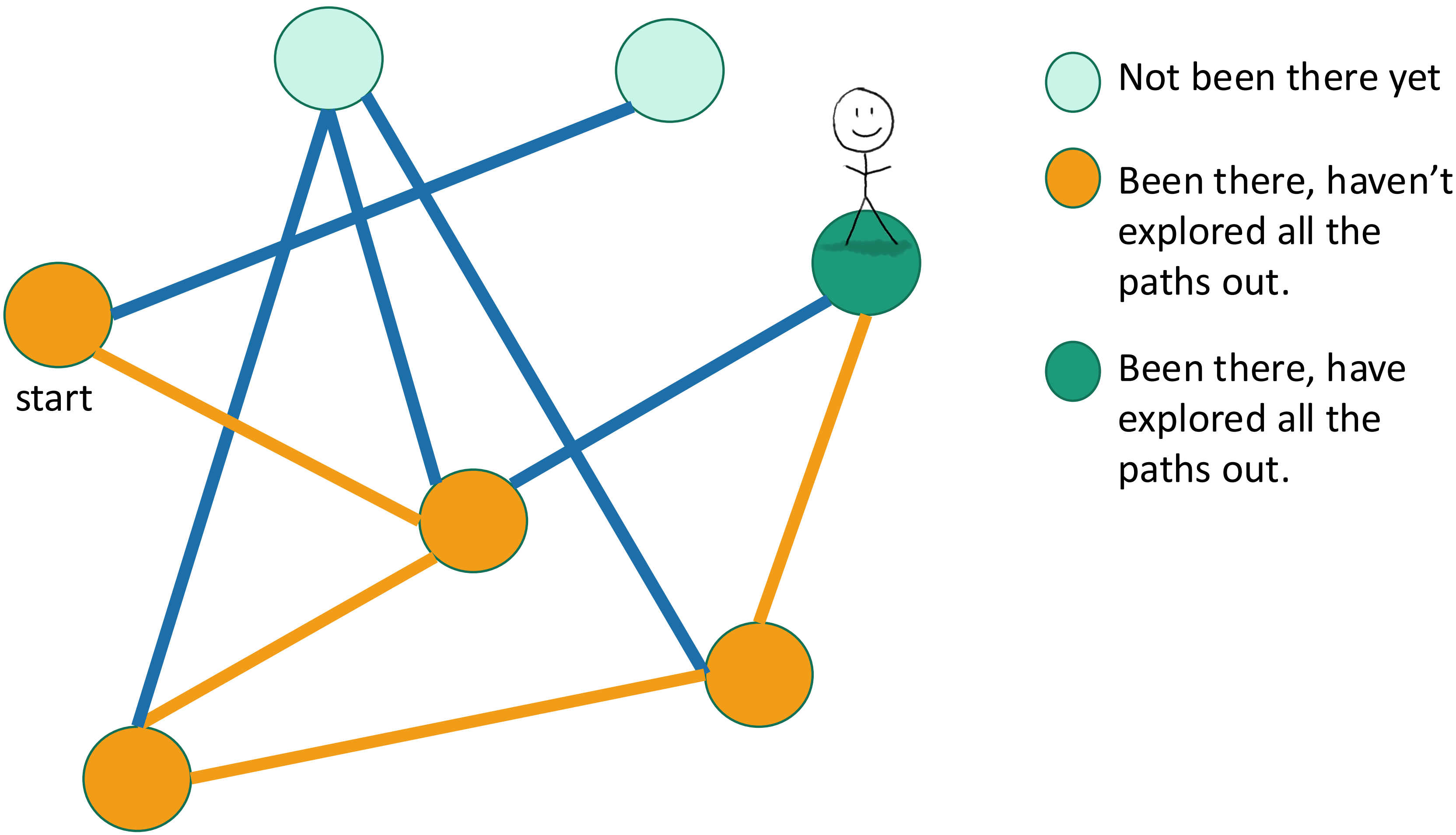

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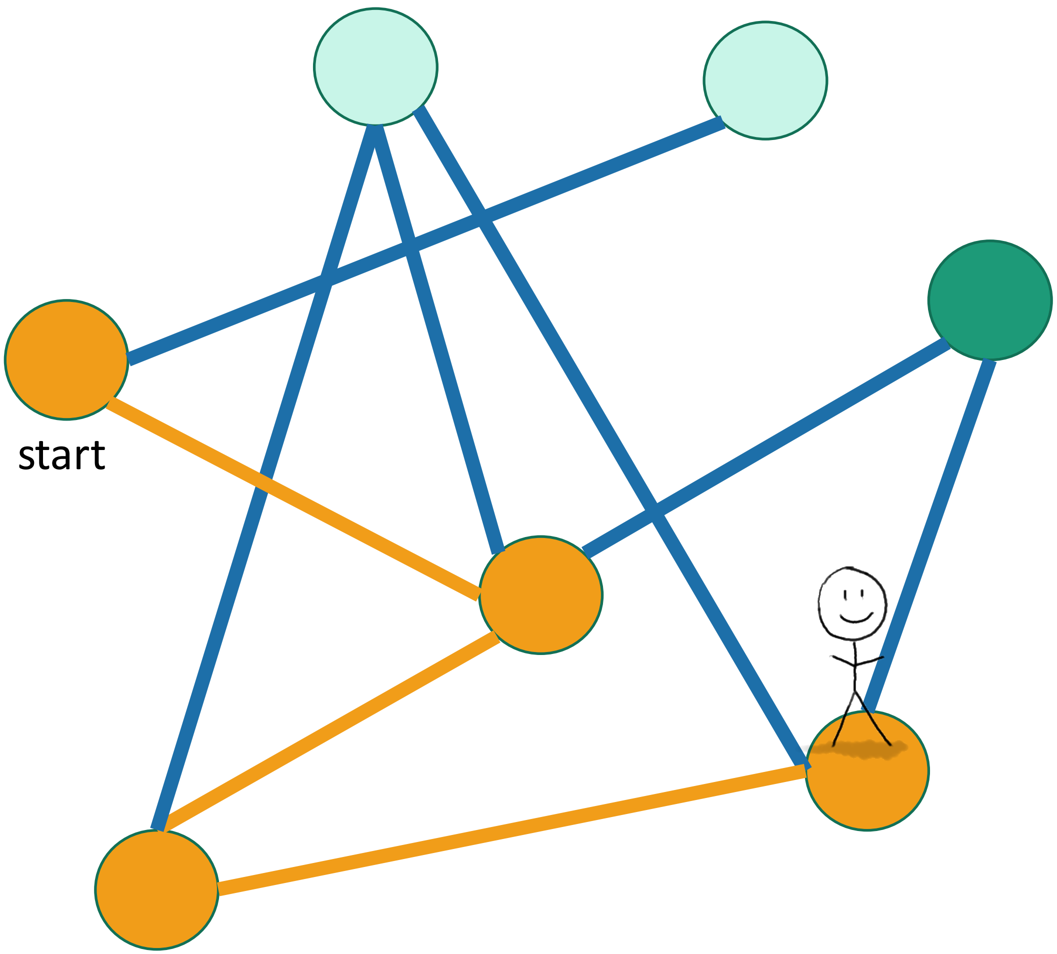

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Depth First Search

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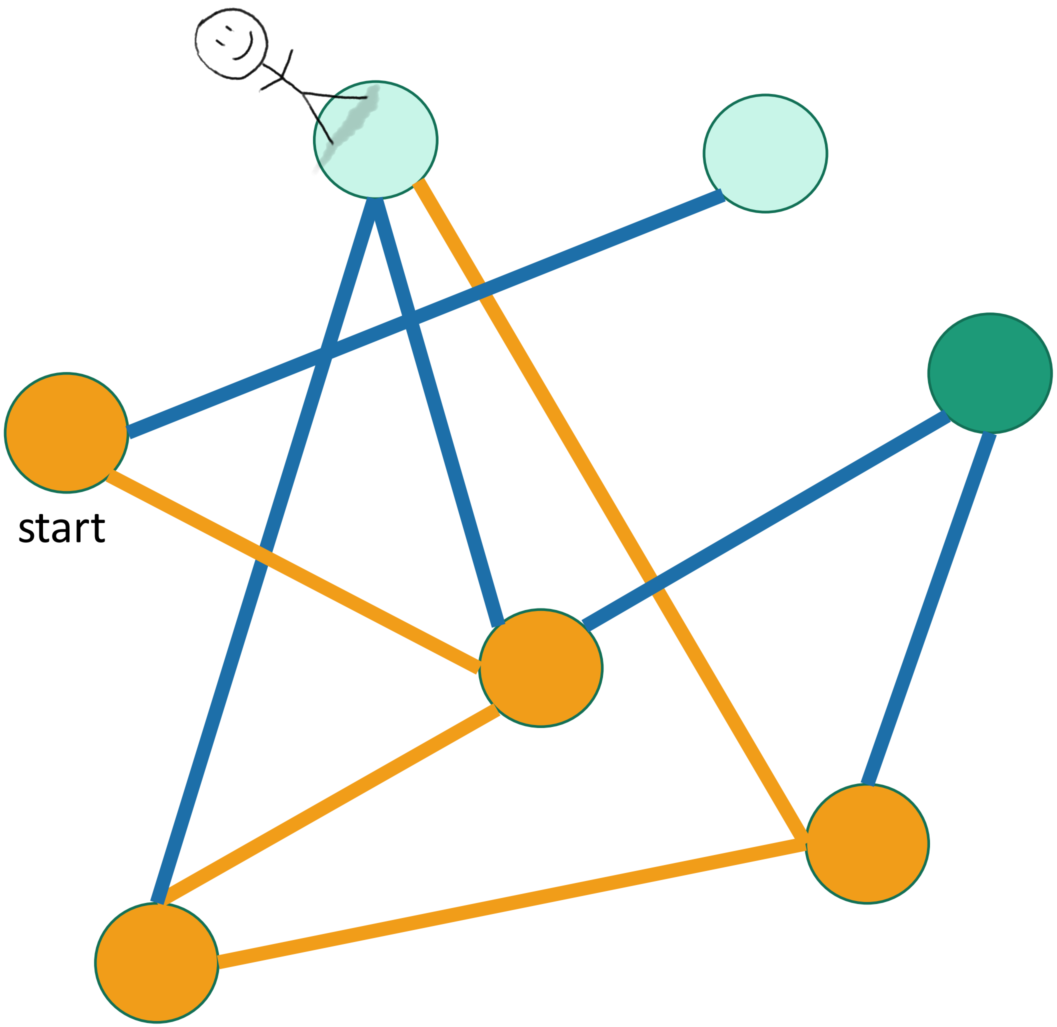

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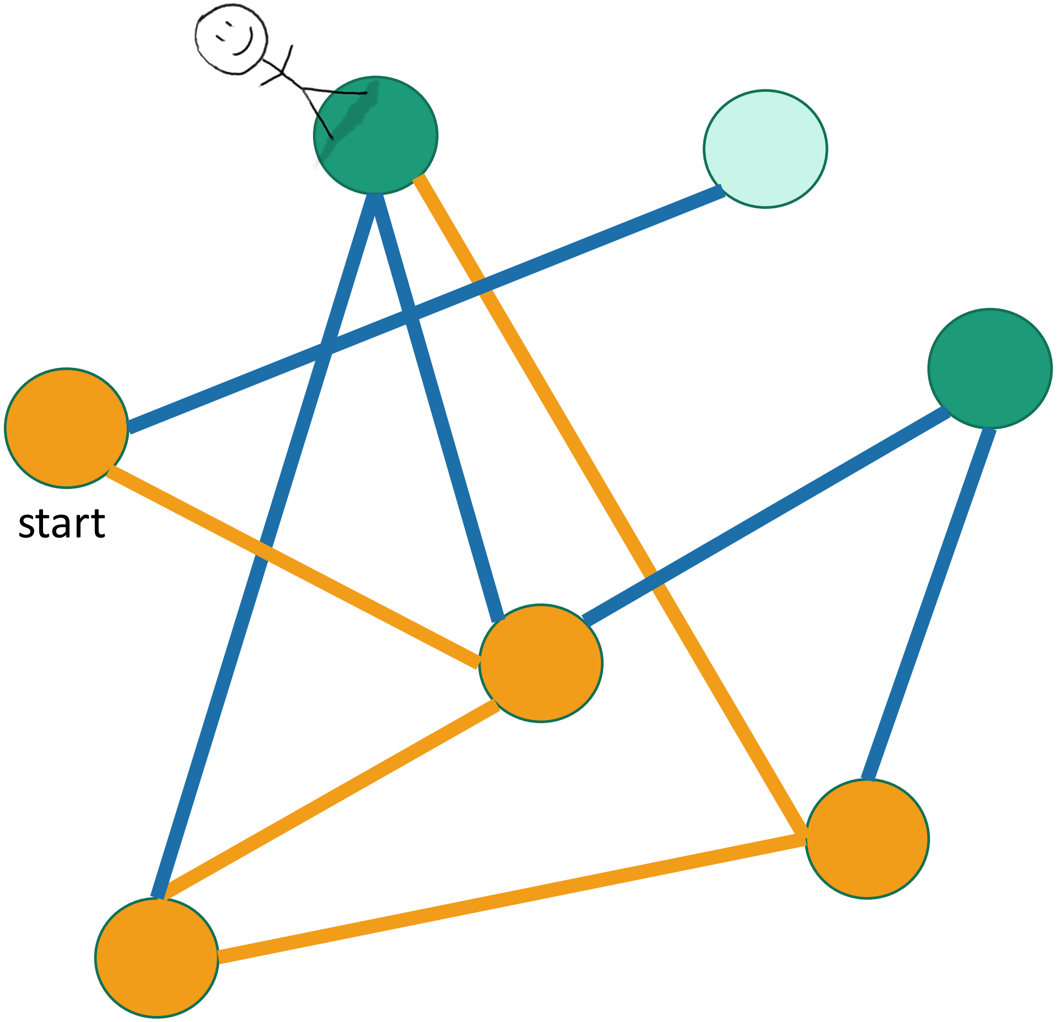

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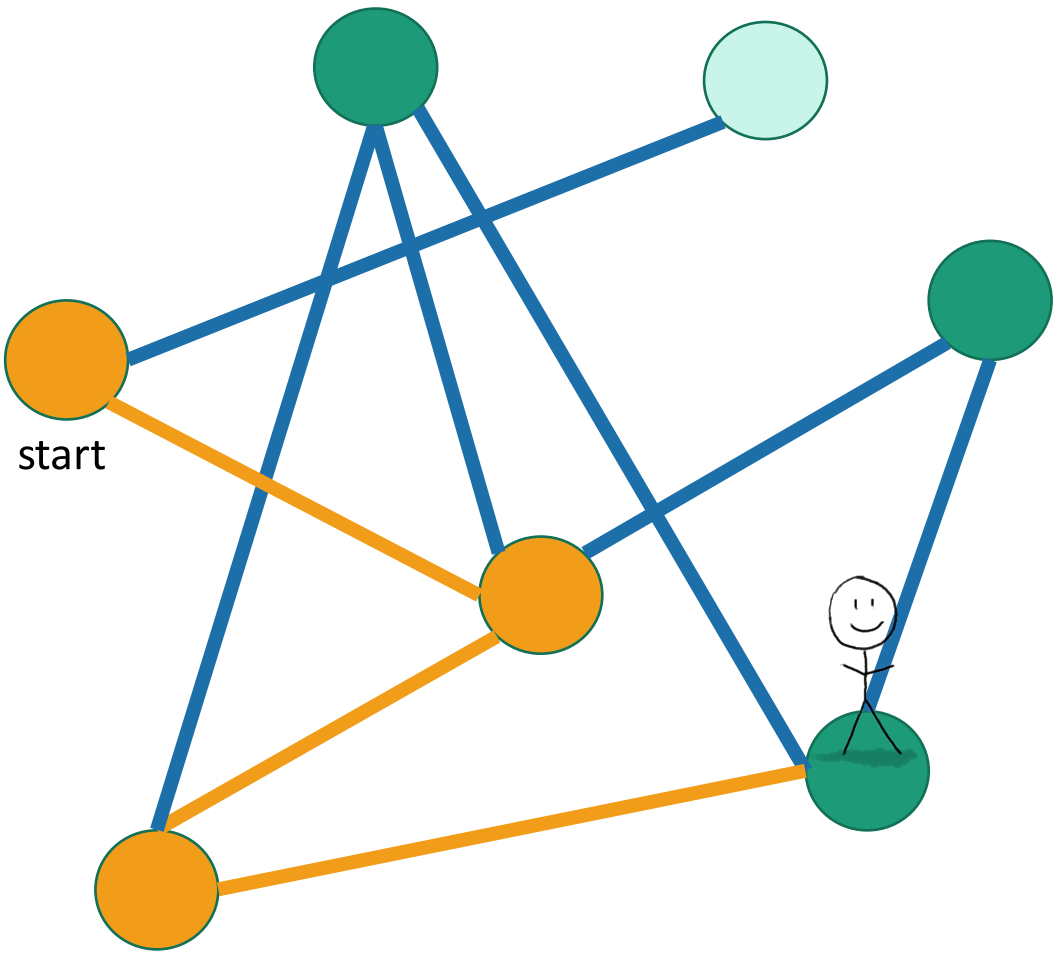

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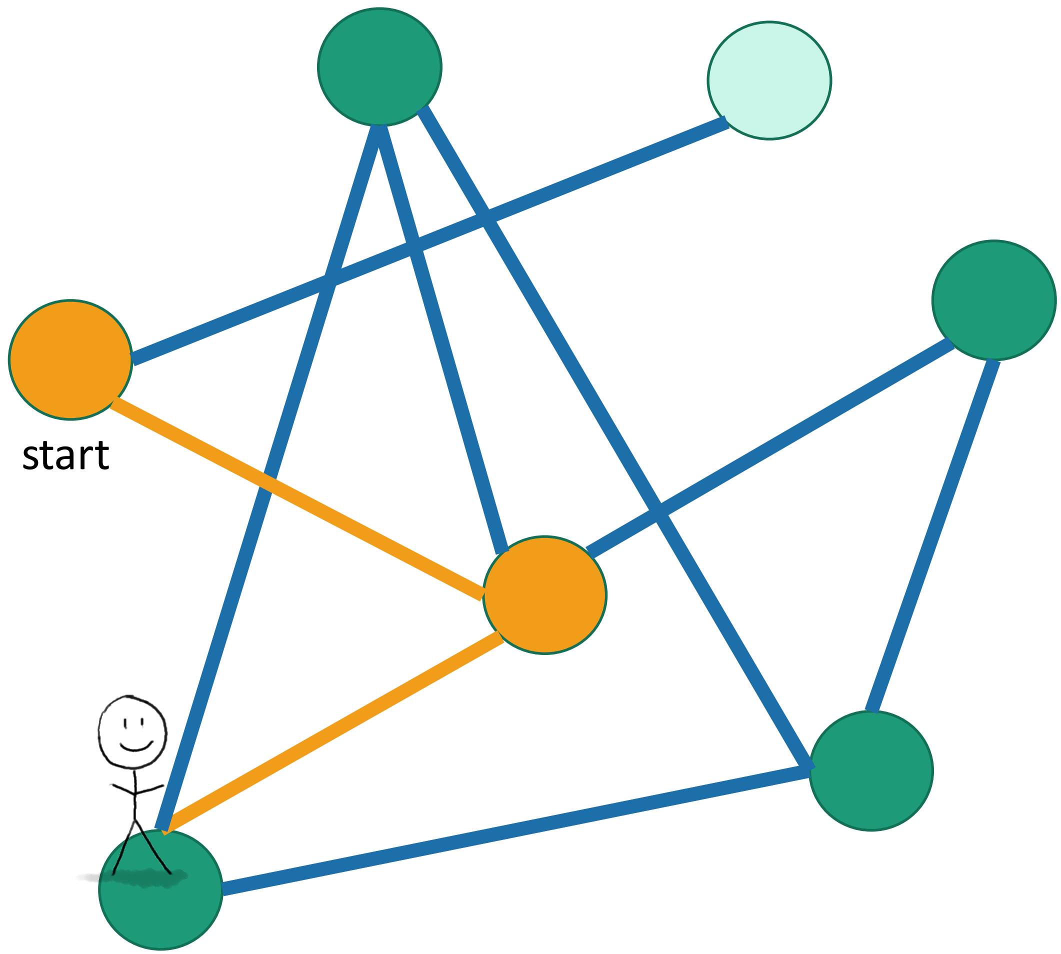

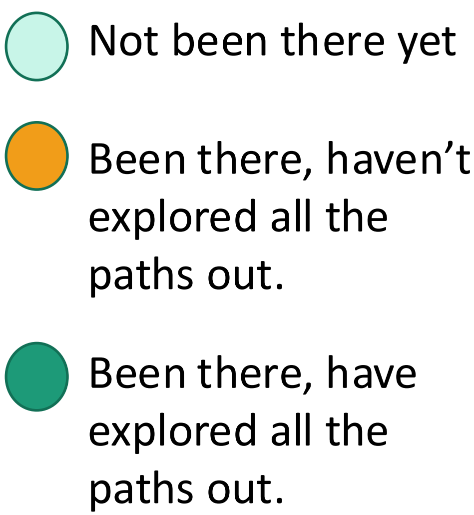

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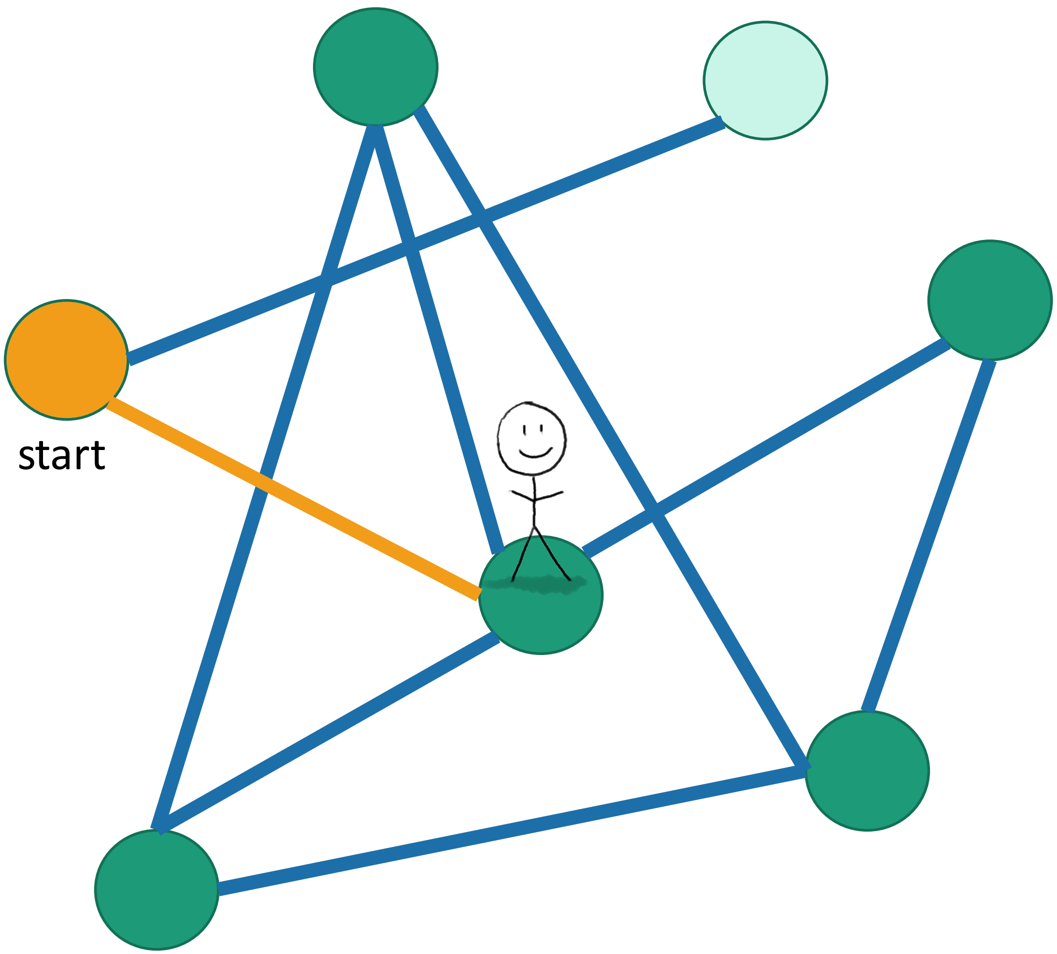

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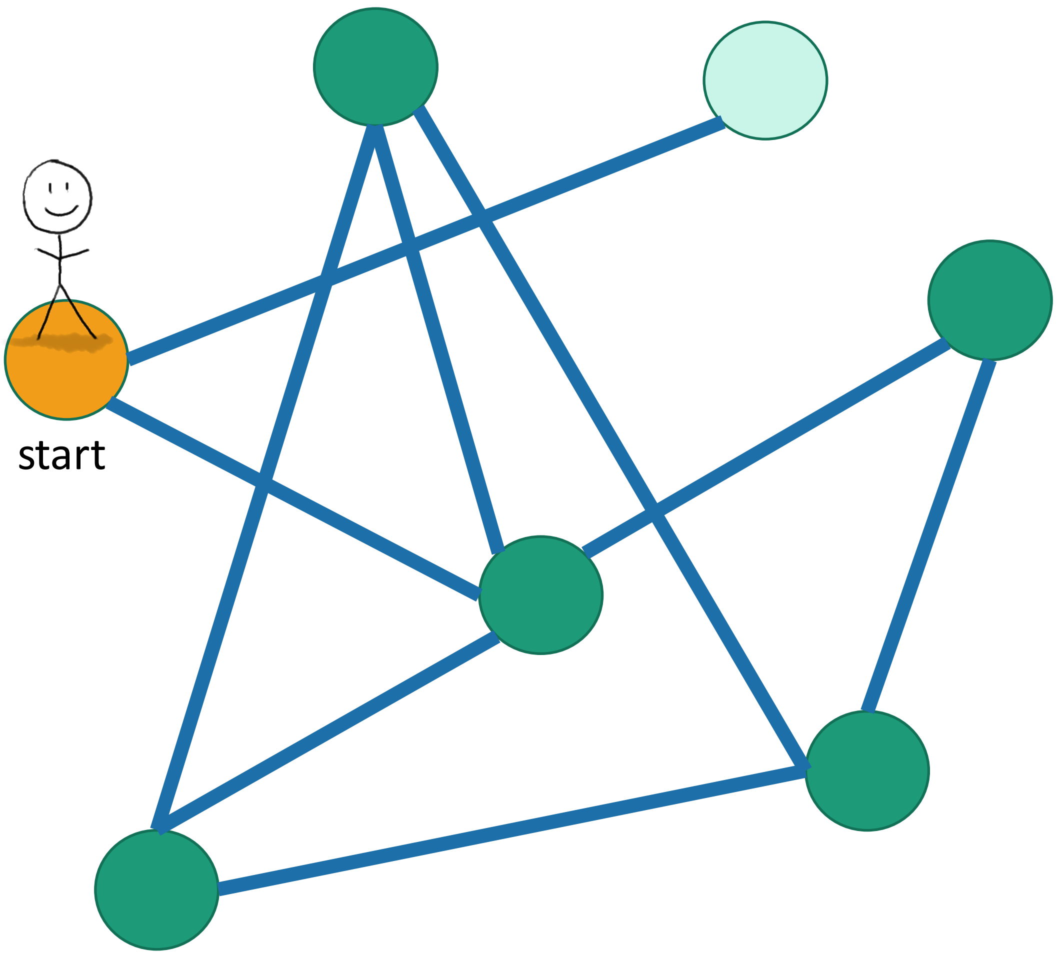

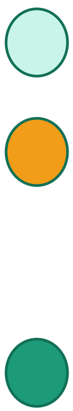

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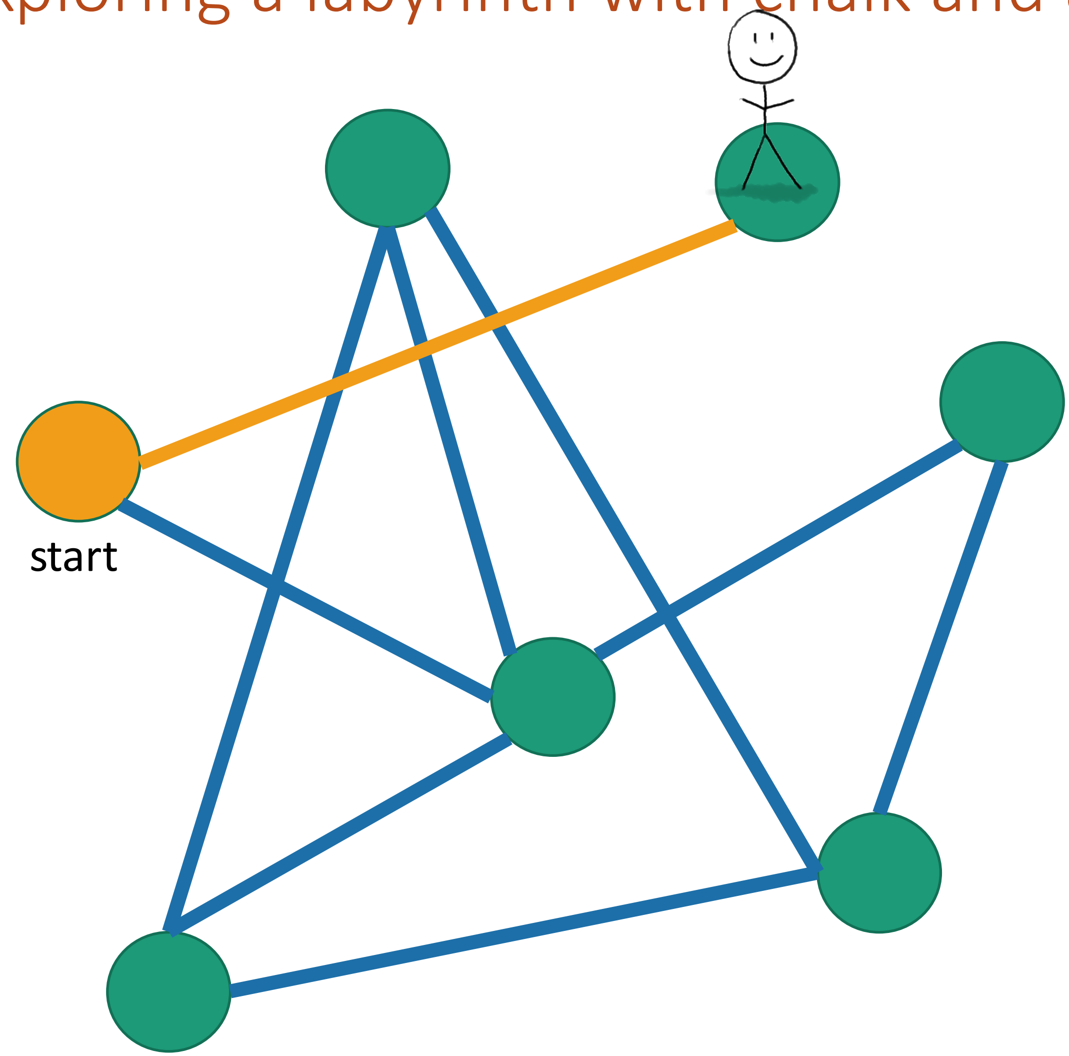

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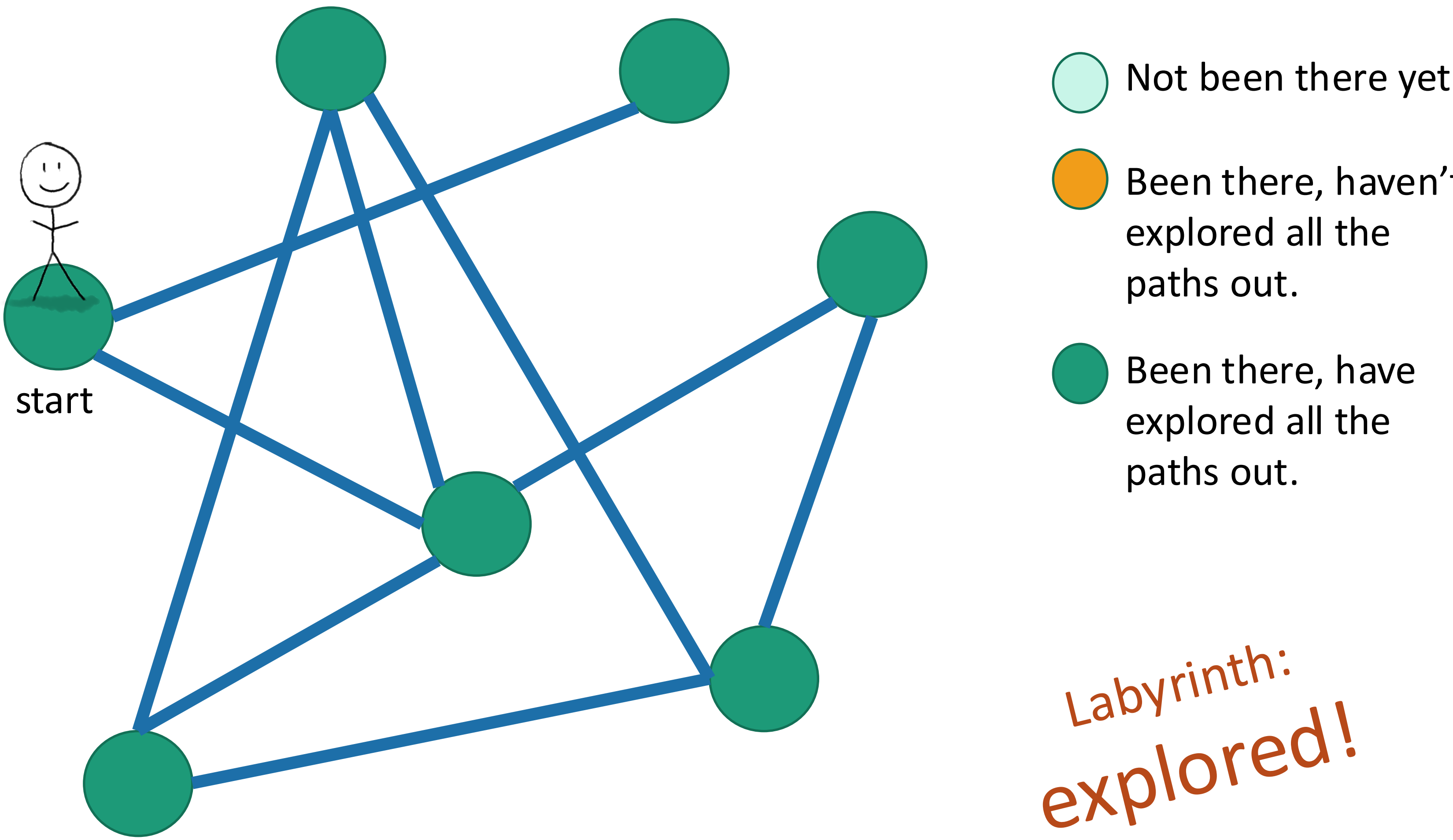


Depth First Search

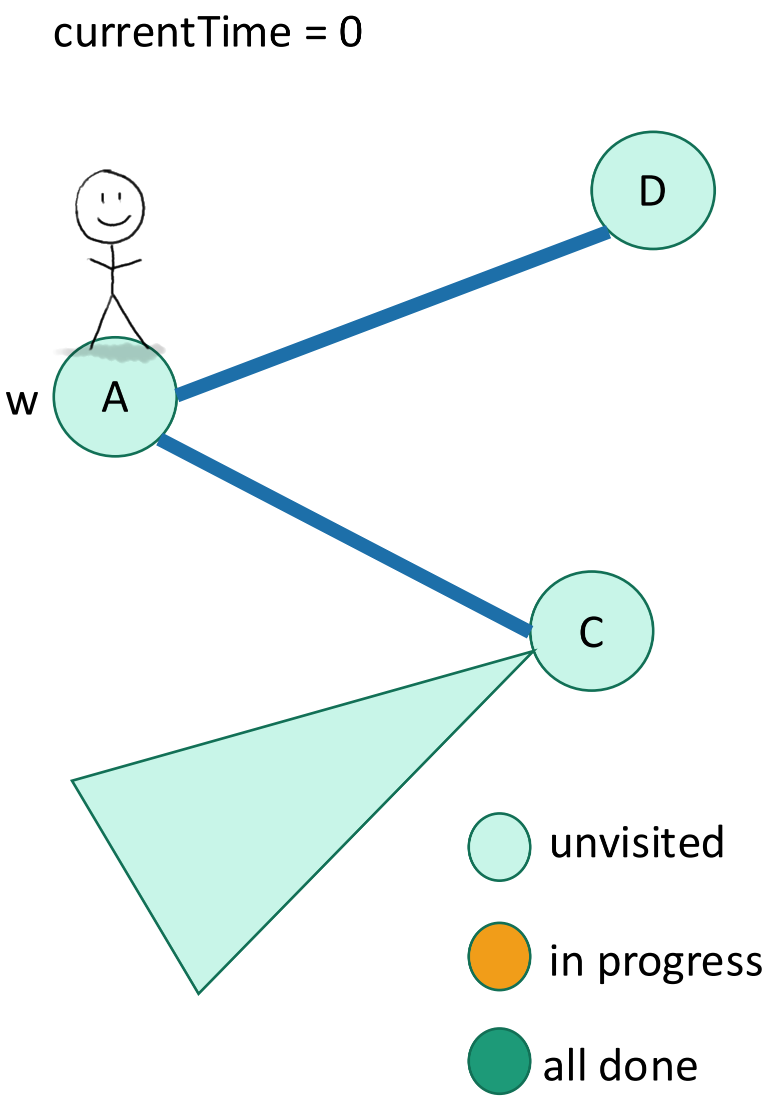
Exploring a labyrinth with pseudocode

- Each vertex keeps track of whether it is:
  - Unvisited
  - In progress
  - All done

- Each vertex will also keep track of:
  - The time we **first enter it**.
  - The time we finish with it and mark it **all done**.

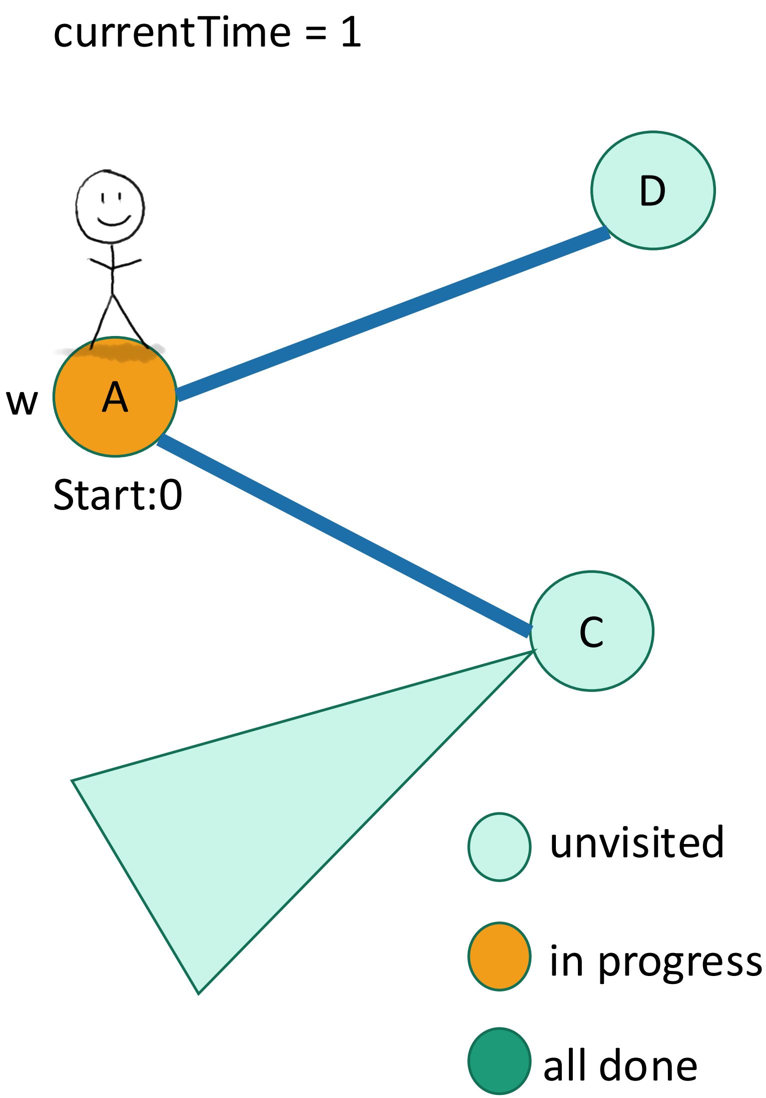
You might have seen other ways to implement DFS than what we are about to go through. This way has more bookkeeping – the bookkeeping will be useful later!

Depth First Search



- **DFS**(w, currentTime):
  - w.startTime = currentTime
  - currentTime ++
  - Mark w as **in progress**.
  - **for** v in w.neighbors:
    - **if** v is **unvisited**:
      - currentTime = **DFS**(v, currentTime)
      - currentTime ++
  - w.finishTime = currentTime
  - Mark w as **all done**
  - **return** currentTime

Depth First Search



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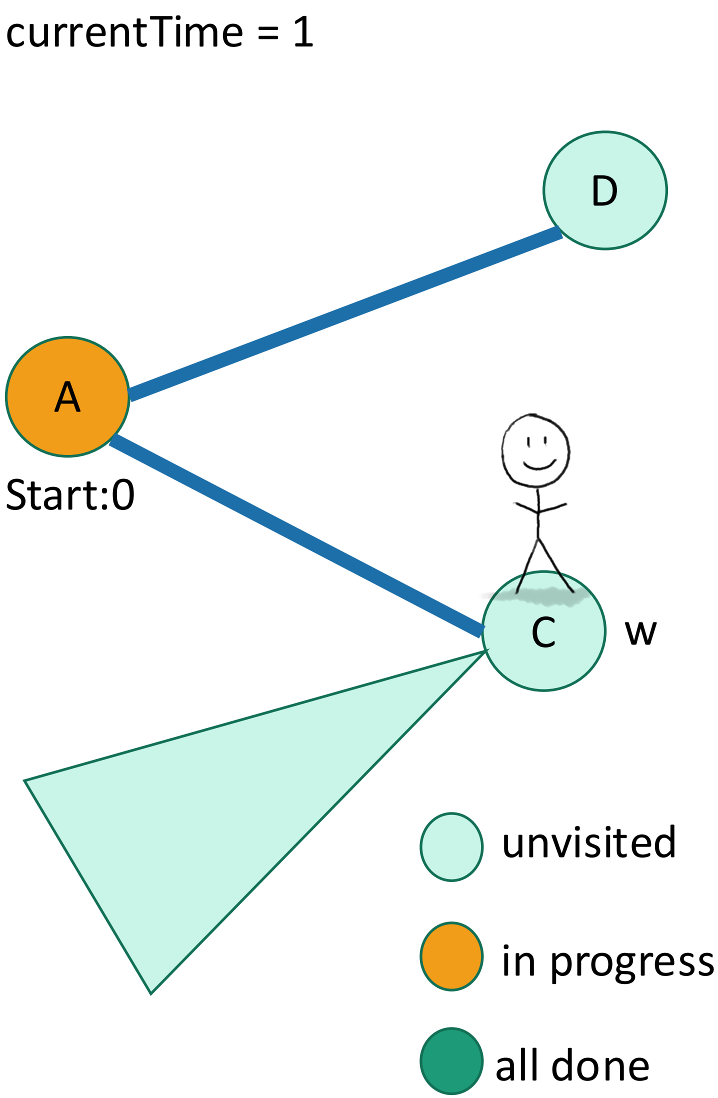
Depth First Search



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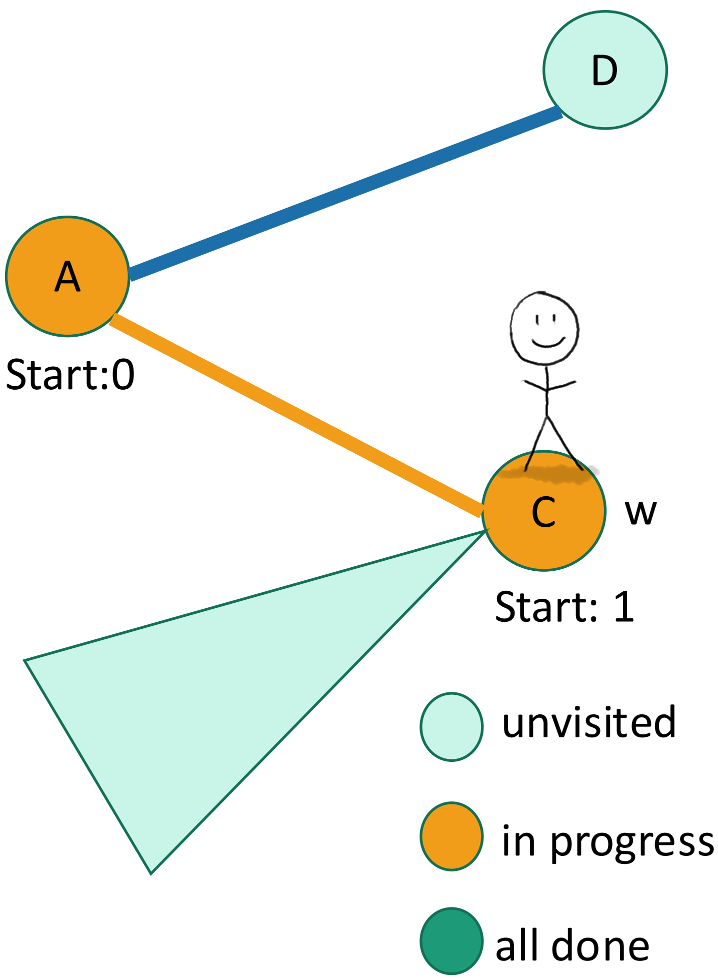
  - currentTime ++
  - Mark w as **in progress**.
  - **for** v in w.neighbors:
    - **if** v is **unvisited**:
      - currentTime
        = **DFS**(v, currentTime)
      - currentTime ++
  - w.finishTime = currentTime
  - Mark w as **all done**
  - **return** currentTime

Depth First Search

currentTime = 2



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  - w.startTime = currentTime
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  - Mark w as **in progress**.
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    - **if** v is **unvisited**:
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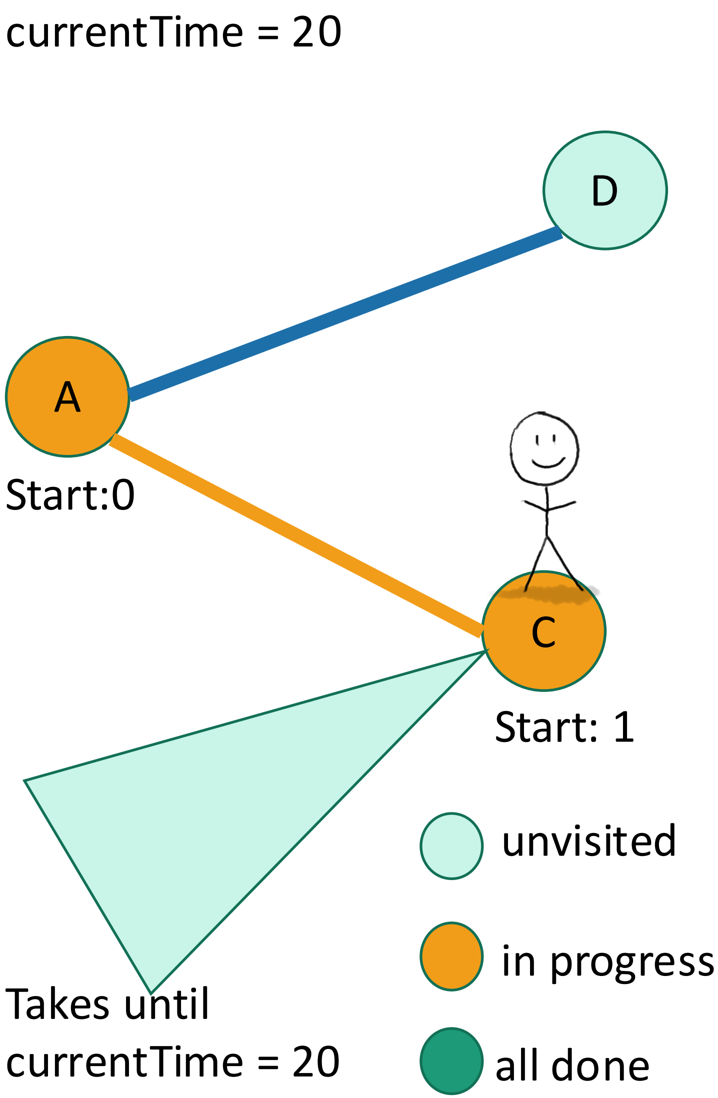
      - currentTime ++
  - w.finishTime = currentTime
  - Mark w as **all done**
  - **return** currentTime

Depth First Search



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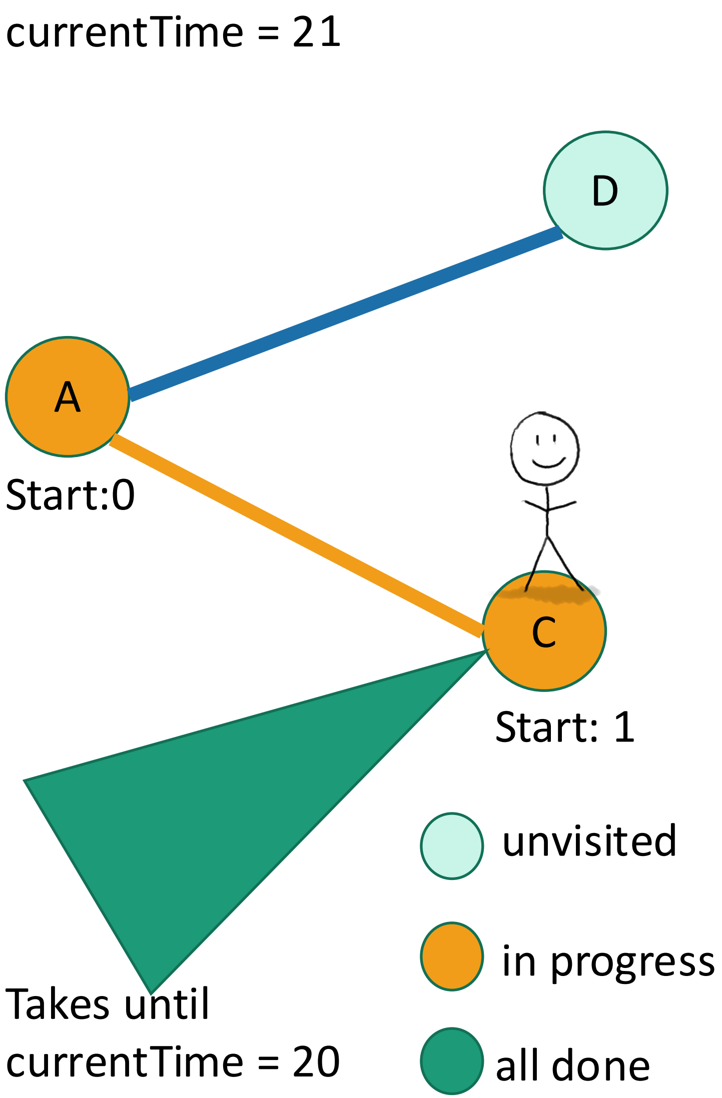
  - w.startTime = currentTime
  - currentTime ++
  - Mark w as **in progress**.
  - **for** v in w.neighbors:
    - **if** v is **unvisited**:
      - currentTime
        = **DFS**(v, currentTime)
      - currentTime ++
  - w.finishTime = currentTime
  - Mark w as **all done**
  - **return** currentTime

Depth First Search



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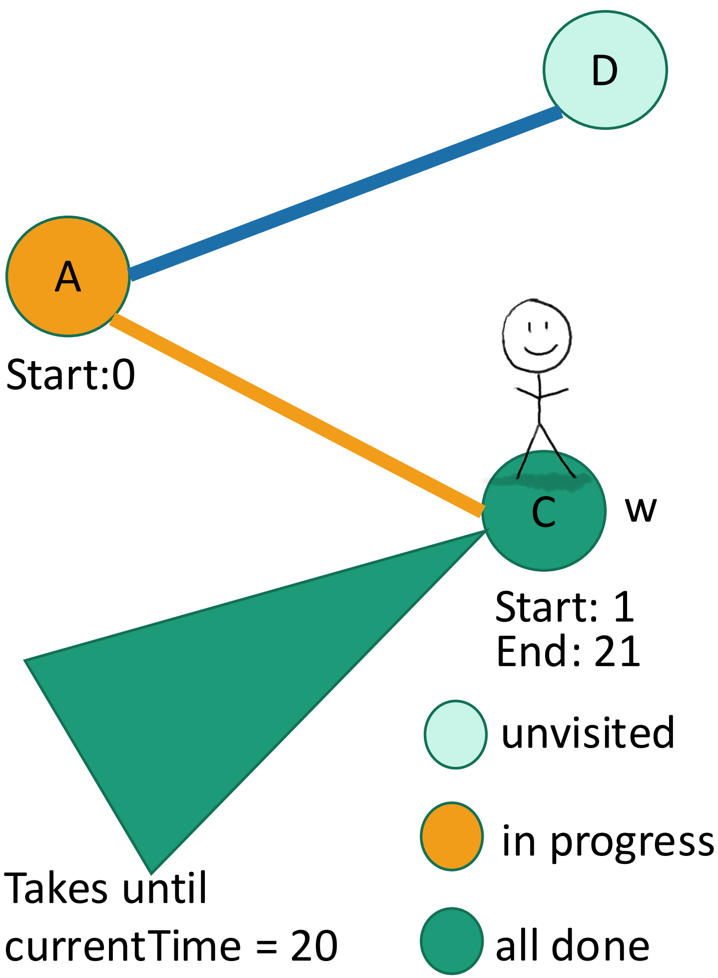
      - currentTime ++
  - w.finishTime = currentTime
  - Mark w as **all done**
  - **return** currentTime

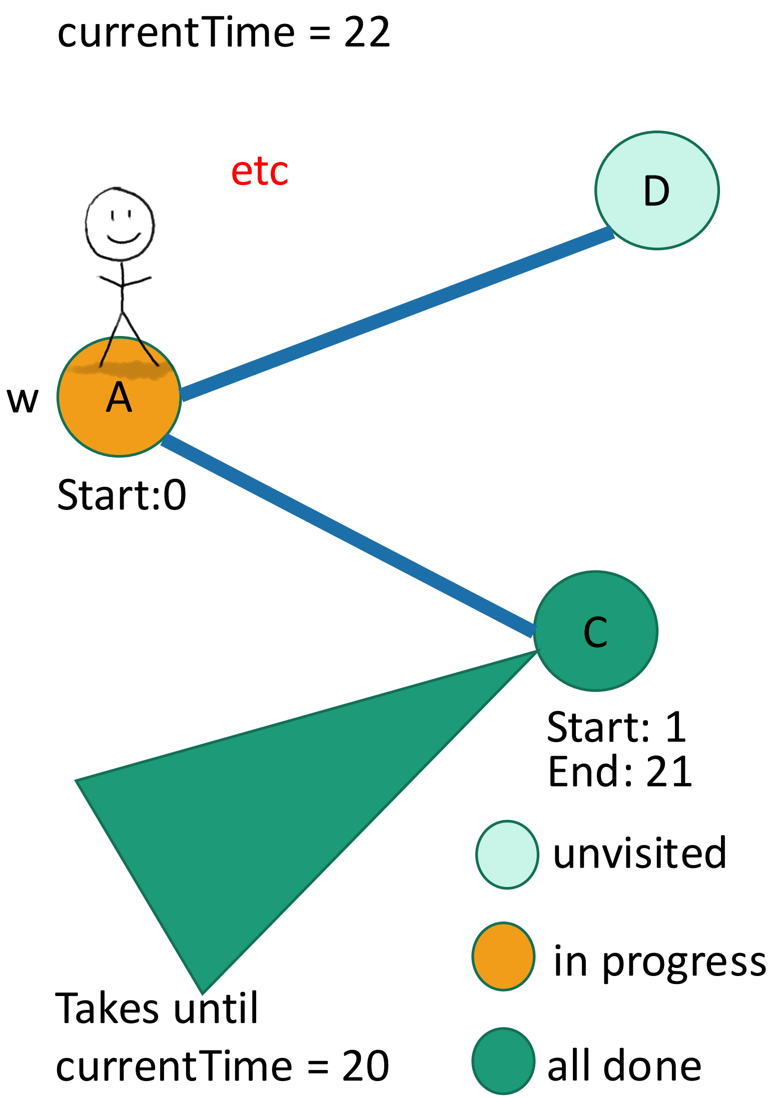
Depth First Search

currentTime = 21



- **DFS**(w, currentTime):
  - w.startTime = currentTime
  - currentTime ++
  - Mark w as **in progress**.
  - **for** v in w.neighbors:
    - **if** v is **unvisited**:
      - currentTime
        = **DFS**(v, currentTime)
      - currentTime ++
  - w.finishTime = currentTime
  - Mark w as **all done**
  - **return** currentTime

Depth First Search



- **DFS**(w, currentTime):
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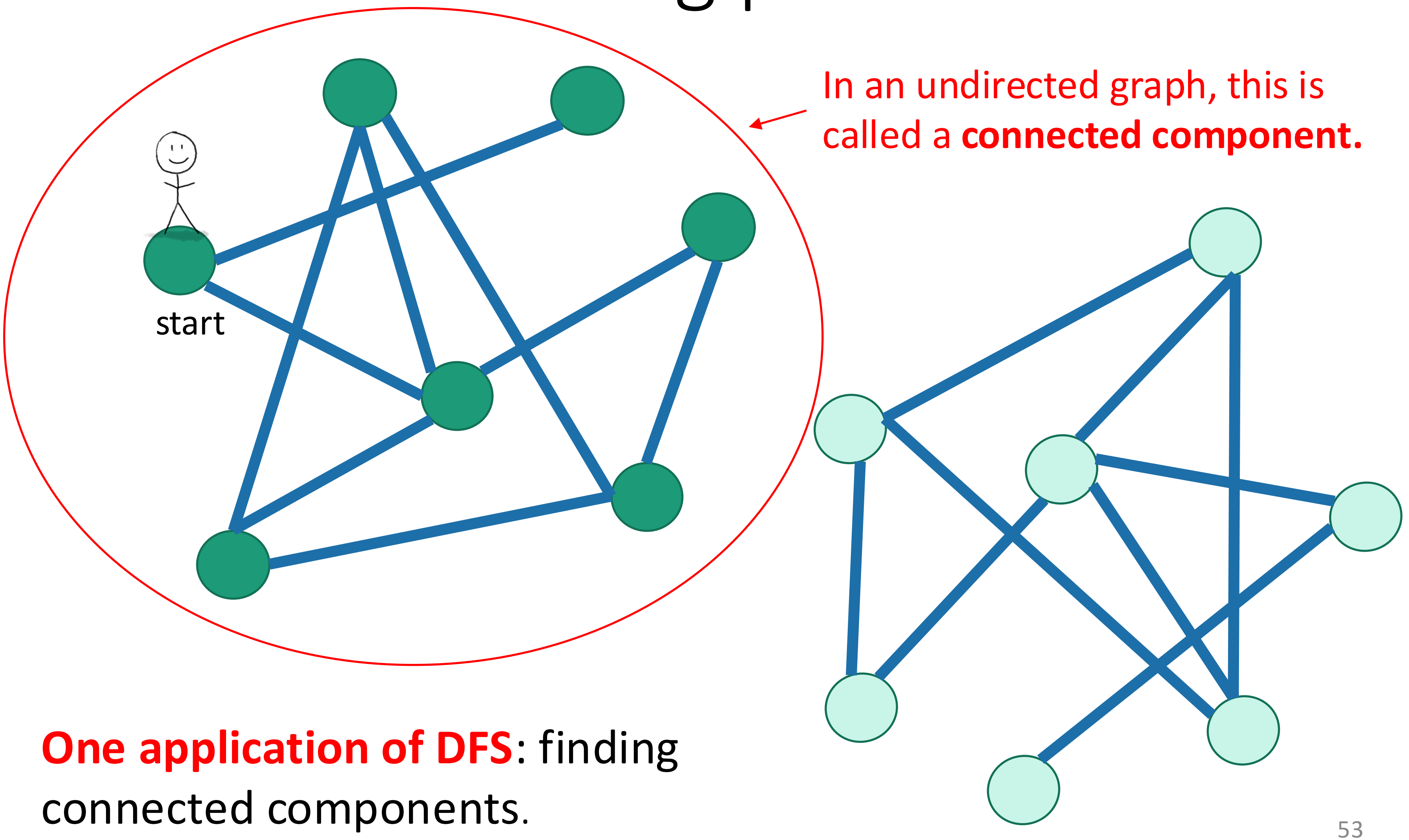
      - currentTime
        = **DFS**(v, currentTime)
      - currentTime ++
  - w.finishTime = currentTime
  - Mark w as **all done**
  - **return** currentTime

This is not the only way to write DFS!

- See the textbook for an iterative version.
- (And/or figure out how to do it yourself!)

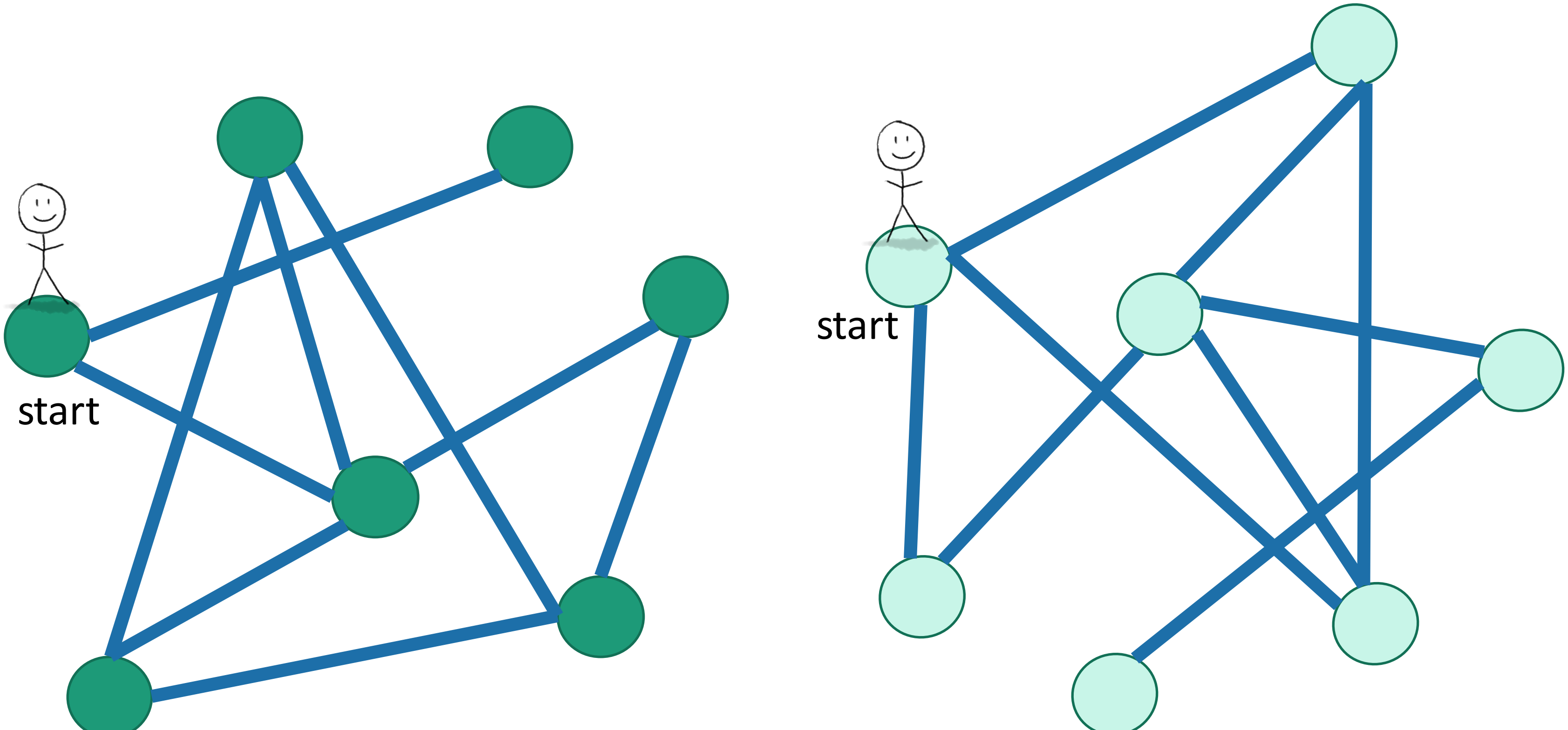
DFS finds all the nodes reachable from the starting point



One application of DFS: finding connected components.

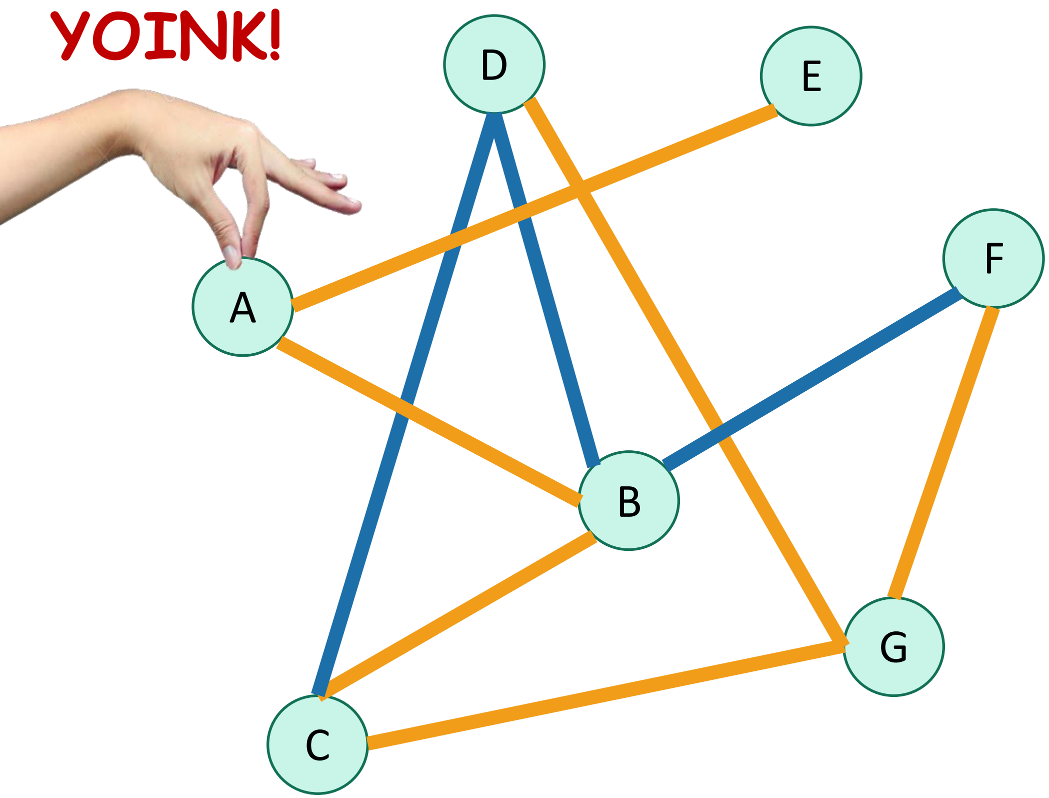
To explore the whole graph

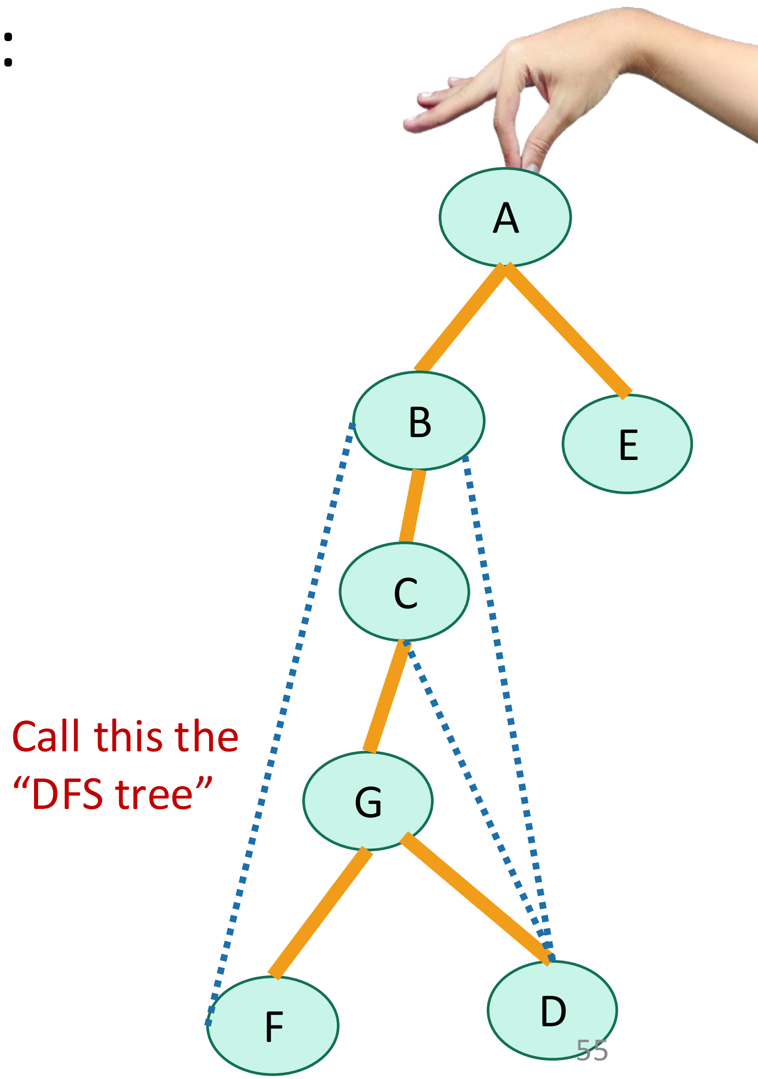
- Do it repeatedly!



Why is it called depth-first?

- We are implicitly building a tree:





- First, we go as deep as we can.

Running time

- We look at each edge at most twice.
  - Once from each of its endpoints
- We visit each vertex at most once
- And basically we don't do anything else.
- So…

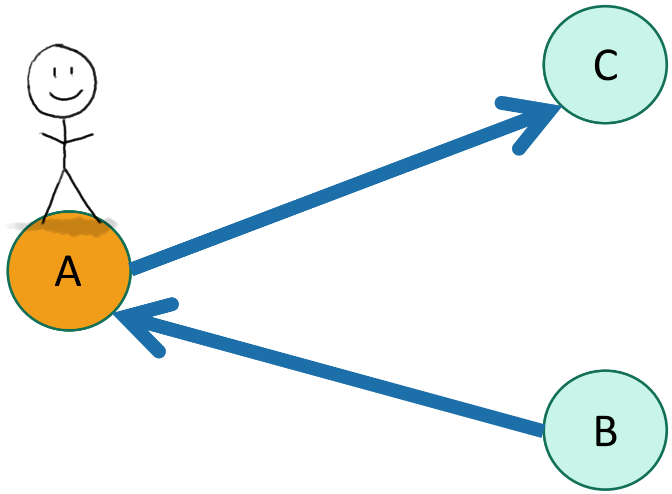
$$O(m+n)$$

Running time

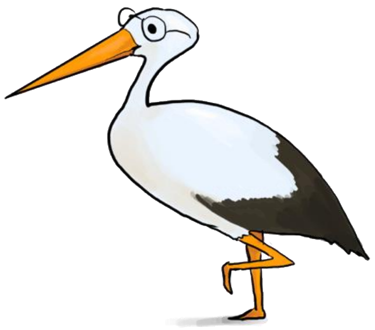
- Assume we are using the linked-list format for G=(V,E).
- We visit each vertex in G exactly once.
  - Here, "visit" means "call DFS on"
- At each vertex w, we:
  - Do some book-keeping: O(1)
  - Loop over w's neighbors and check if they are visited (and then potentially make a recursive call): O(1) per neighbor or O(deg(w)) total.
- Total time:
  - $\sum_{w \in V} (O(\deg(w)) + O(1))$
  - $= O(|E| + |V|) = \boldsymbol{O(n+m)}$

You check:

DFS works fine on directed graphs too!



Only walk to C, not to B.



Siggi the studious stork

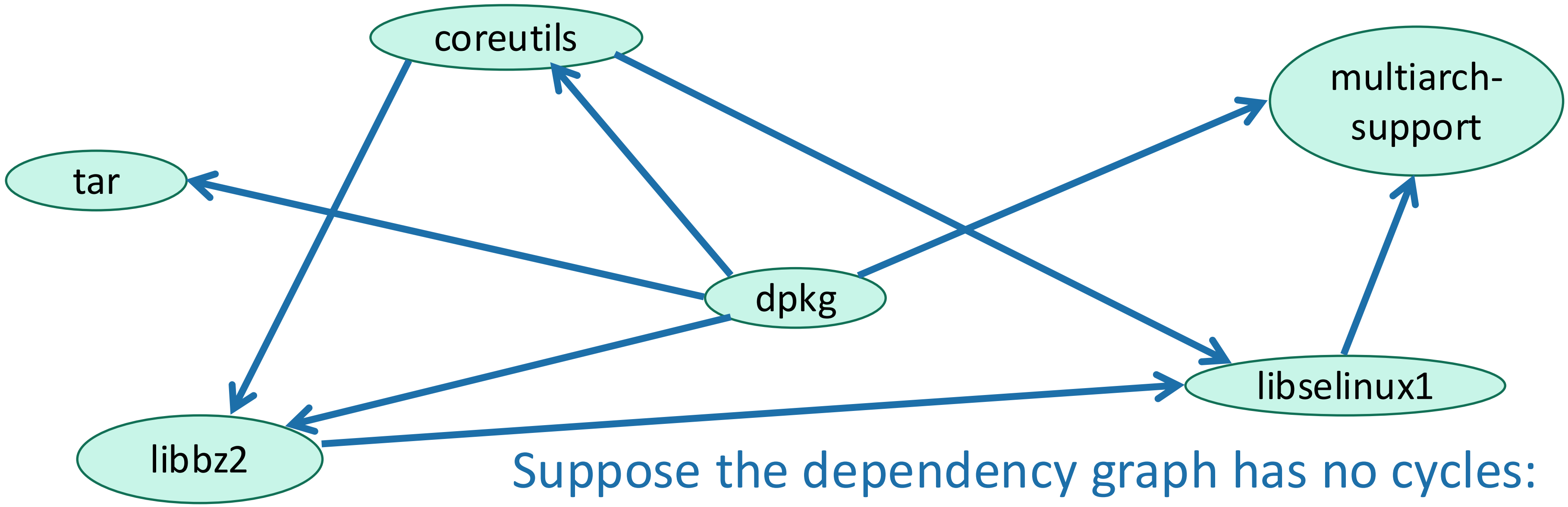
Pre-lecture exercise

- How can you sign up for classes so that you never violate the pre-req requirements?

- More practically, how can you install packages without violating dependency requirements?

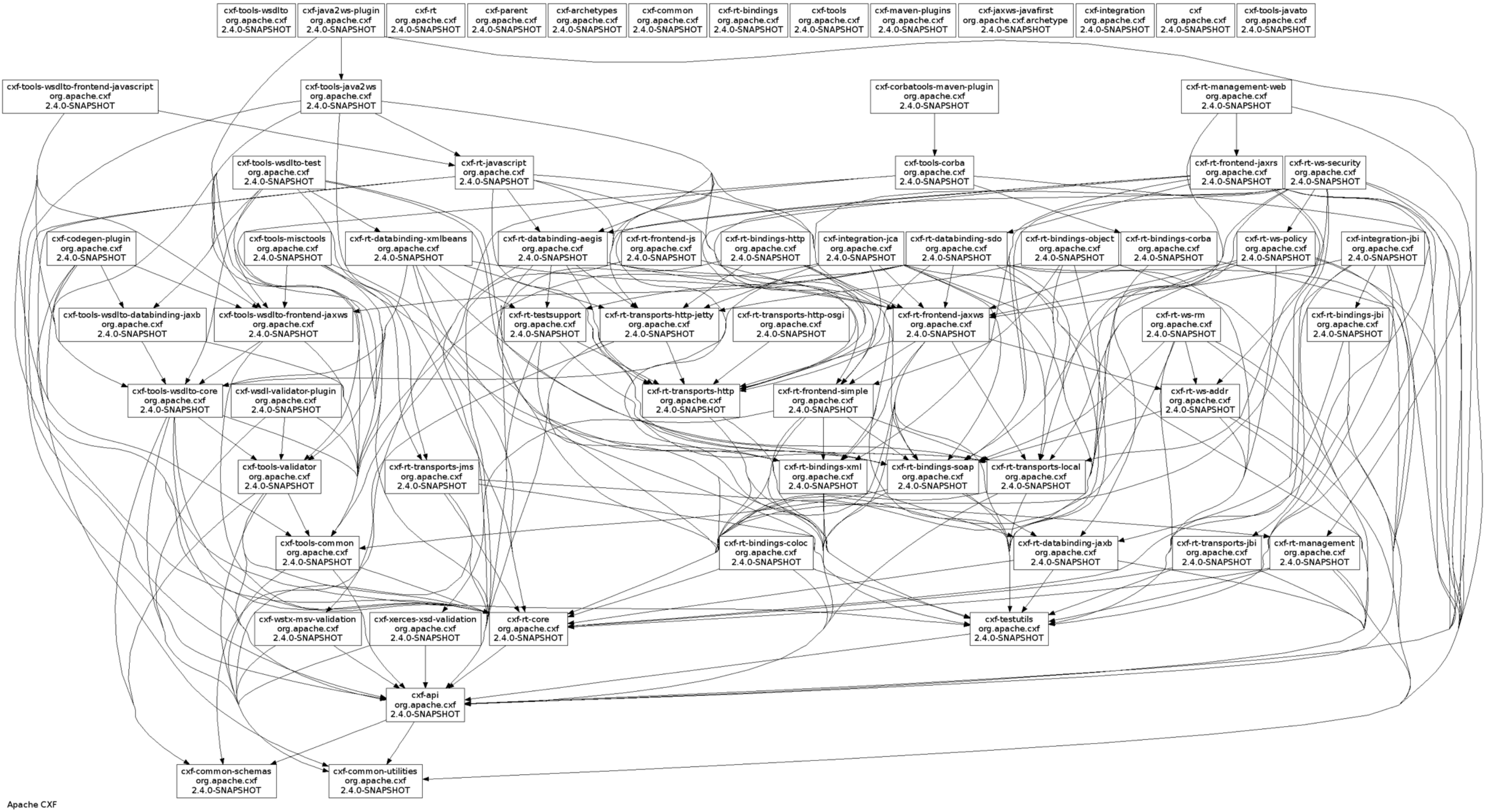
Application of DFS: topological sorting

- Find an ordering of vertices so that all of the dependency requirements are met.
  - Aka, if v comes before w in the ordering, there is not an edge from w to v.



Suppose the dependency graph has no cycles: it is a **Directed Acyclic Graph (DAG)**

Can’t always eyeball it.

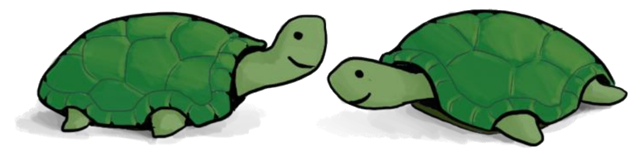


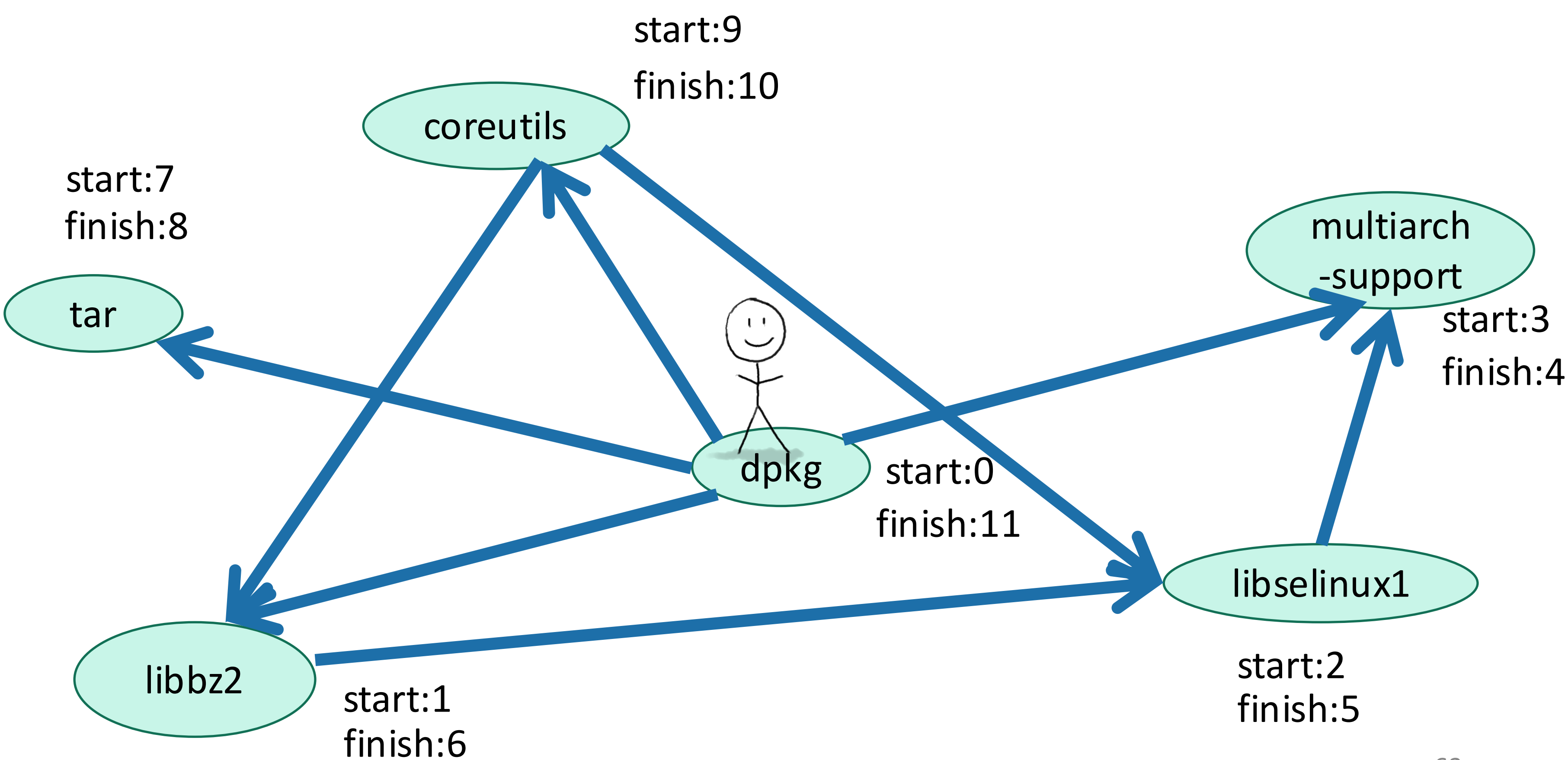
Let’s do DFS

What do you notice about the finish times? Any ideas for how we should do topological sort?

1 minute think

1 minute pair+share

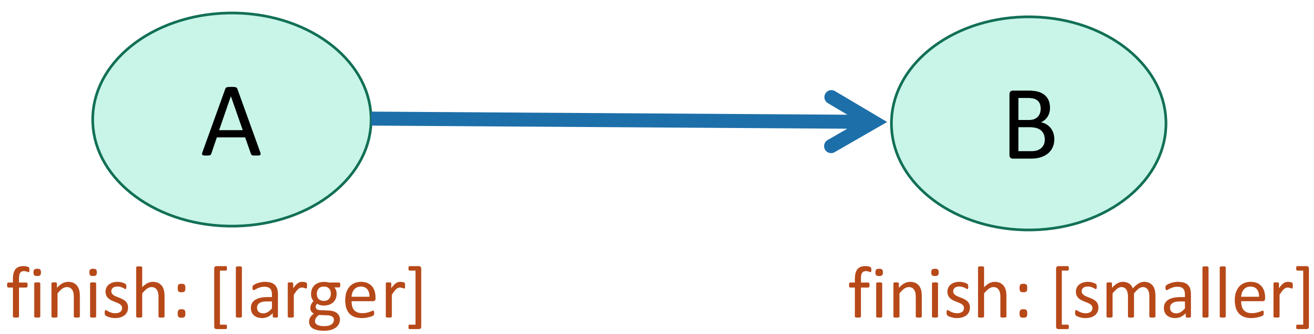




Suppose the underlying graph has no cycles

Finish times seem useful

Claim: In general, we'll always have:



To understand why, let's go back to that DFS tree.

A more general statement

(this holds even if there are cycles)

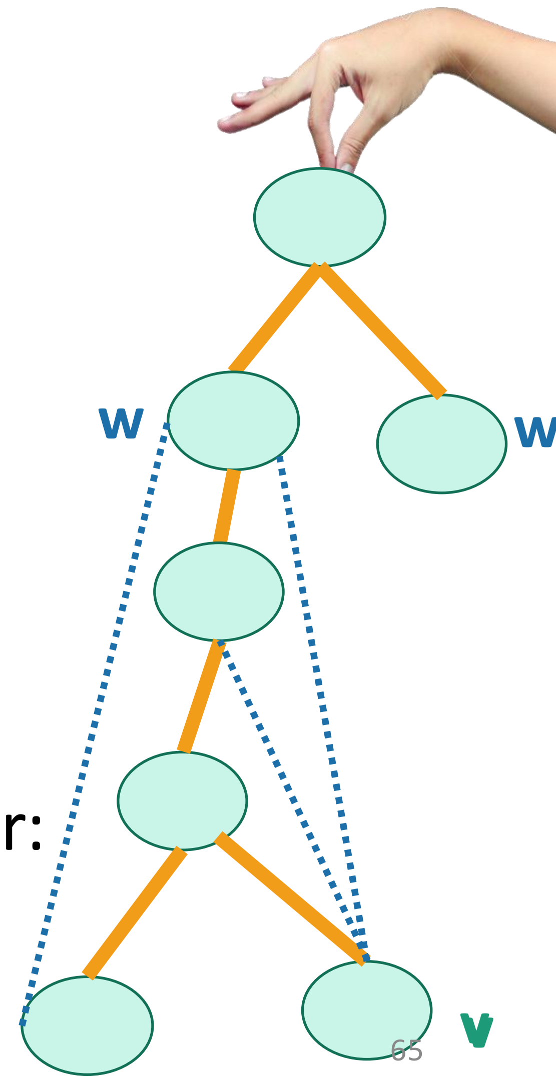
(check this statement carefully!)

- If v is a descendant of w in this tree:

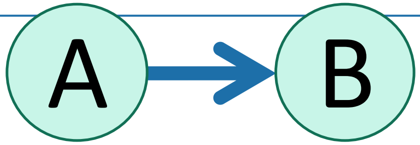
- If w is a descendant of v in this tree:

- If neither are descendants of each other:

(or the other way around)

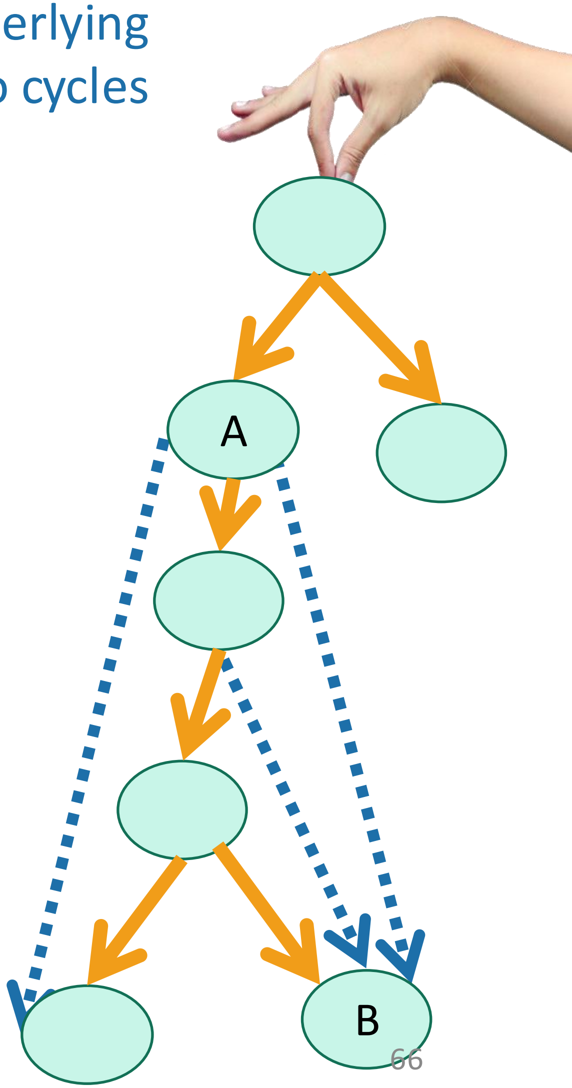


Proof of this →

If 

Then B.finishTime < A.finishTime

Suppose the underlying graph has no cycles

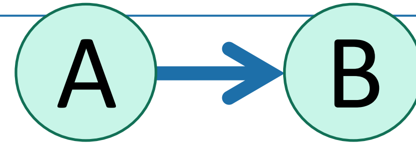


- **Case 1**: B is a descendant of A in the DFS tree.

- Then

- aka, B.finishTime < A.finishTime.

Proof of this →

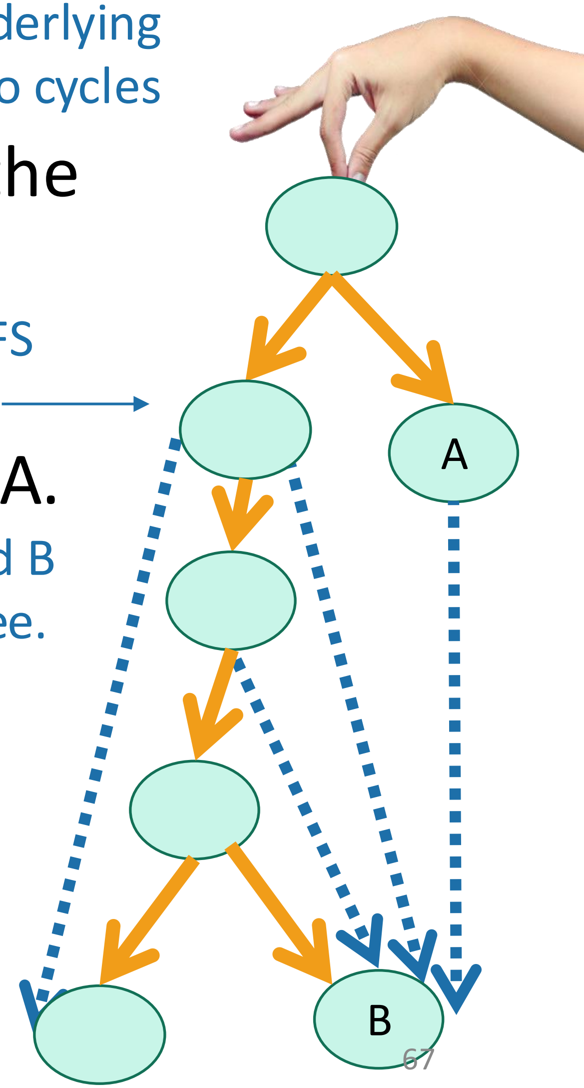
If 

Then B.finishTime < A.finishTime

Suppose the underlying graph has no cycles

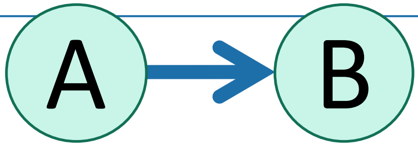
- **Case 2**: B is a NOT descendant of A in the DFS tree.
  - Notice that A can't be a descendant of B in the DFS tree or else there'd be a cycle; so it looks like this →
- Then we must have explored B before A.
  - Otherwise we would have gotten to B from A, and B would have been a descendant of A in the DFS tree.
- Then

- aka, B.finishTime < A.finishTime.



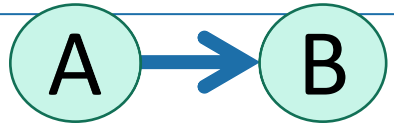
Theorem

- If we run DFS on a directed acyclic graph,

If 

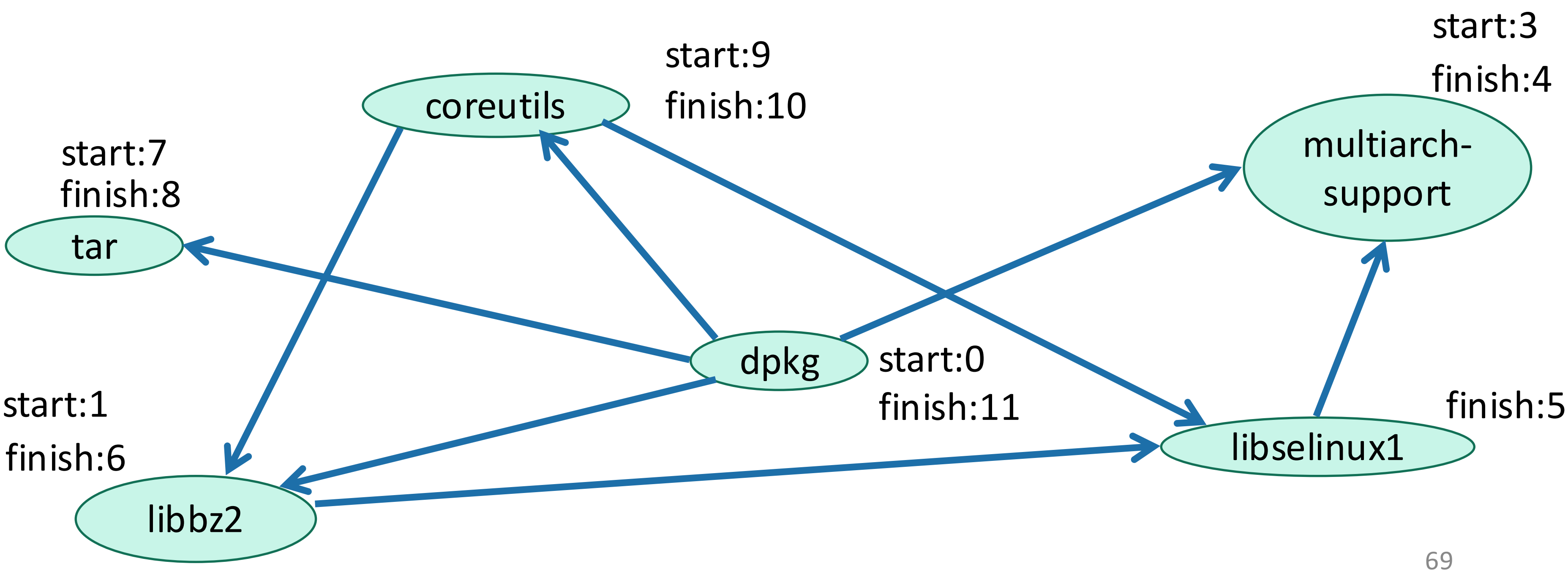
Then B.finishTime < A.finishTime

Back to topological sorting

If 

Then B.finishTime < A.finishTime

- In what order should I install packages?
- In reverse order of finishing time in DFS!
  - Then, the **theorem** says we'll never have a "backward" edge

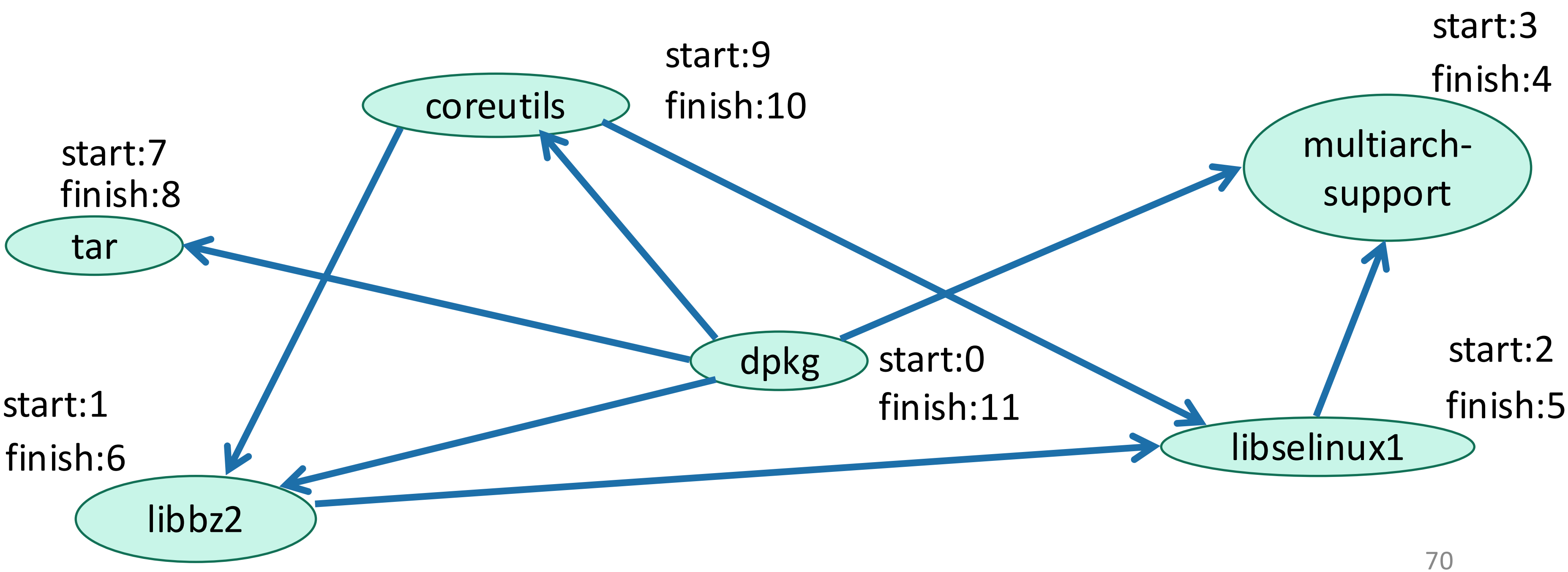


Topological Sorting (on a DAG)

Check out iPython notebook for an implementation!

- Do DFS
- When you mark a vertex as **all done**, put it at the **beginning** of the list.

- `dpkg`
- `coreutils`
- `tar`
- `libbz2`
- `libselinux1`
- `multiarch_support`



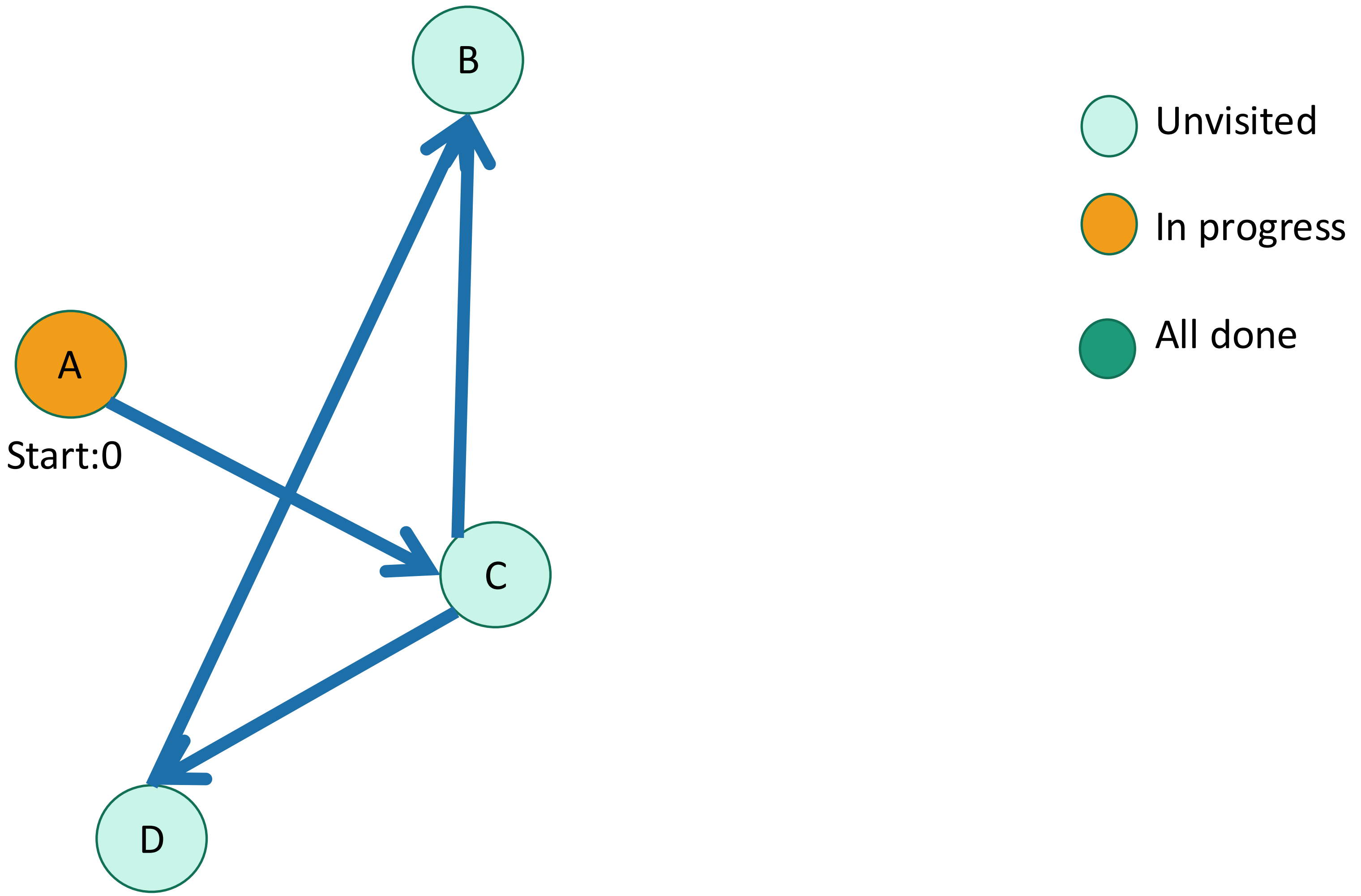
What have we learned?

- DFS can help you solve the **topological sorting problem**
  - That’s the fancy name for the problem of finding an ordering that respects all the dependencies

- Thinking about the DFS tree is helpful.

Example:

This example skipped in class – here for reference.



Example

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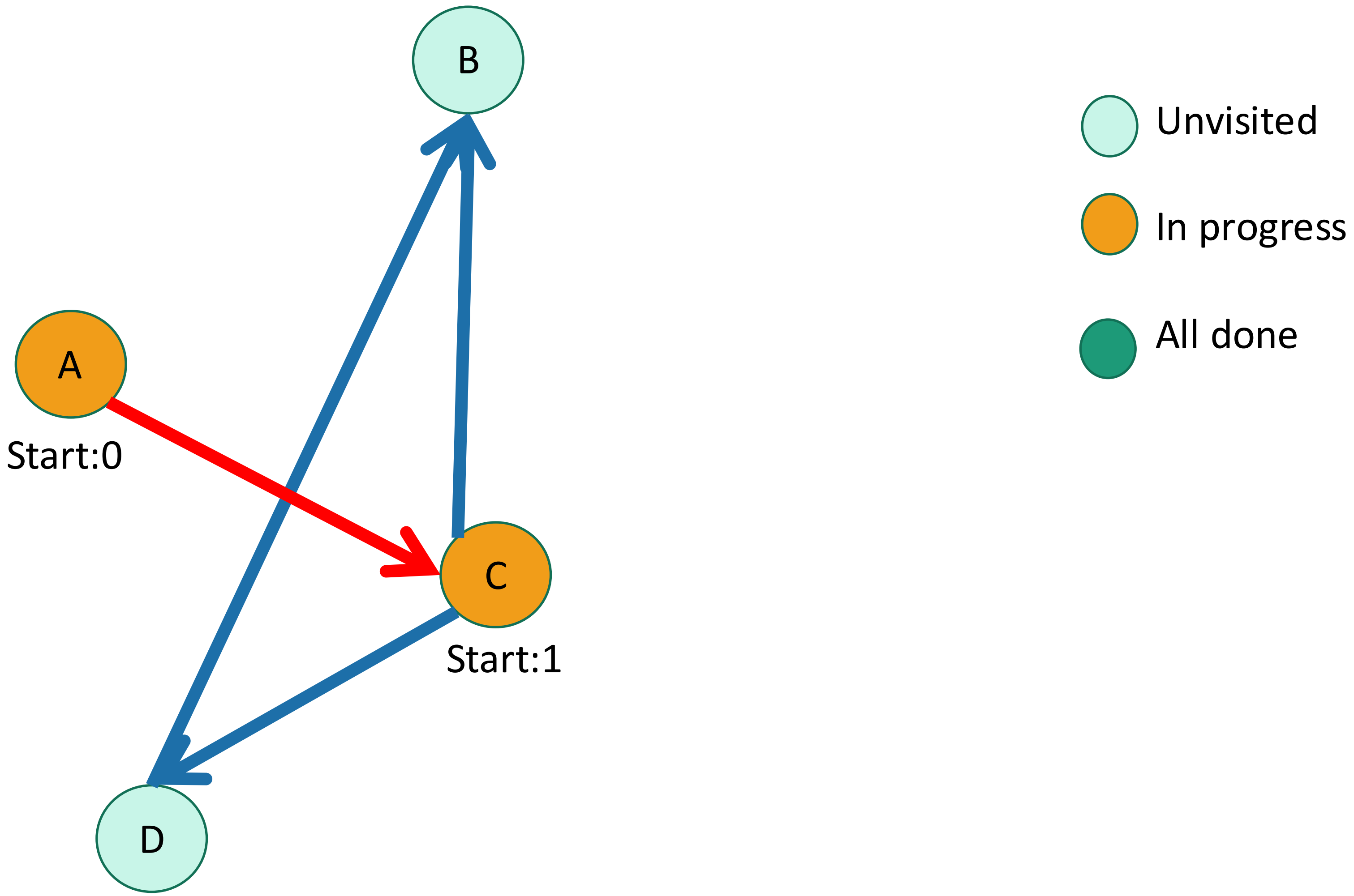

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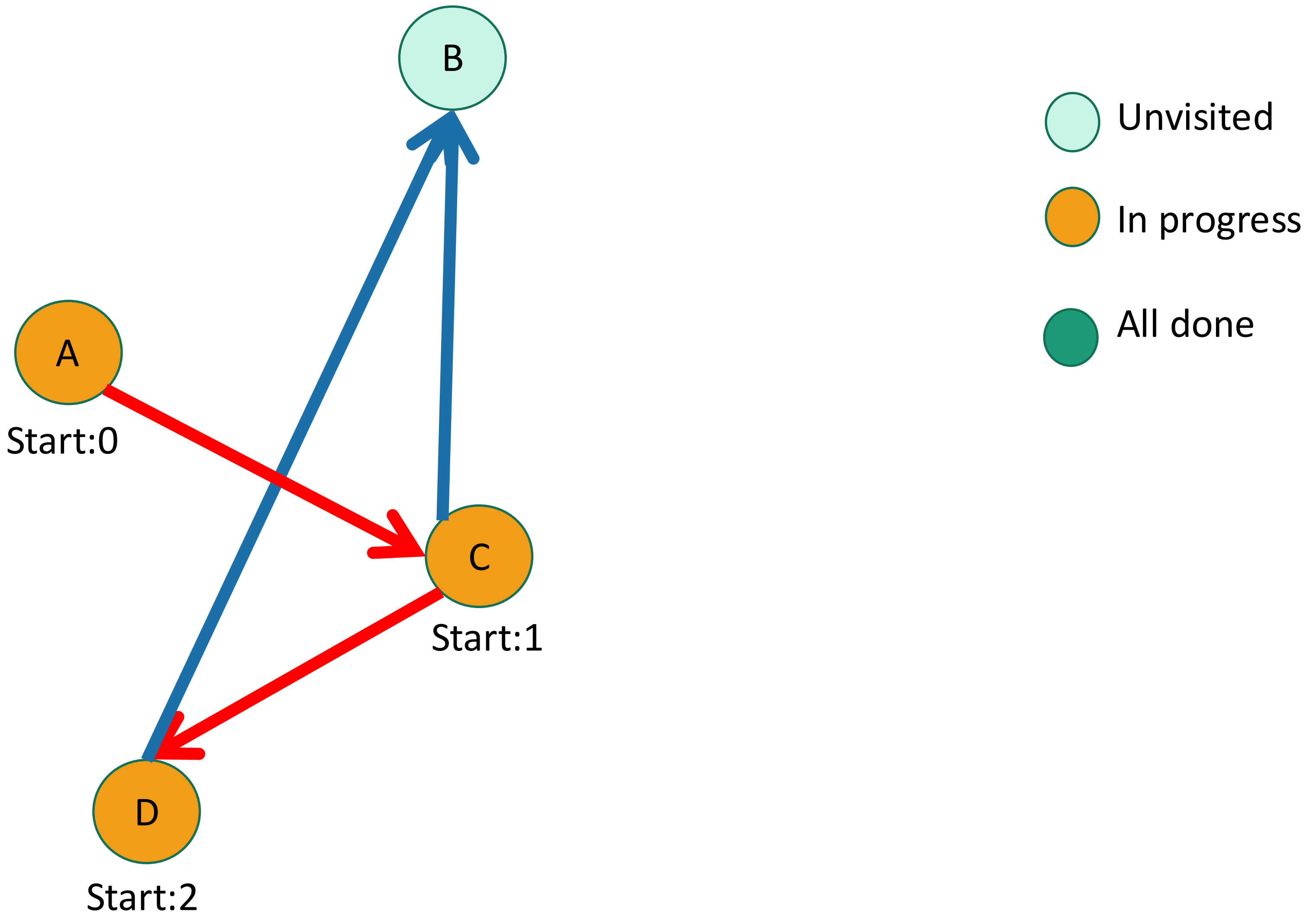

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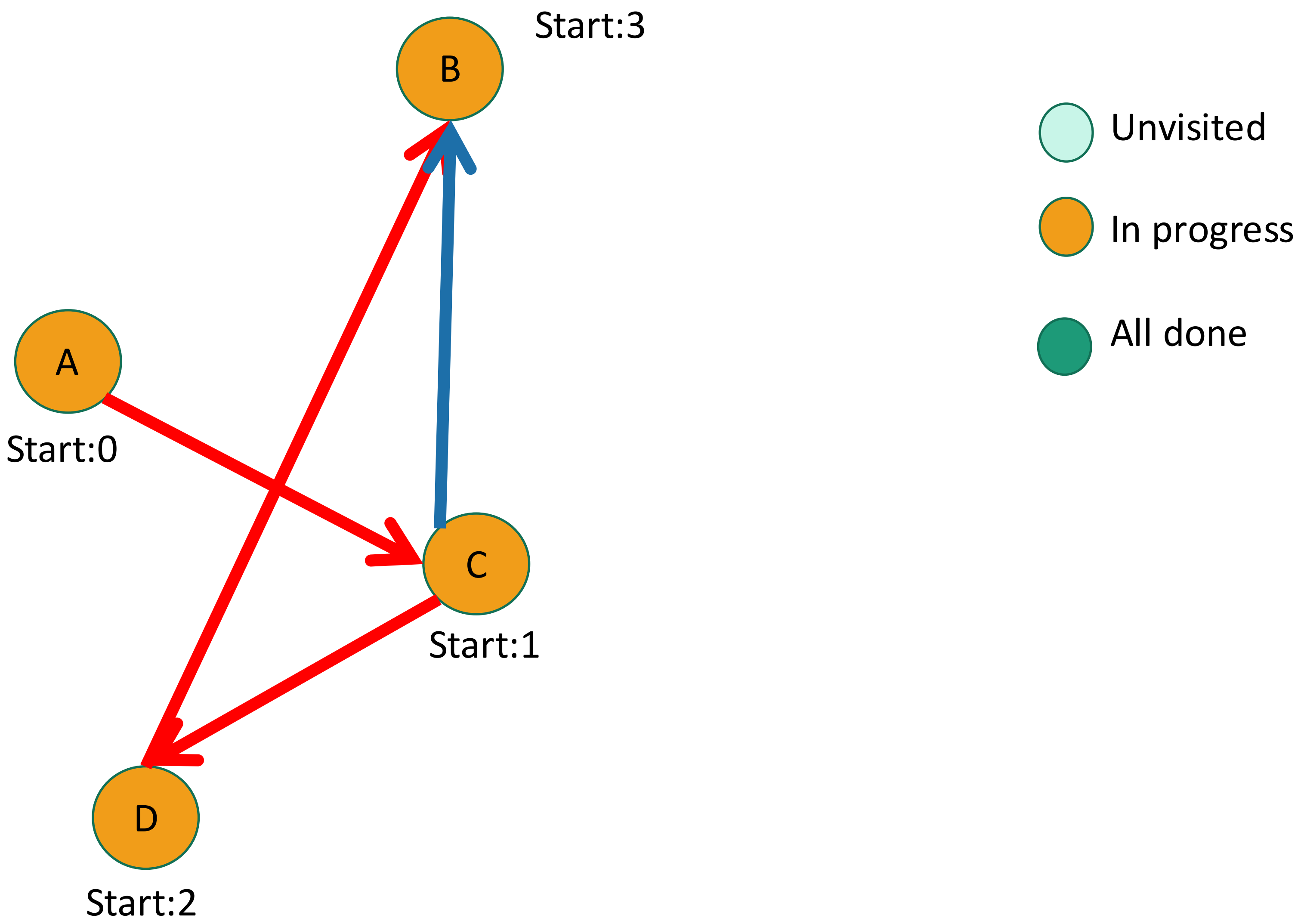

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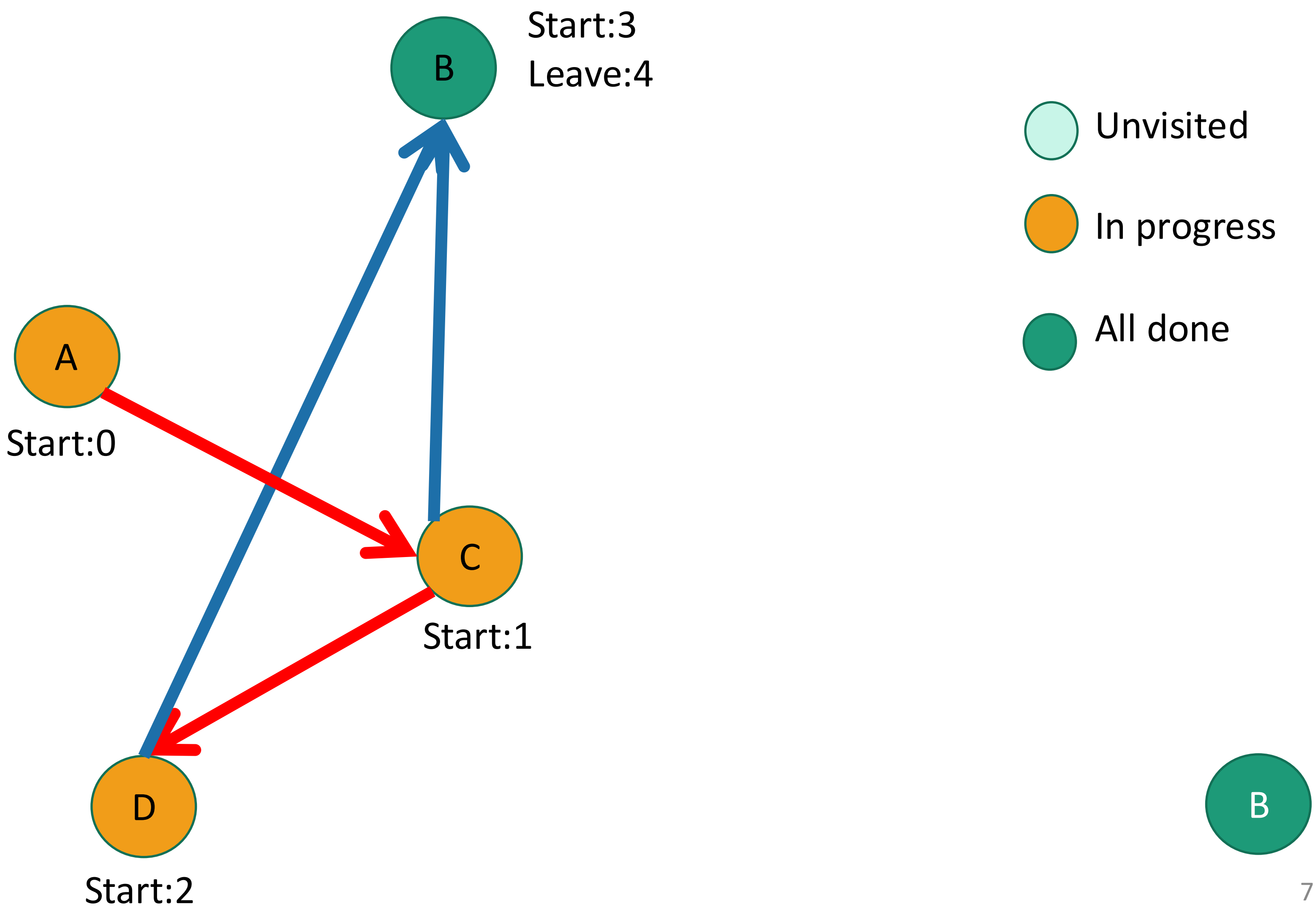

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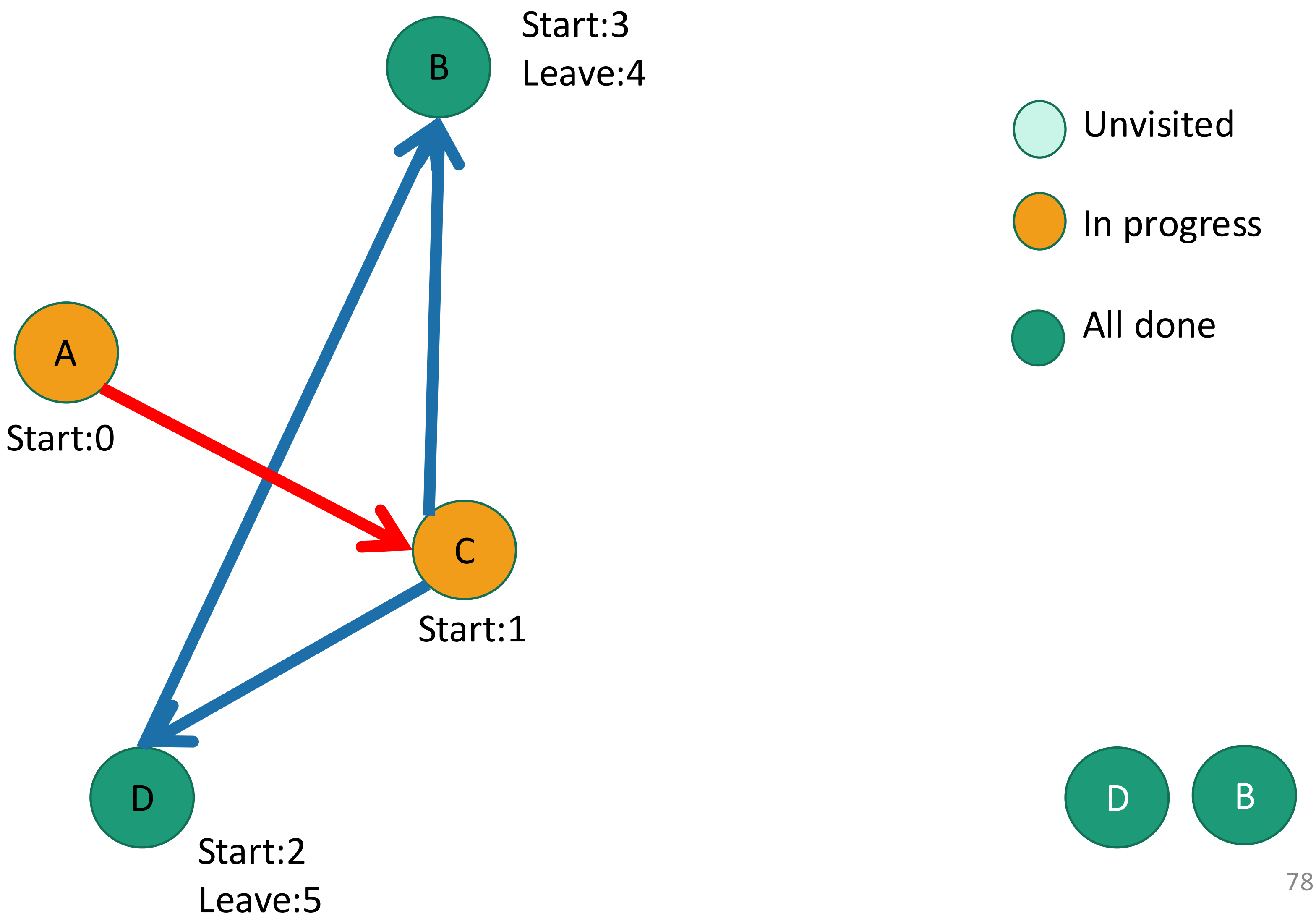

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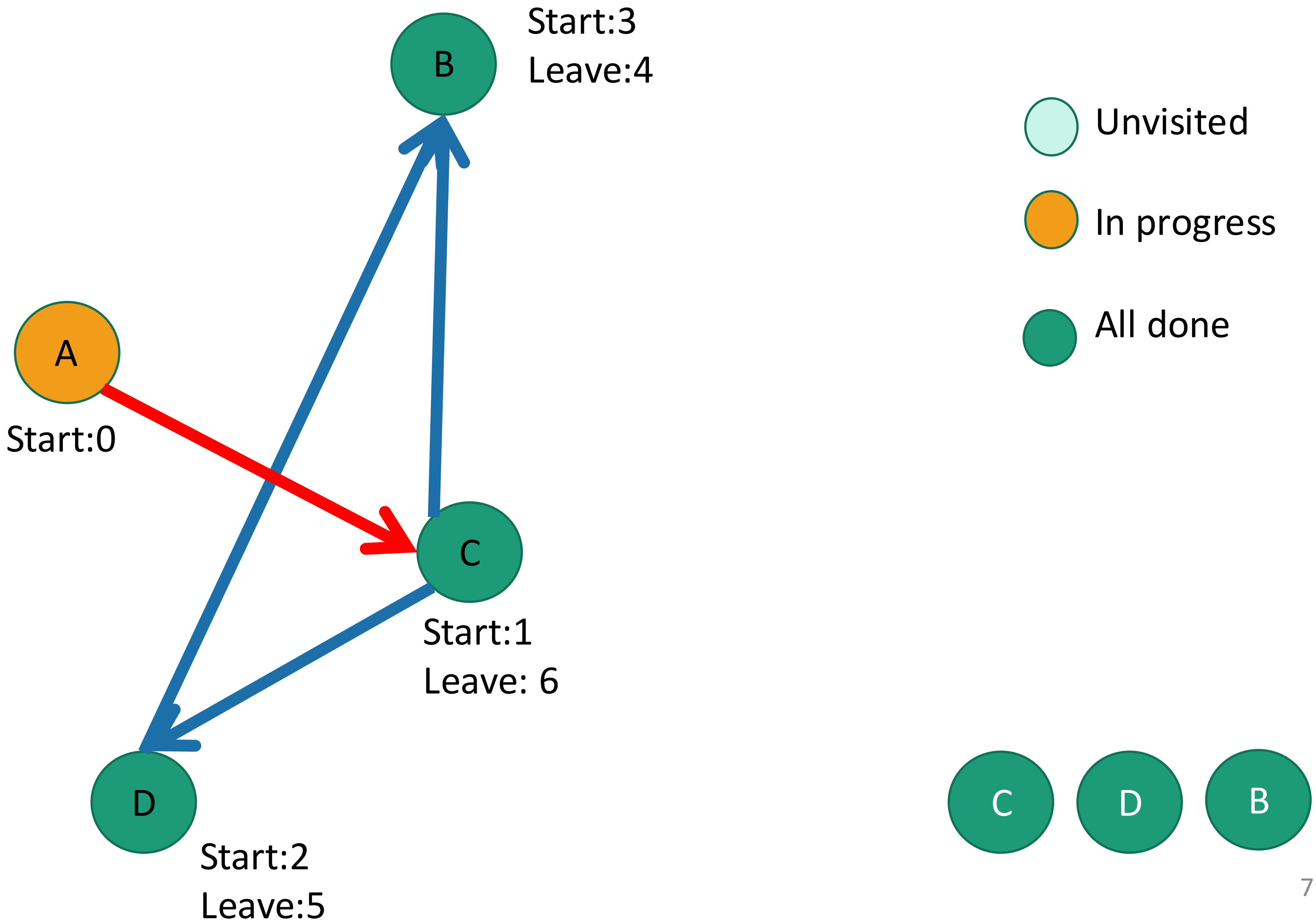

Example

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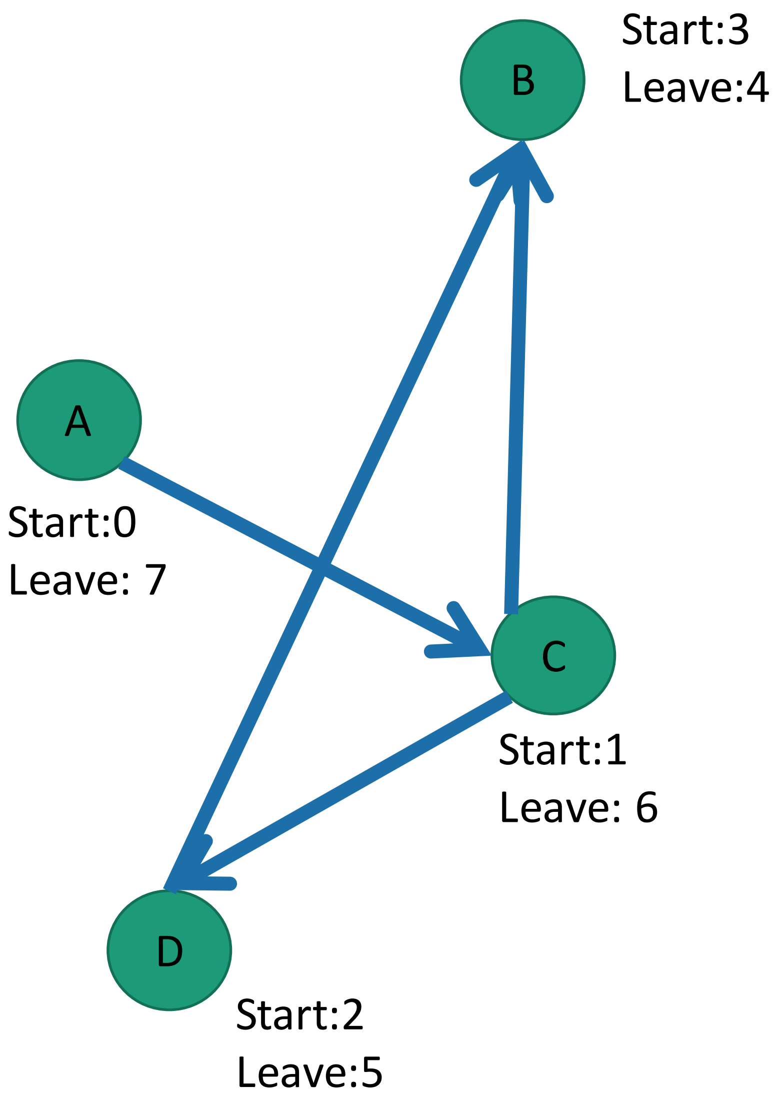
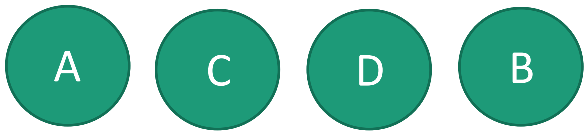


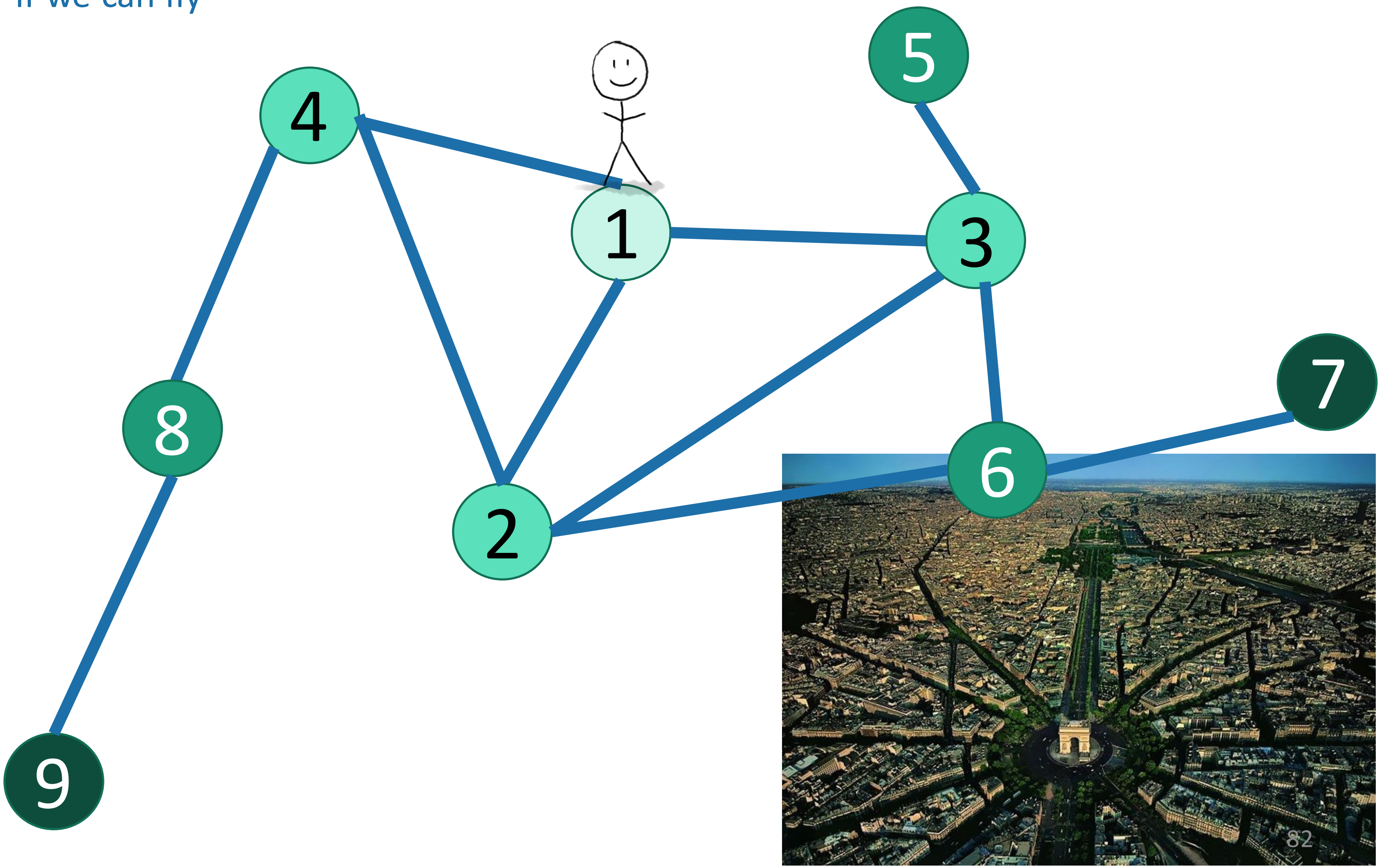
Do them in this order:



Part 2: breadth-first search

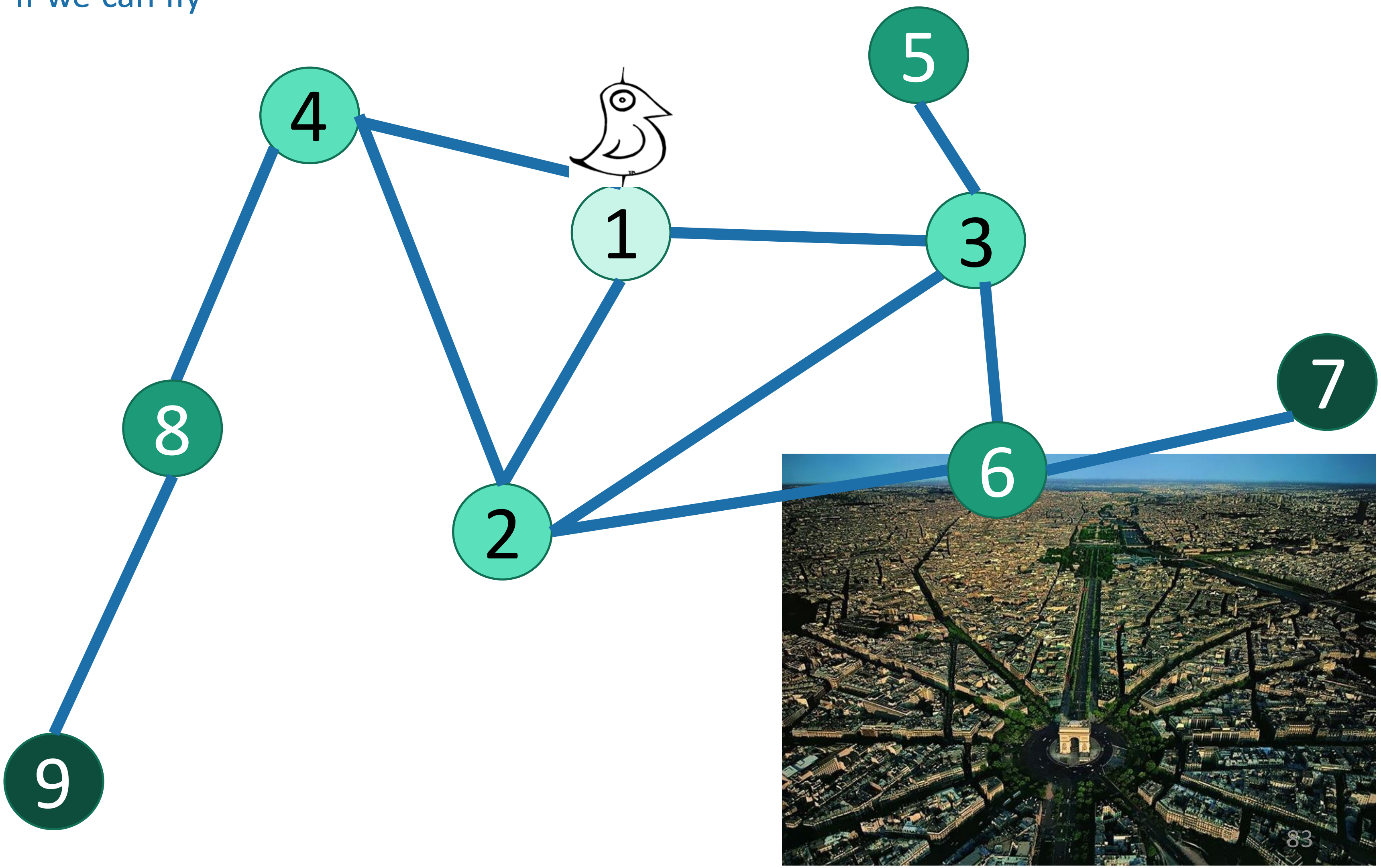
How do we explore a graph?

If we can fly



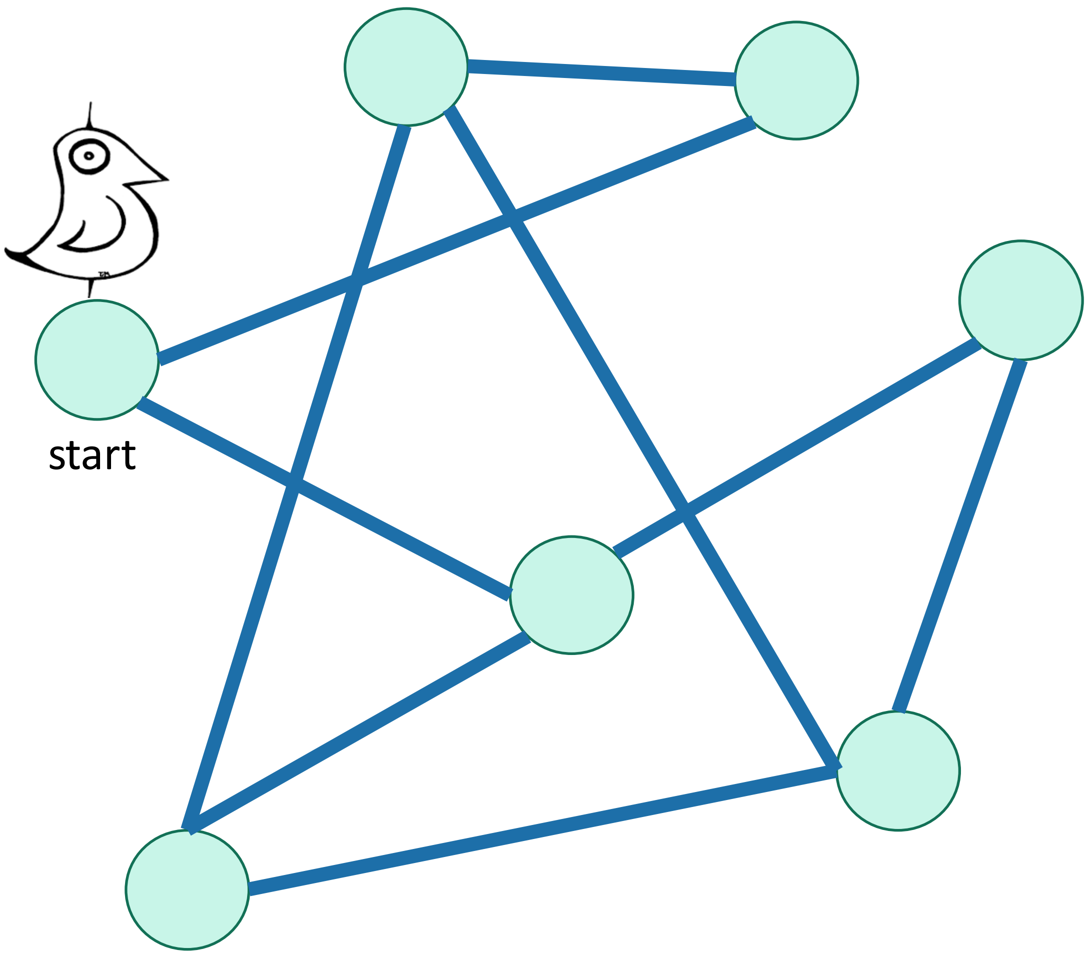
How do we explore a graph?

If we can fly



Breadth-First Search

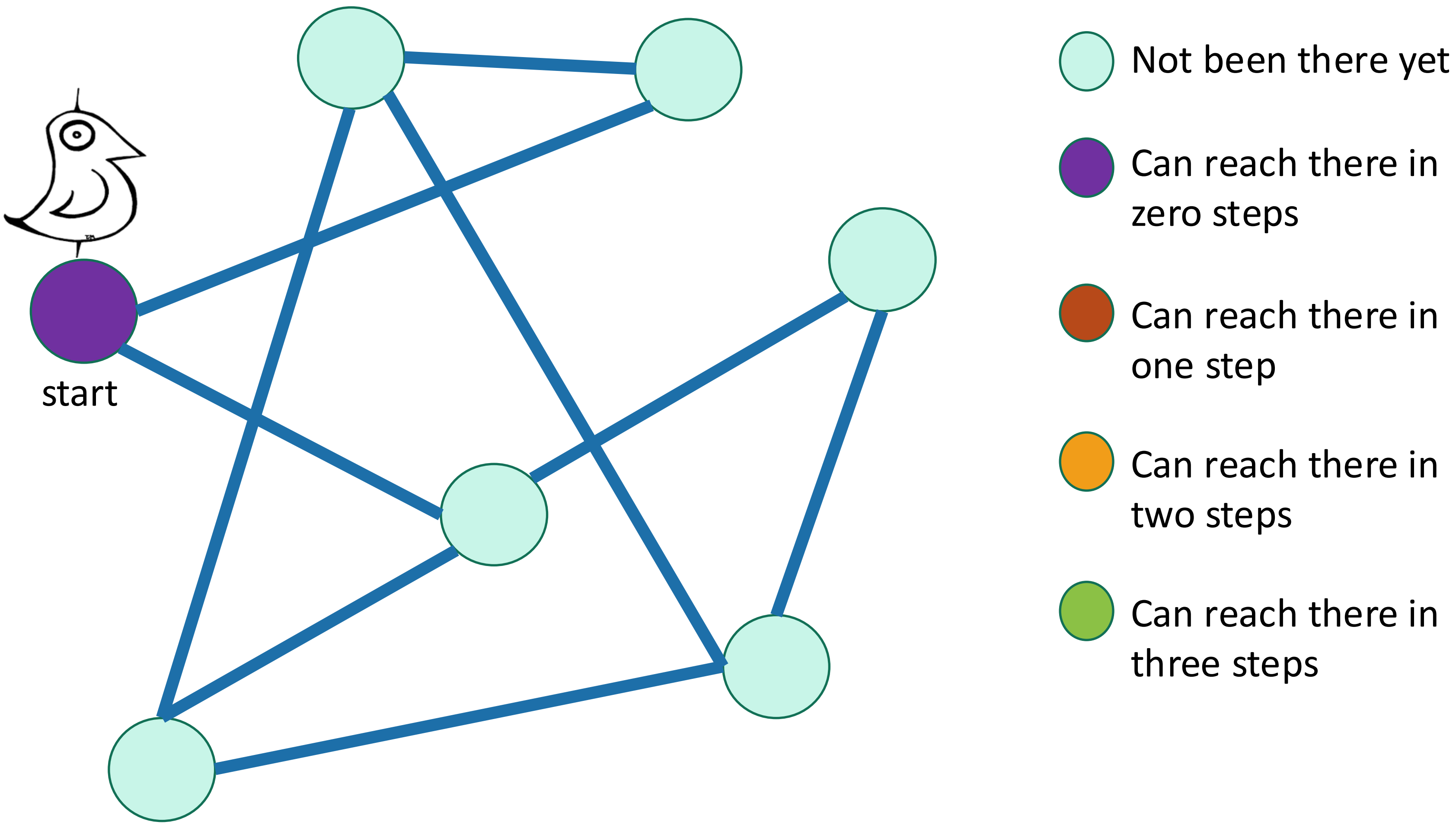
Exploring the world with a bird's-eye view



- Not been there yet
- Can reach there in zero steps
- Can reach there in one step
- Can reach there in two steps
- Can reach there in three steps

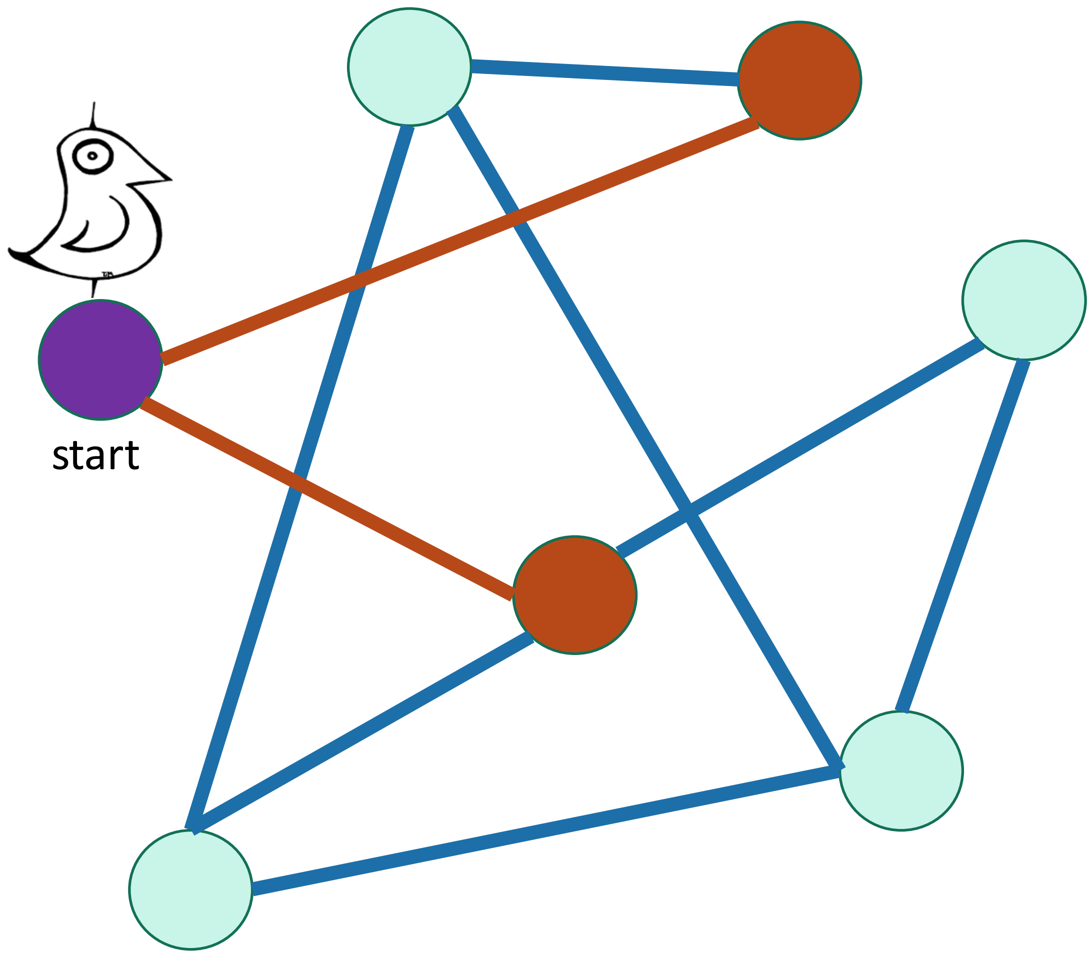
Breadth-First Search

Exploring the world with a bird's-eye view



Breadth-First Search

Exploring the world with a bird's-eye view



- Not been there yet
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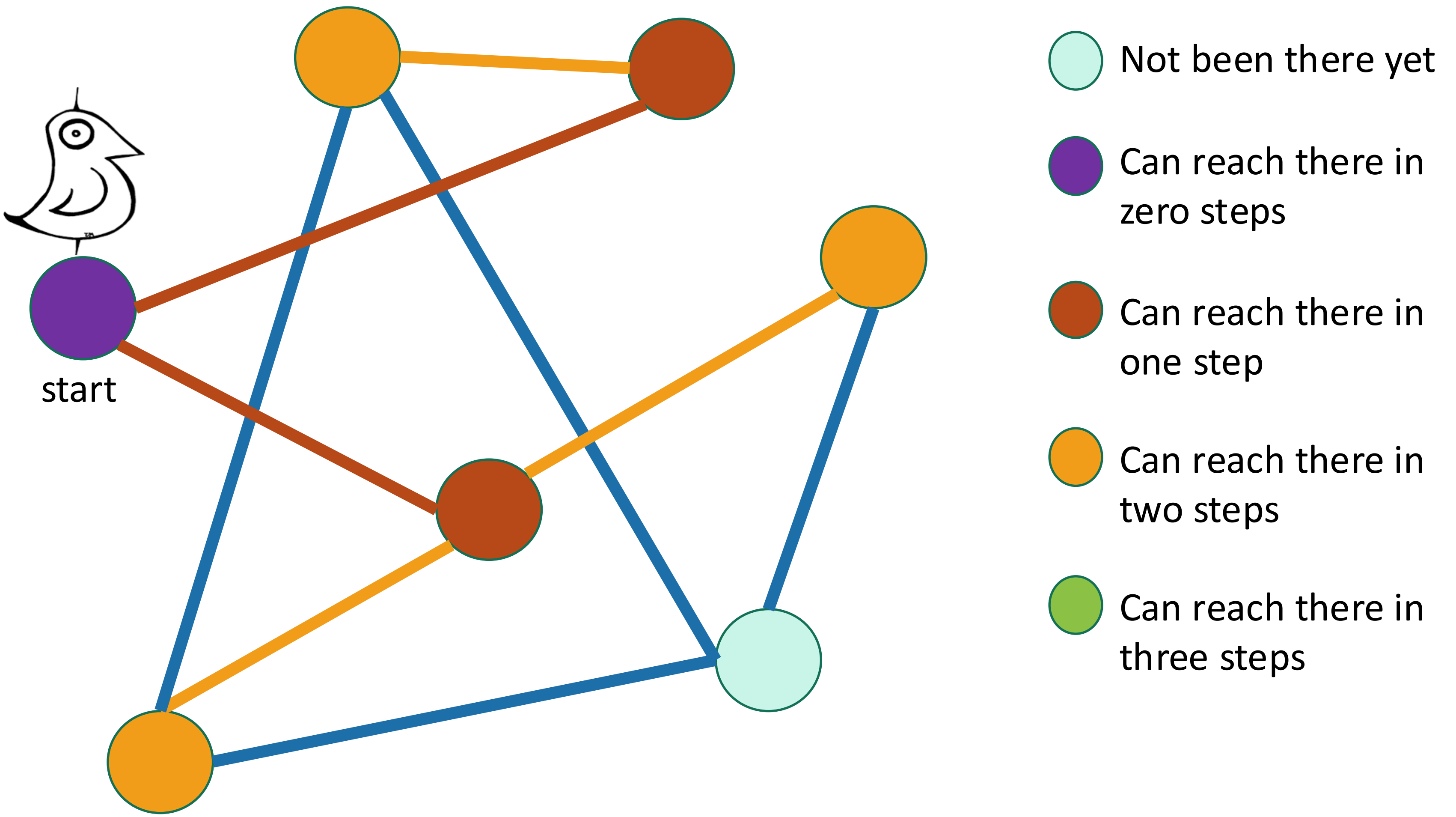
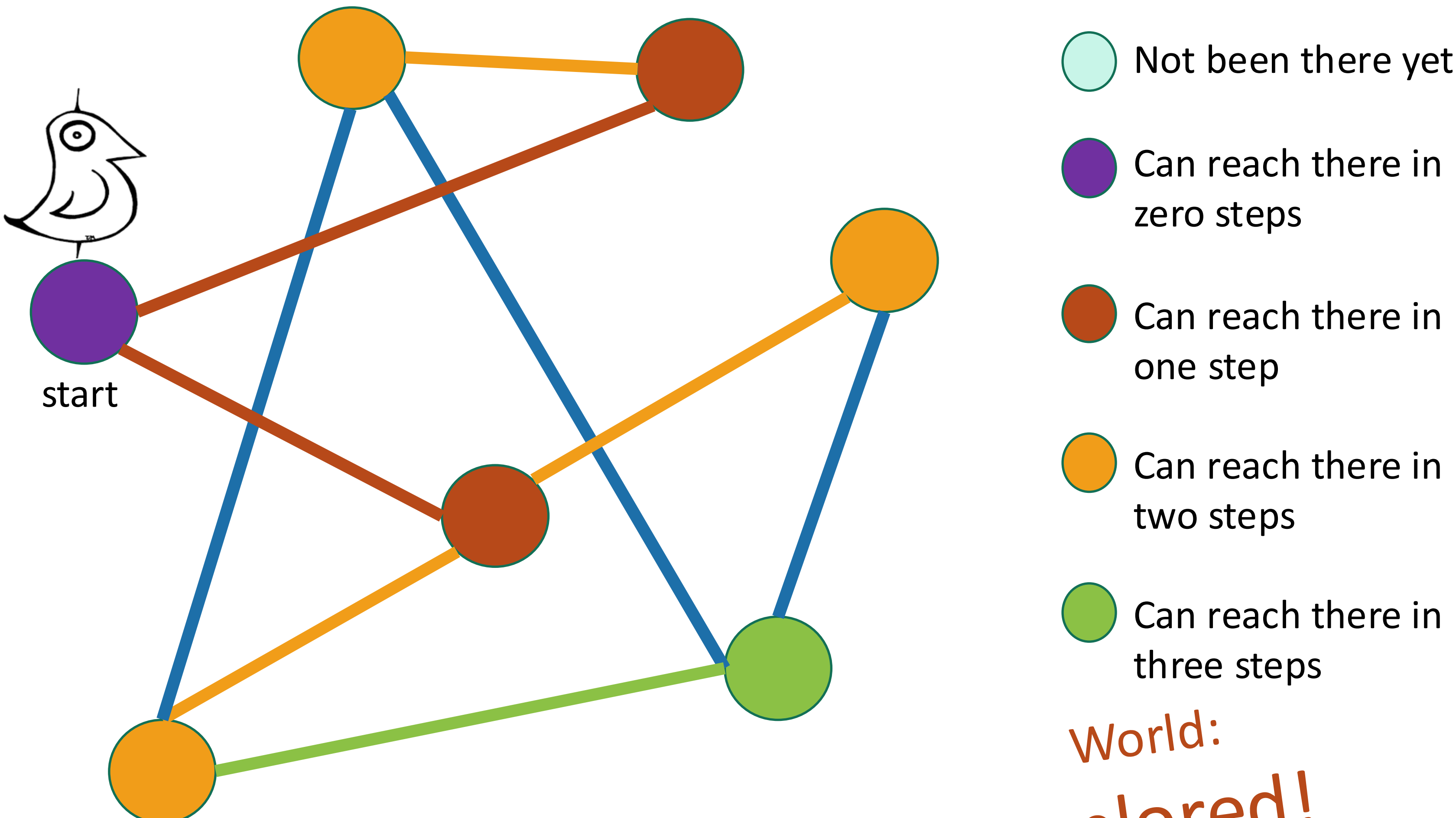
Breadth-First Search

Exploring the world with a bird's-eye view



Breadth-First Search

Exploring the world with a bird's-eye view



World: explored!

Same disclaimer as for DFS: you may have seen other ways to implement this, this will be convenient for us.

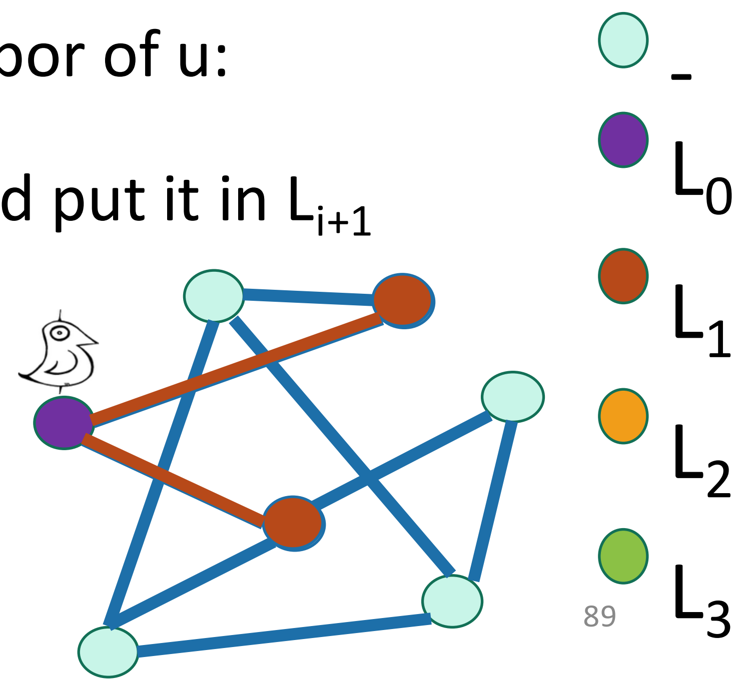
Breadth-First Search

Exploring the world with pseudocode

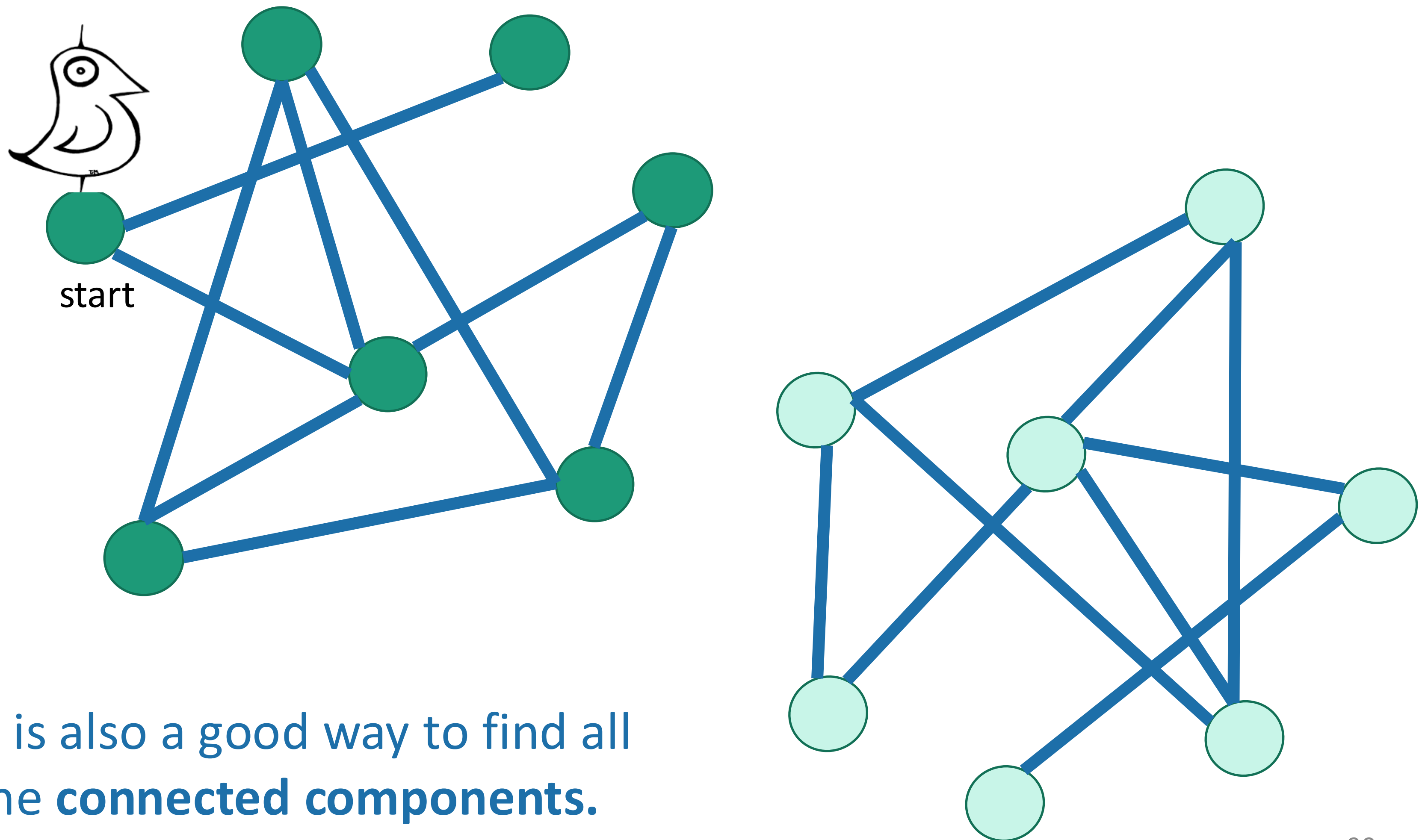
- Set L_i = [] for i=1,…,n
- L_0 = [w], where w is the start node
- Mark w as visited
- **For** i = 0, …, n-1:
  - **For** u in L_i:
    - **For** each v which is a neighbor of u:
      - **If** v isn't yet visited:
        - mark v as visited, and put it in L_{i+1}

L_i is the set of nodes we can reach in i steps from w

Go through all the nodes in L_i and add their unvisited neighbors to L_{i+1}



BFS also finds all the nodes reachable from the starting point

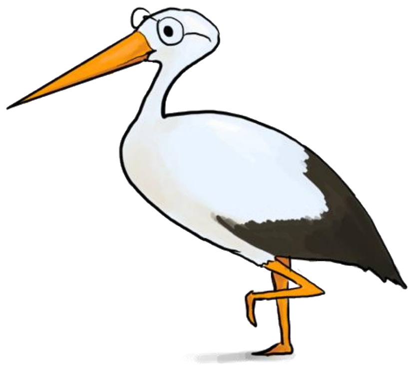


It is also a good way to find all the **connected components.**

Running time and extension to directed graphs

- To explore the whole graph, explore the connected components one-by-one.
  - Same argument as DFS: BFS running time is O(n + m)
- Like DFS, BFS also works fine on directed graphs.

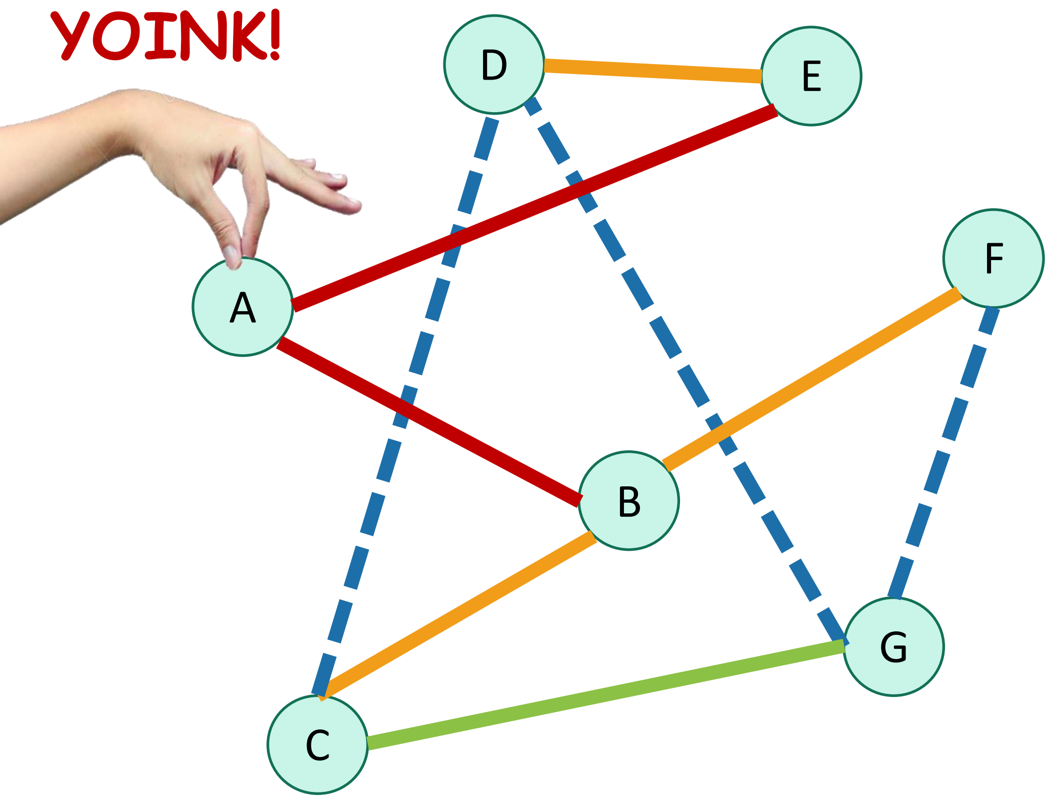
Verify these!

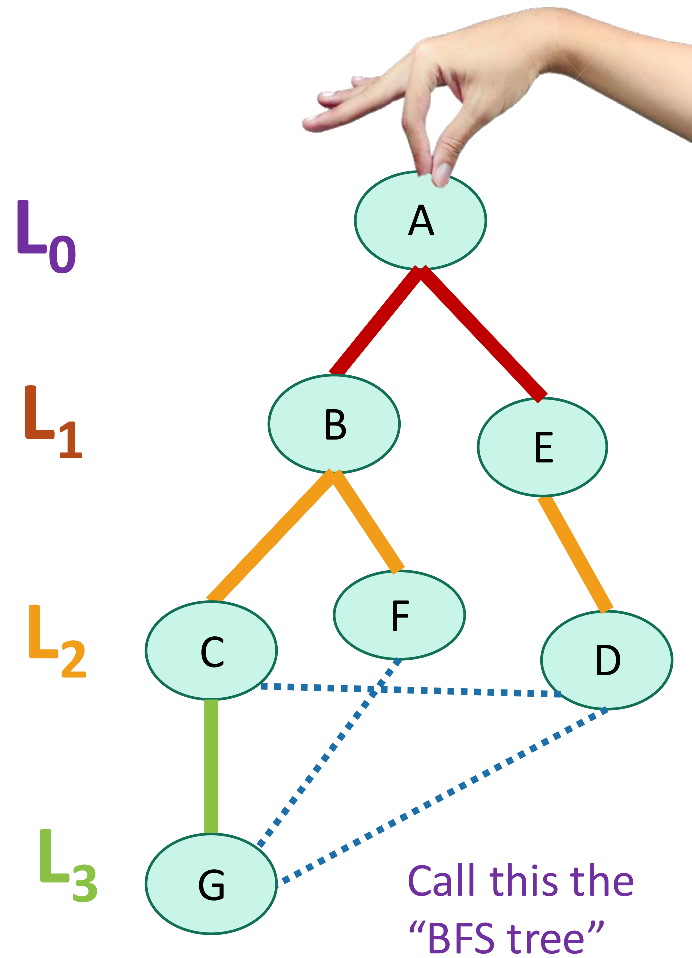


Siggi the Studious Stork

Why is it called breadth-first?

- We are implicitly building a tree:

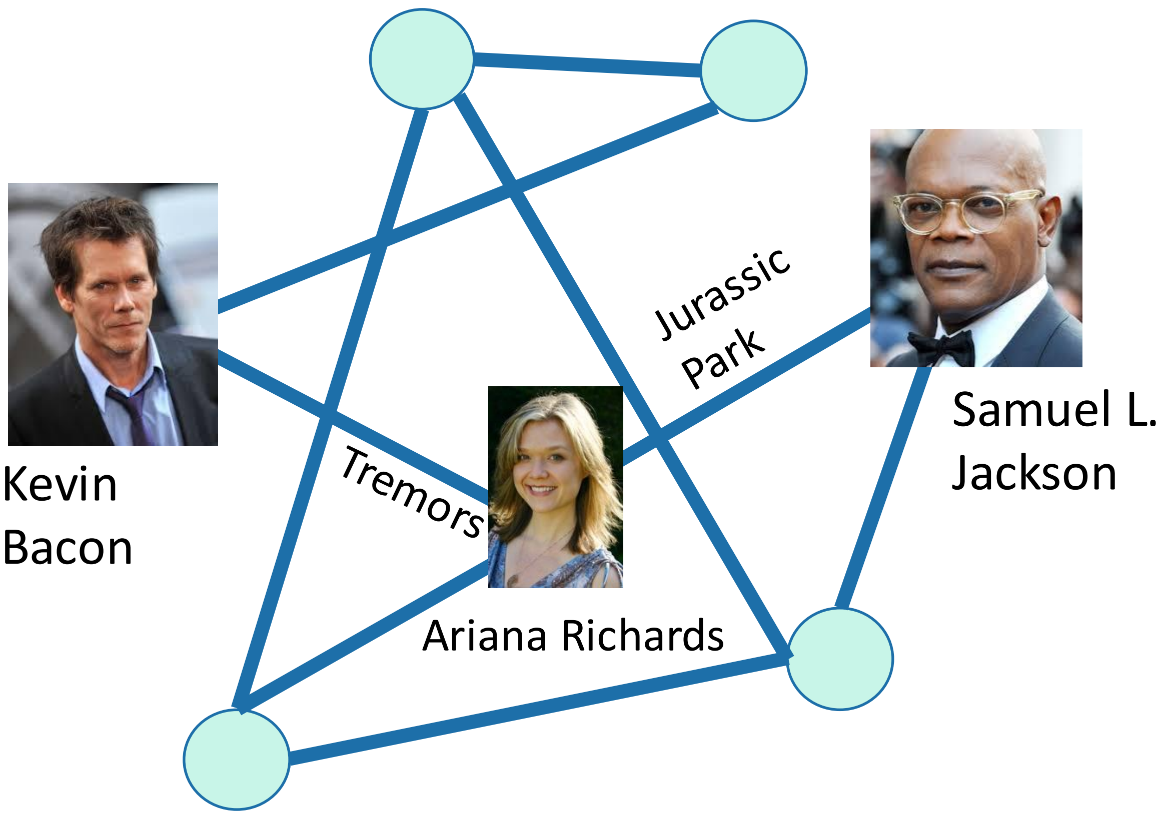




- First we go as broadly as we can.

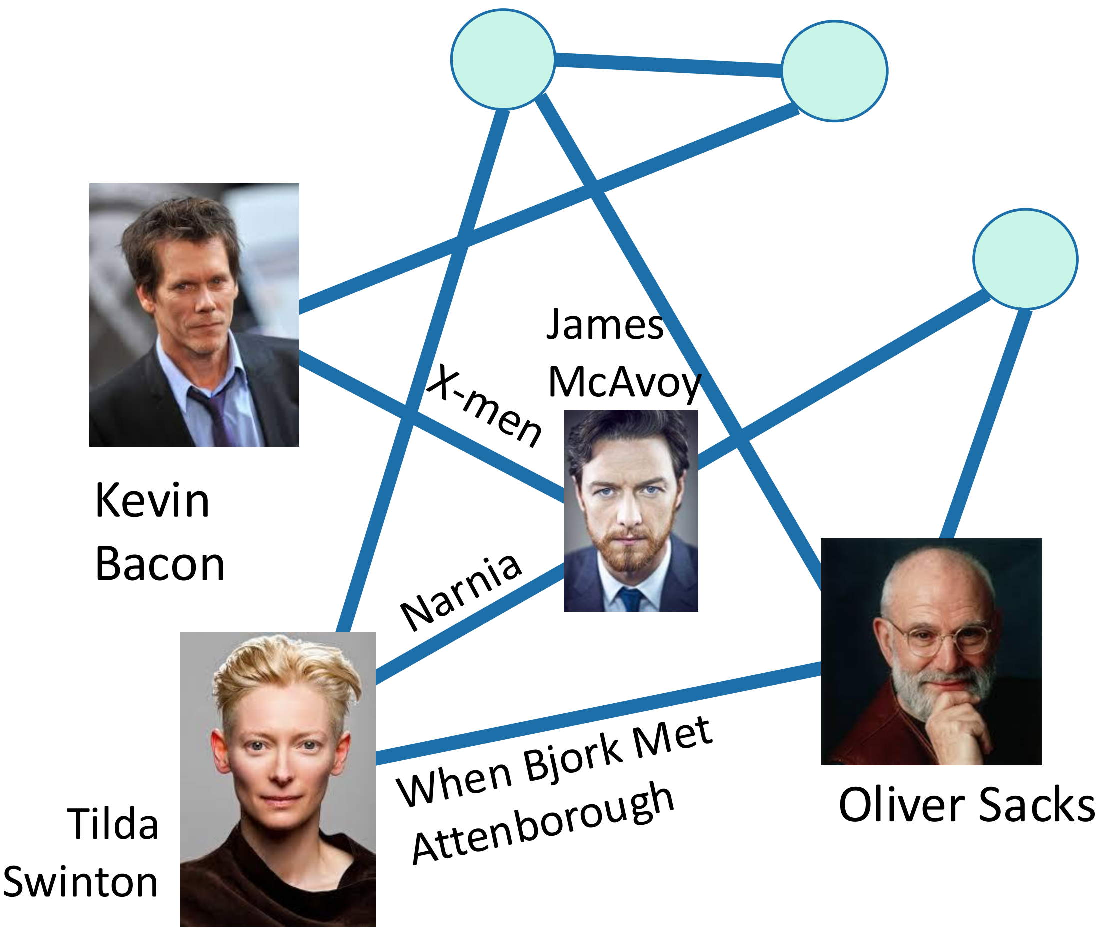
Pre-lecture exercise

- What Samuel L. Jackson's Bacon number?



(Answer: 2)

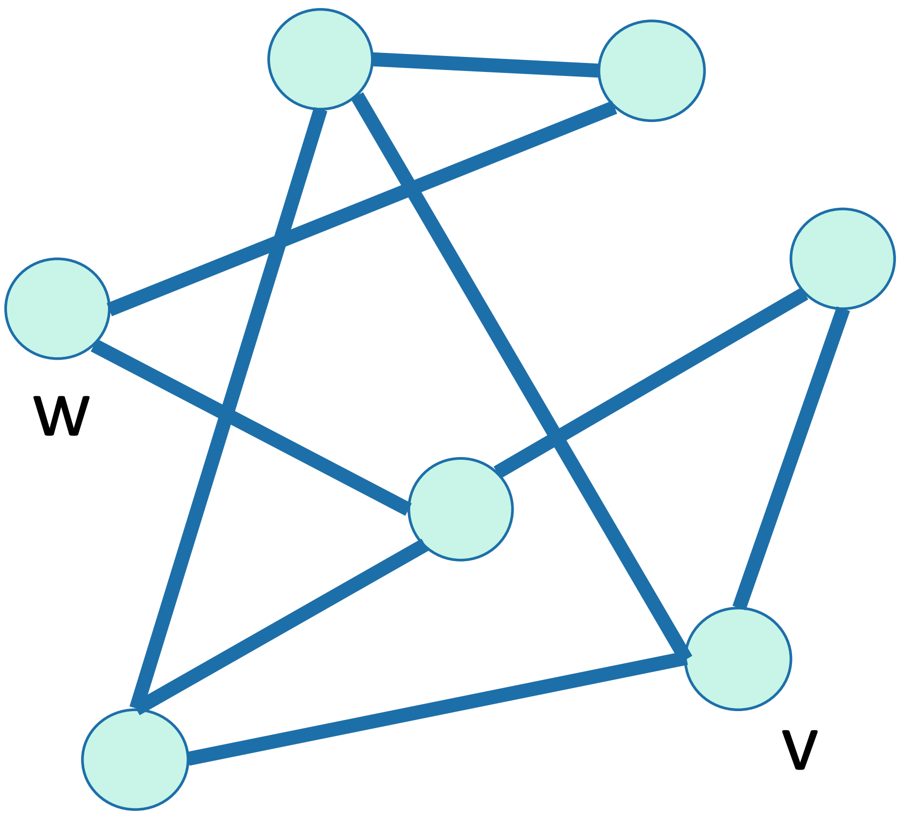
I wrote the pre-lecture exercise before I realized that I really wanted an example with distance 3



It is really hard to find people with Bacon number 3!

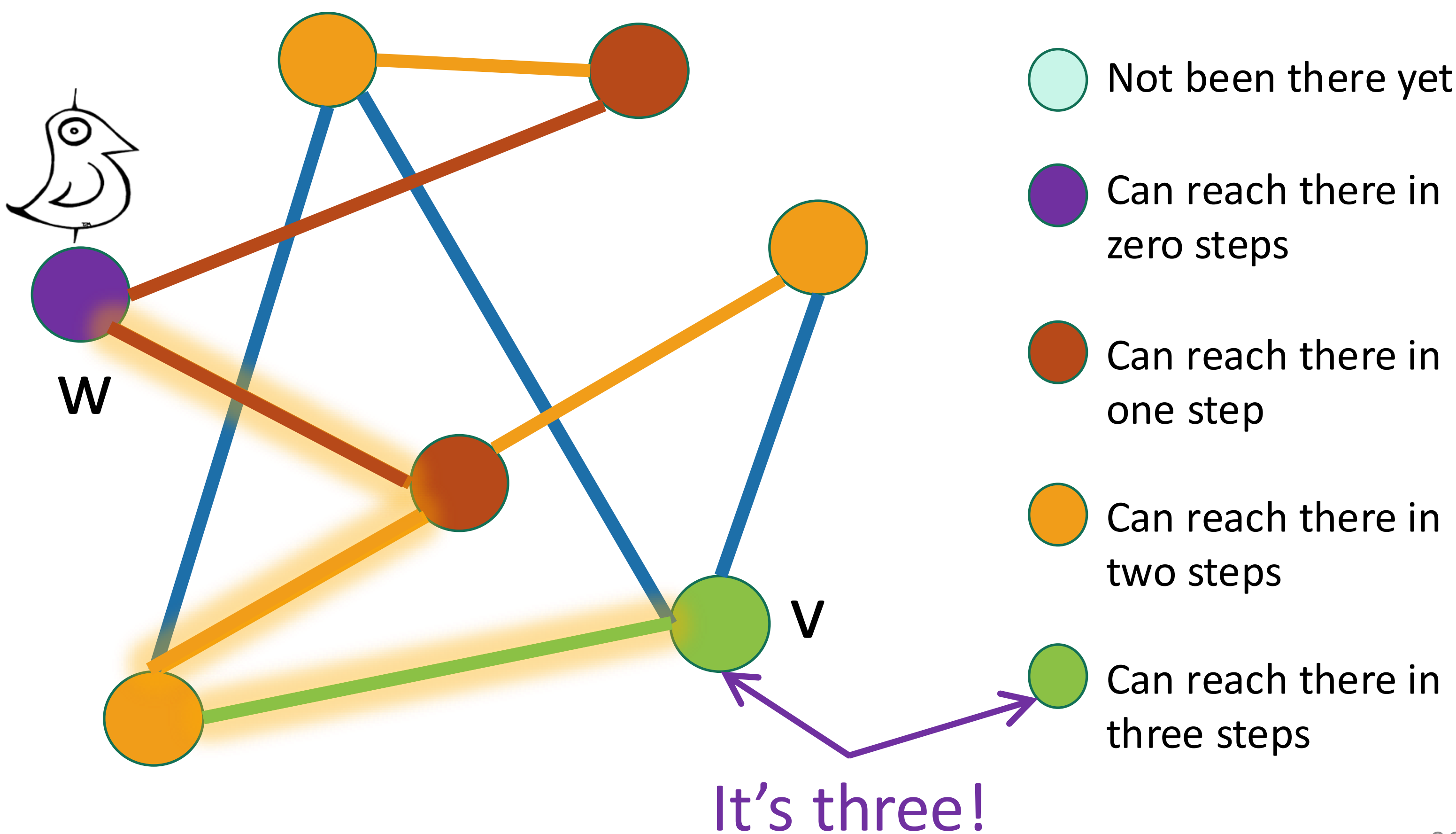
Application of BFS: shortest path

- How long is the shortest path between w and v?



Application of BFS: shortest path

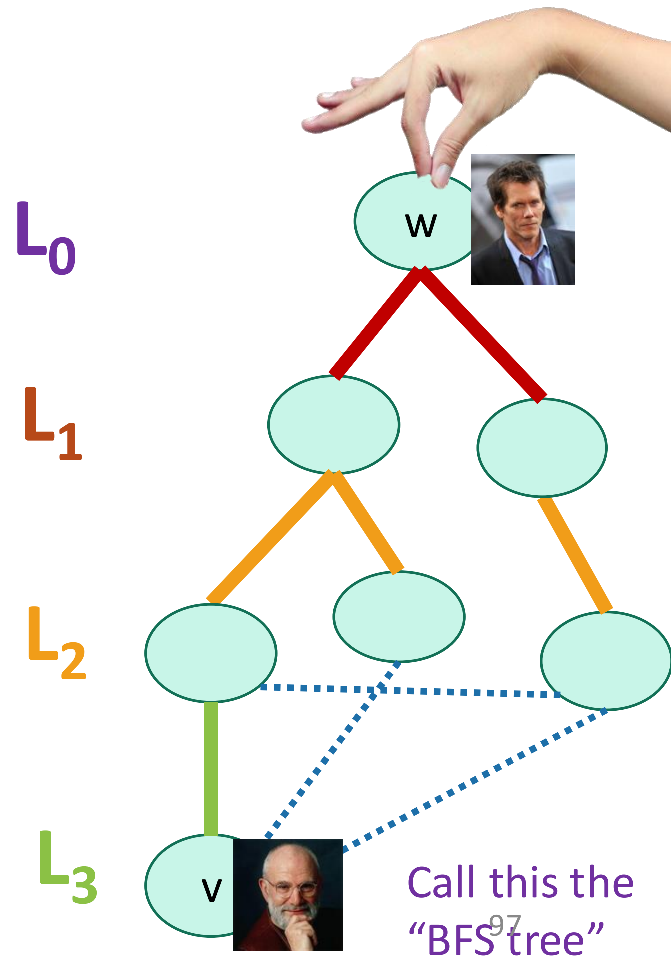
- How long is the shortest path between w and v?



To find the distance between w and all other vertices v

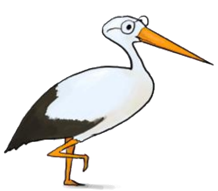
The **distance** between two vertices is the number of edges in the shortest path between them.

- Do a BFS starting at w
- For all v in L_i
  - The shortest path between w and v has length i
  - A shortest path between w and v is given by the path in the BFS tree.
- If we never found v, the distance is infinite.

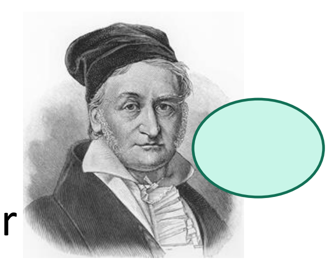


Call this the "BFS tree"

Modify the BFS pseudocode to return shortest paths!
Prove that this indeed returns shortest paths!



Gauss has no Bacon number



What have we learned?

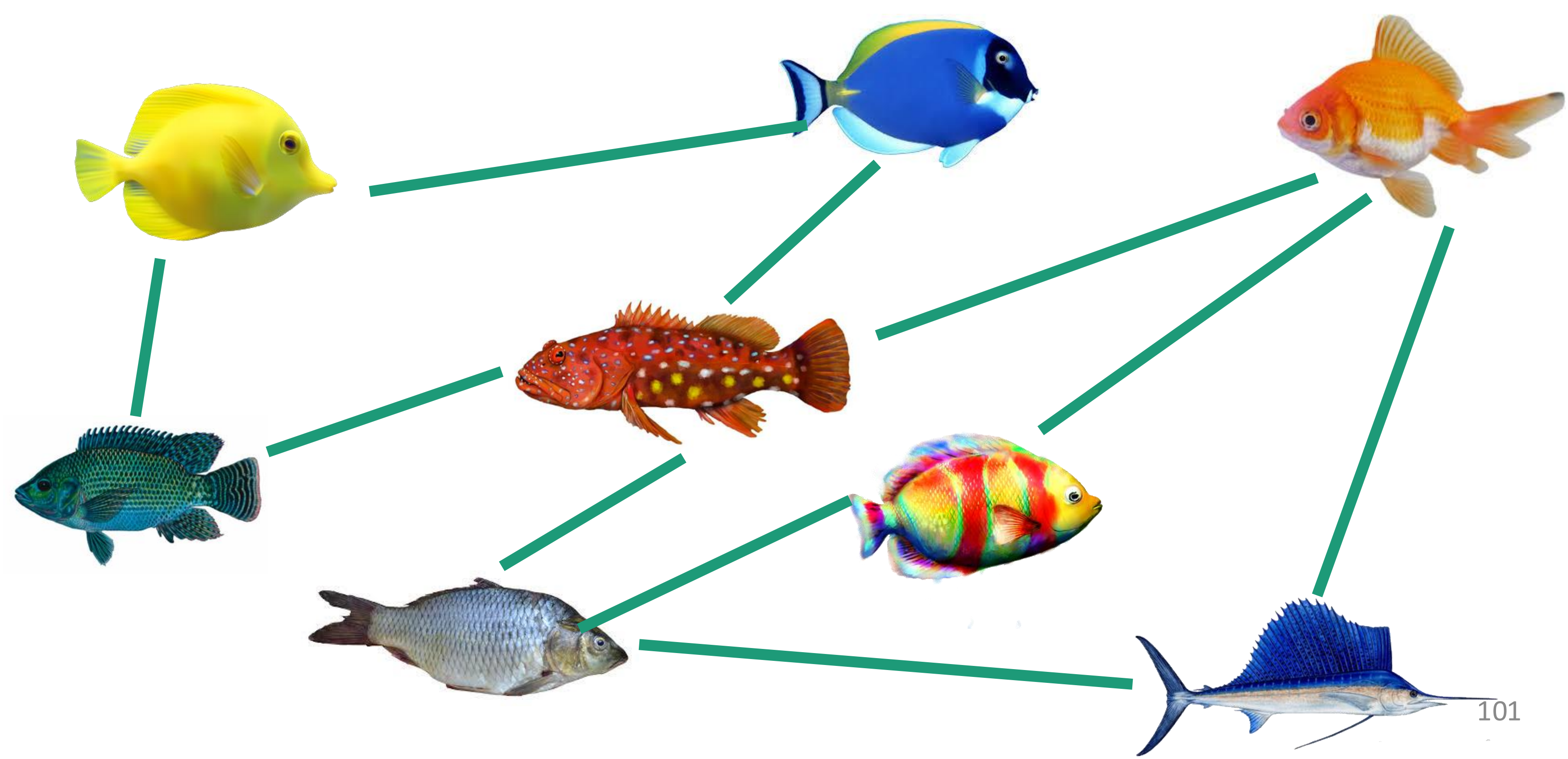
- The BFS tree is useful for computing distances between pairs of vertices.
- We can find the shortest path between u and v in time O(m).

Another application of BFS

- Testing bipartite-ness

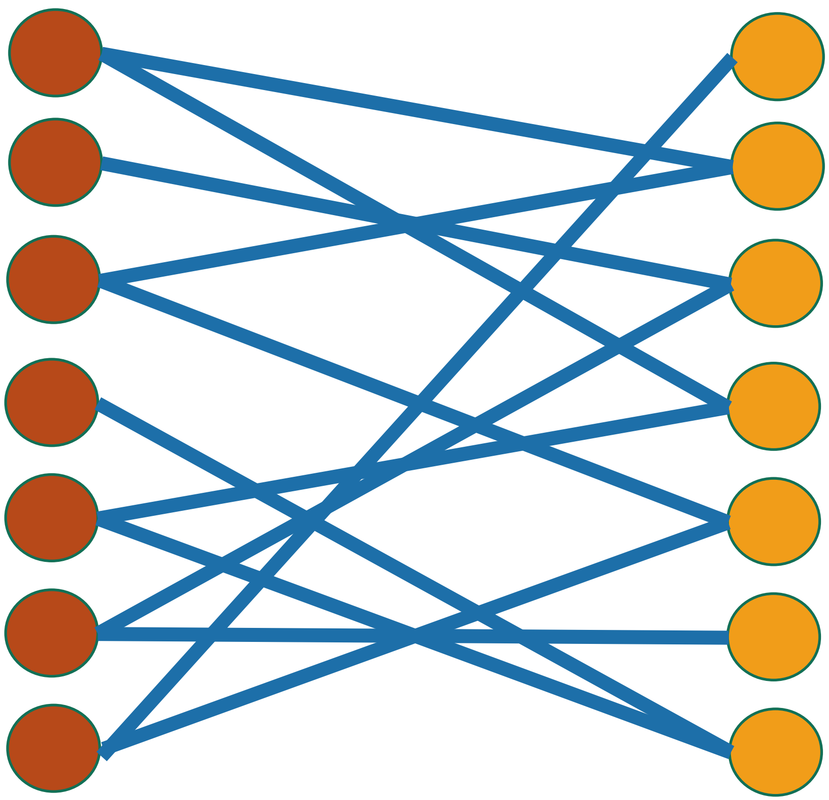
Pre-lecture exercise: fish

- You have a bunch of fish and two fish tanks.
- Some pairs of fish will fight if put in the same tank.
  - Model this as a graph: connected fish will fight.
- Can you put the fish in the two tanks so that there is no fighting?



Bipartite graphs

- A bipartite graph looks like this:



Can color the vertices red and orange so that there are no edges between any same-colored vertices

Example:

are in tank A

are in tank B

if the fish fight

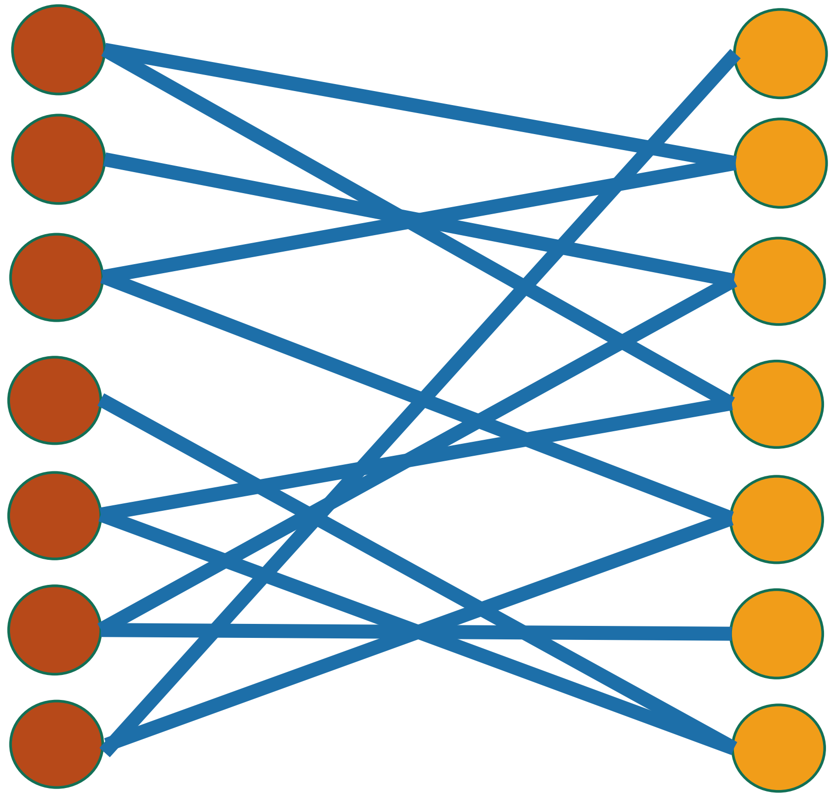
Example:

are students

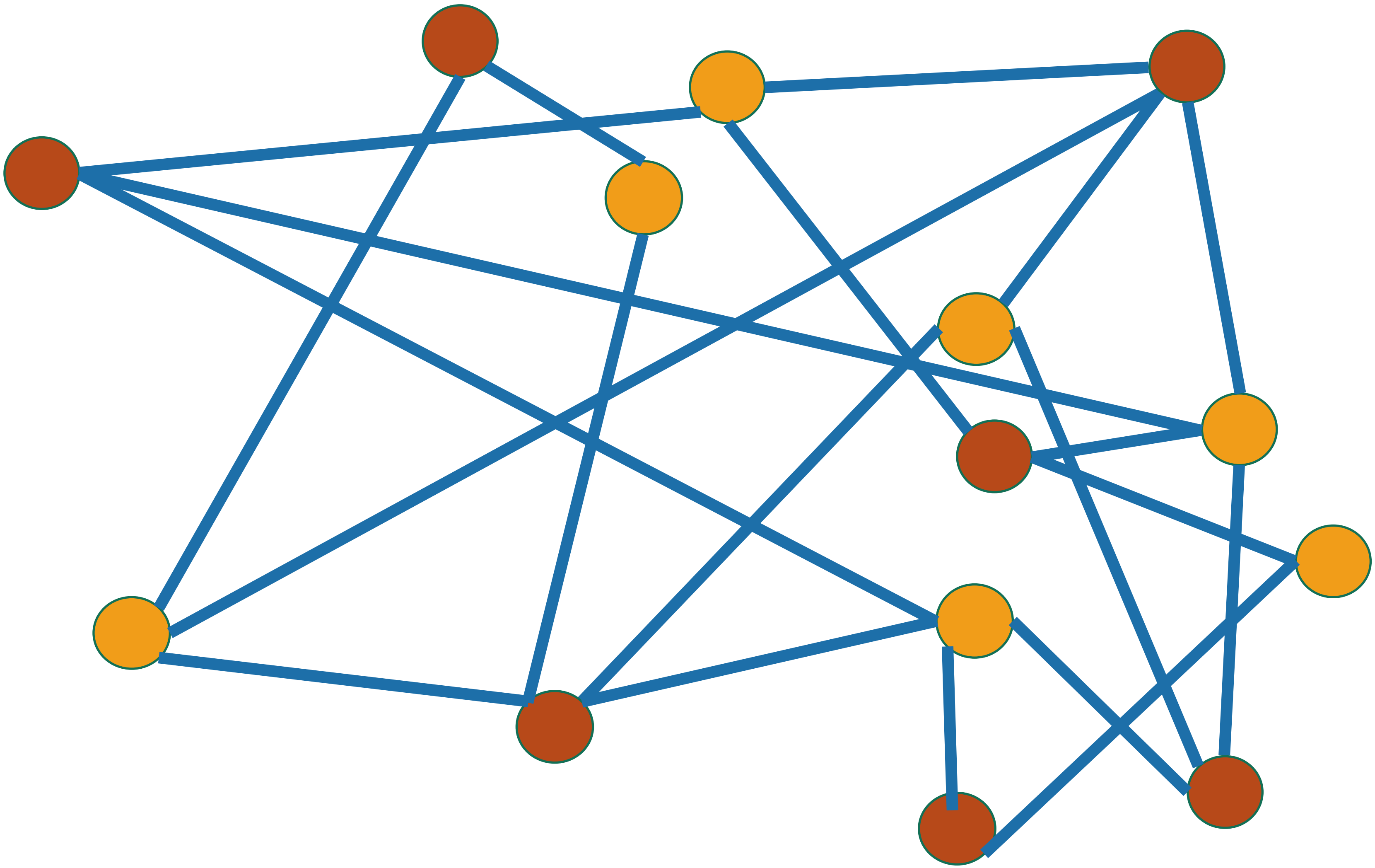
are classes

if the student is enrolled in the class

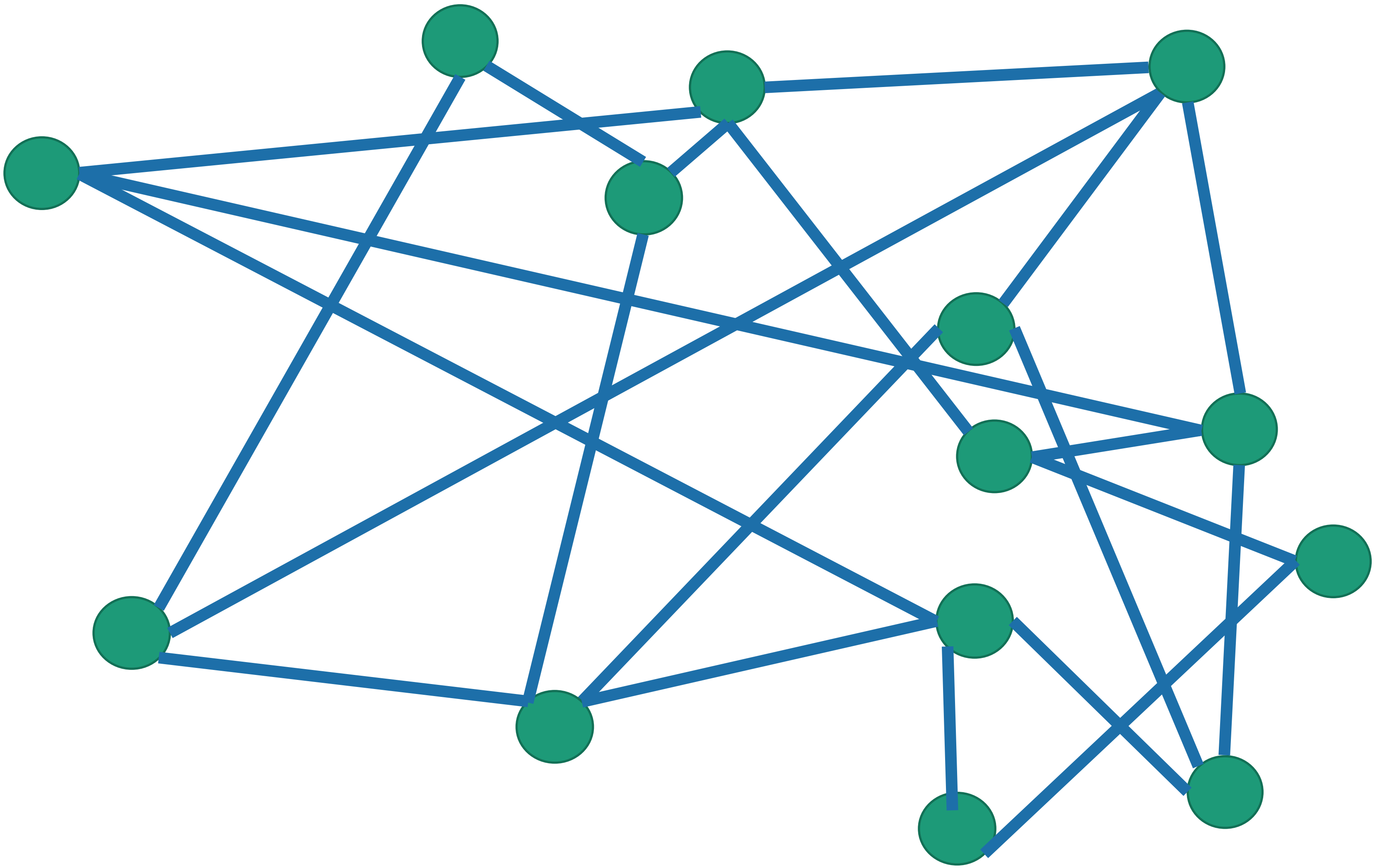
Is this graph bipartite?



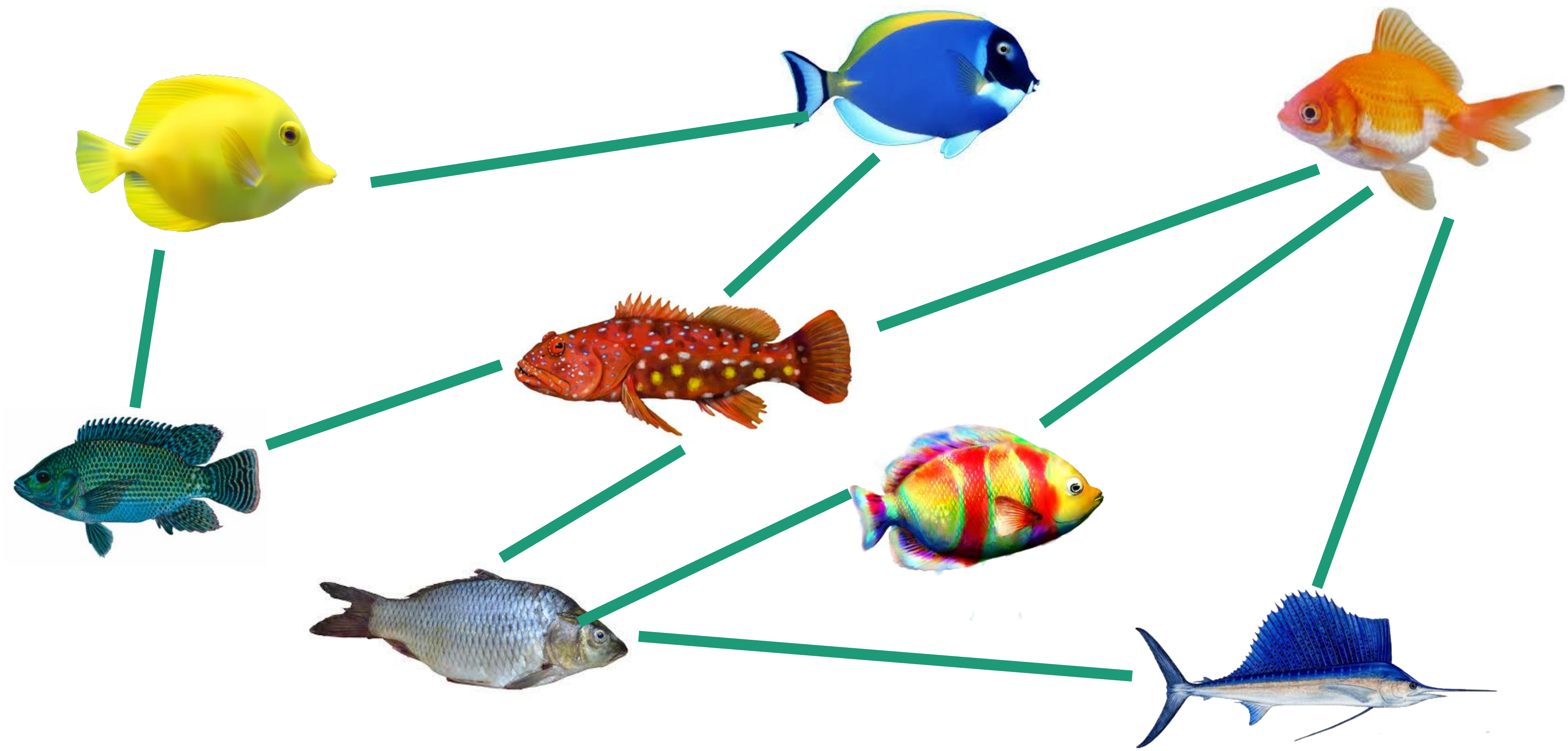
How about this one?



How about this one?

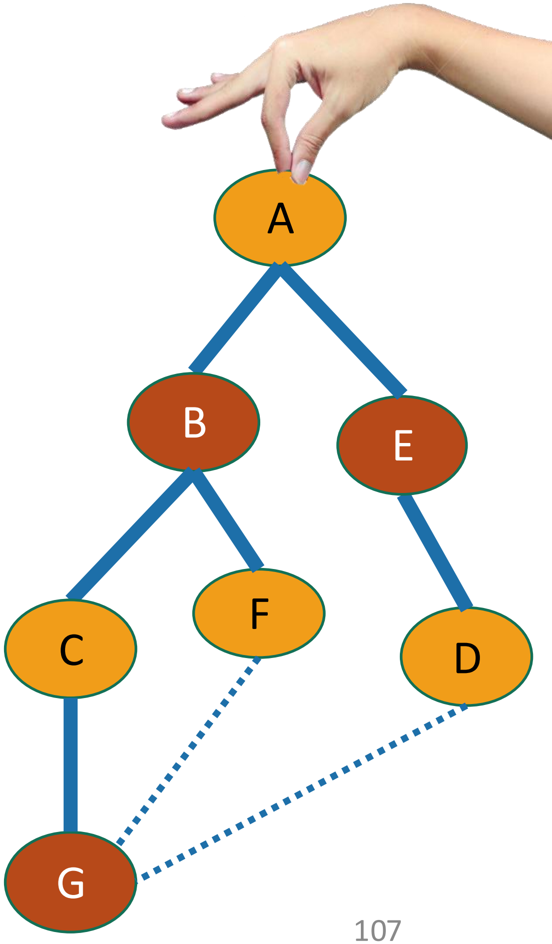


This one?



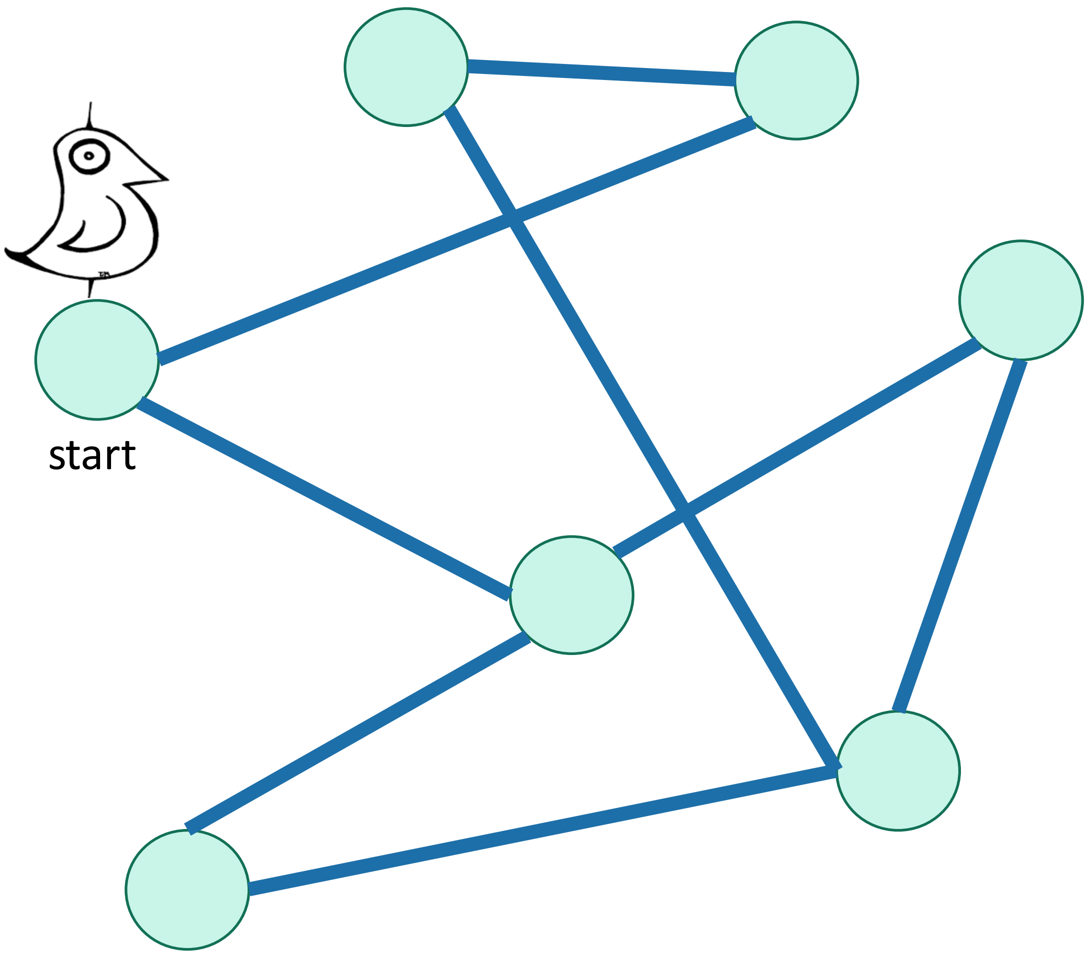
Application of BFS:
Testing Bipartiteness

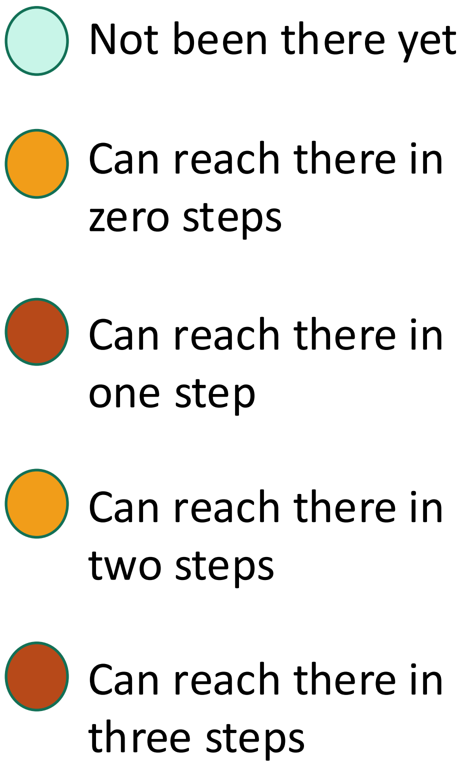
- Color the levels of the BFS tree in alternating colors.
- If you never color two connected nodes the same color, then it is bipartite.
- Otherwise, it’s not.



Breadth-First Search

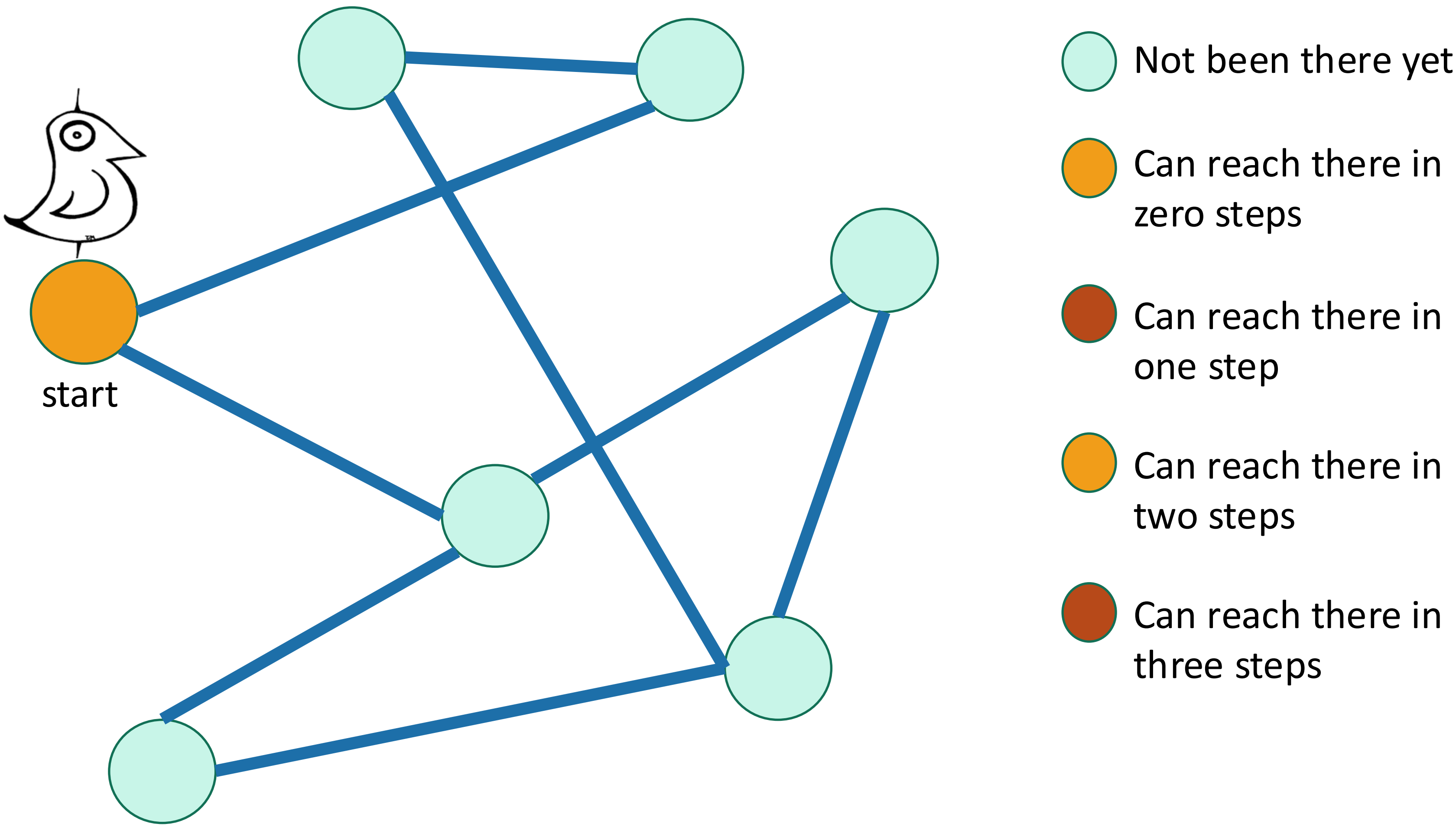
For testing bipartite-ness





Breadth-First Search

For testing bipartite-ness



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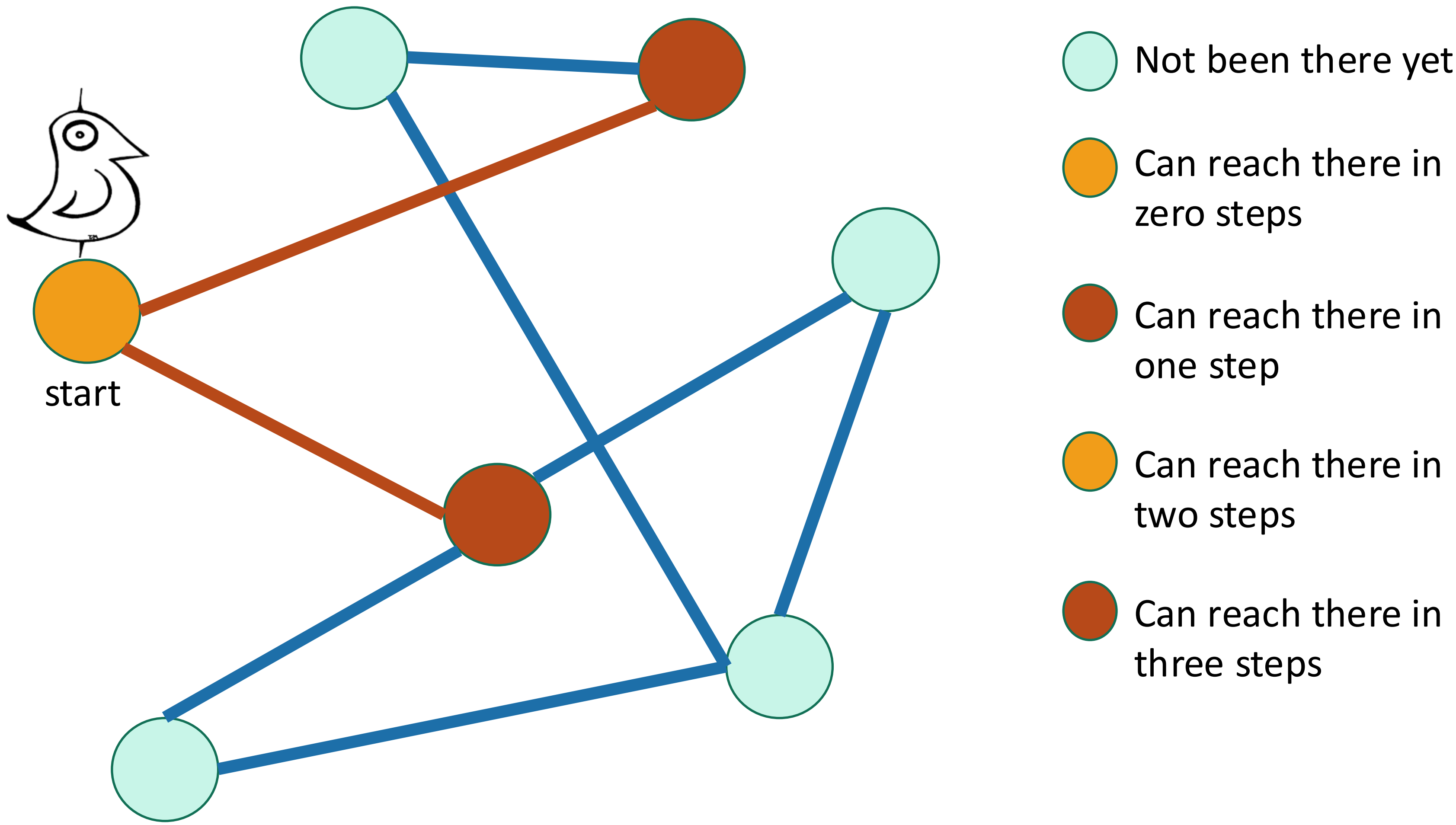
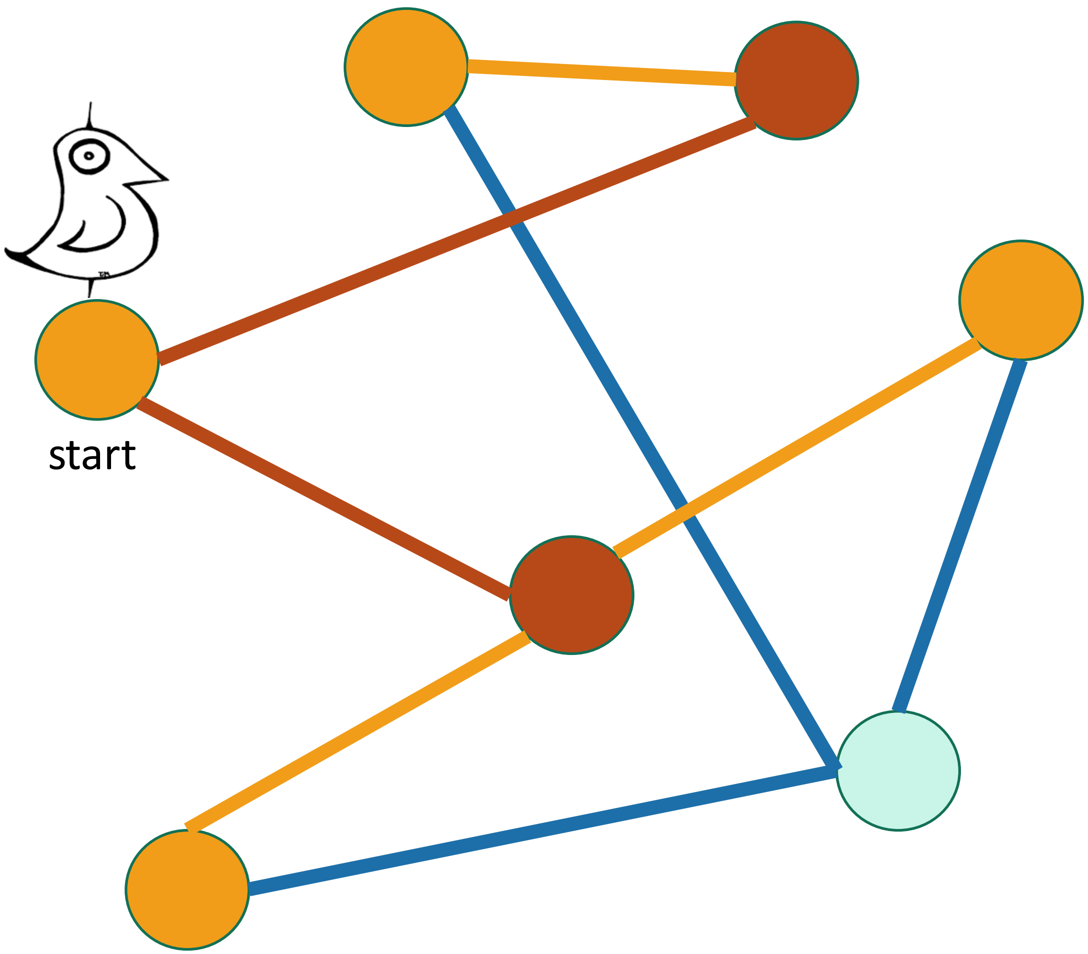
For testing bipartite-ness



Breadth-First Search

For testing bipartite-ness



Not been there yet

Can reach there in zero steps

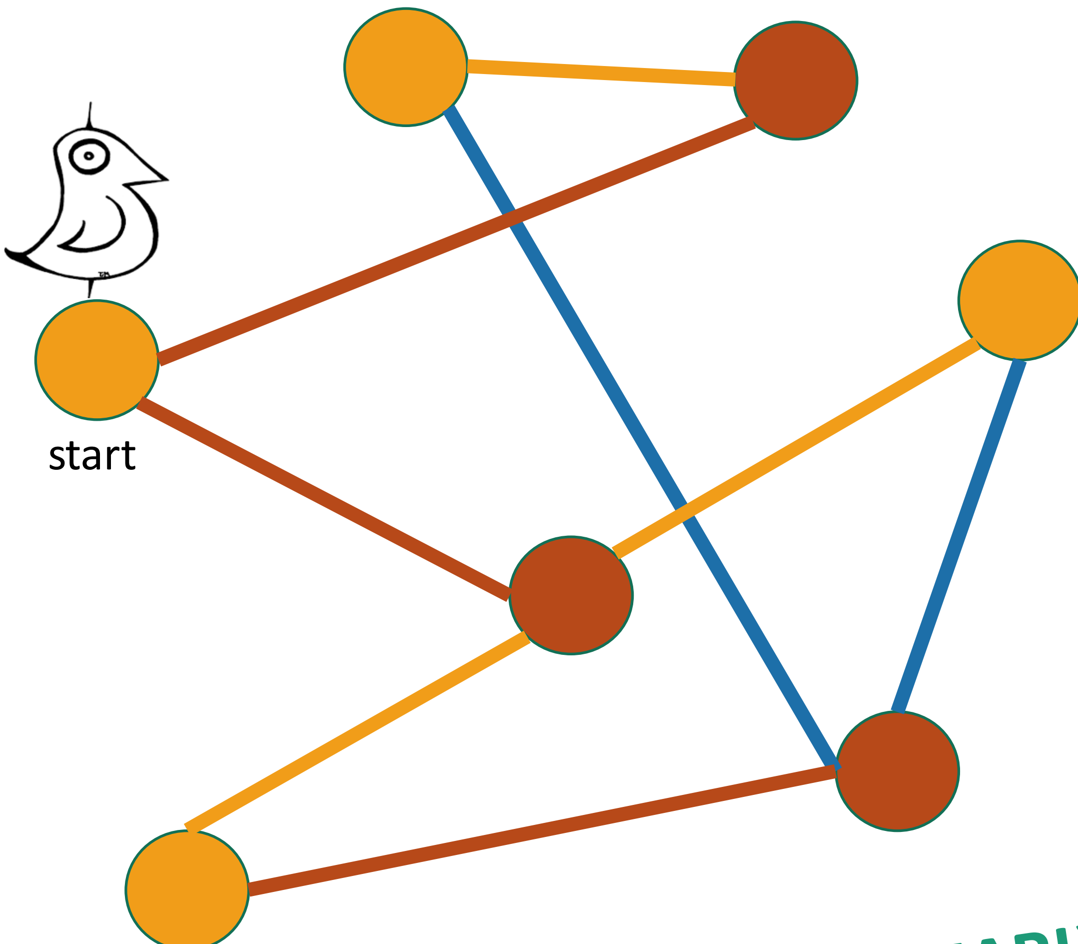
Can reach there in one step

Can reach there in two steps

Can reach there in three steps

Breadth-First Search

For testing bipartite-ness

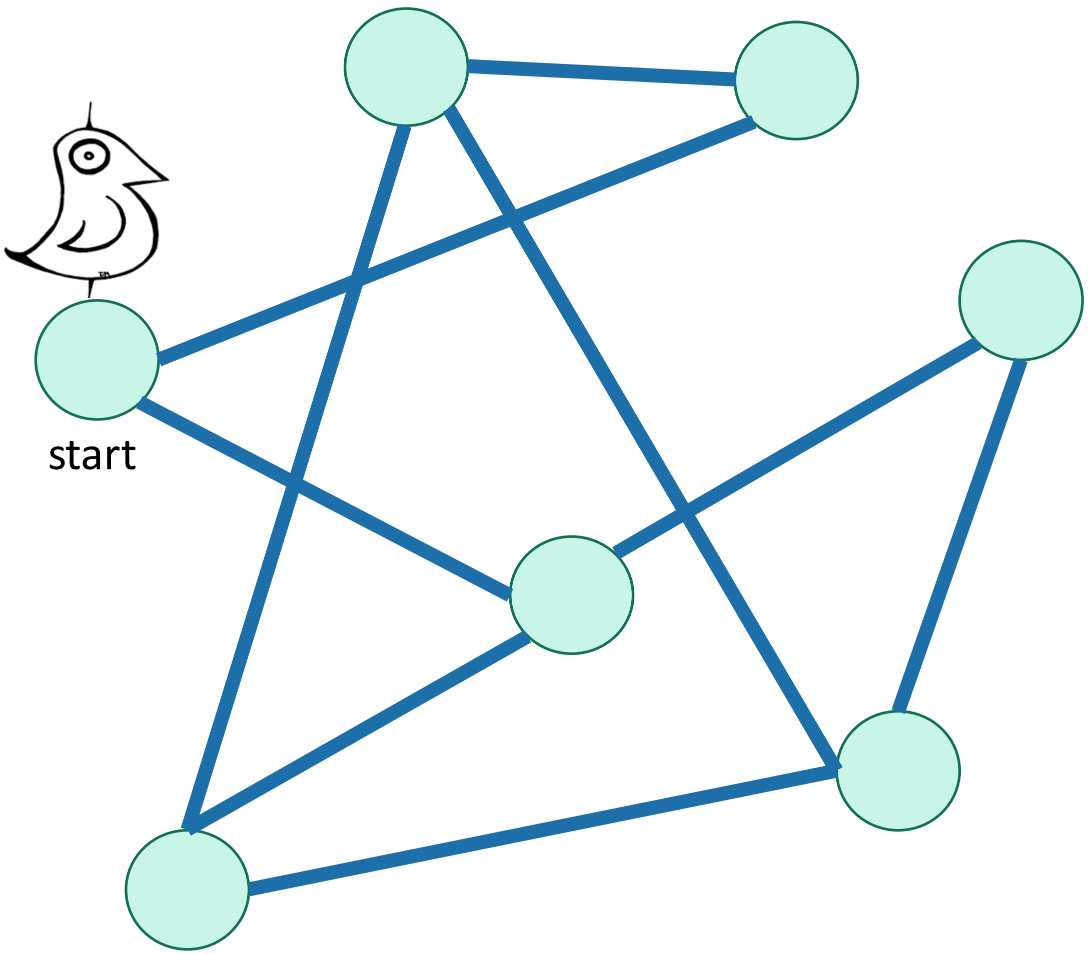


- Not been there yet
- Can reach there in zero steps
- Can reach there in one step
- Can reach there in two steps
- Can reach there in three steps

CLEARLY BIPARTITE!

Breadth-First Search

For testing bipartite-ness



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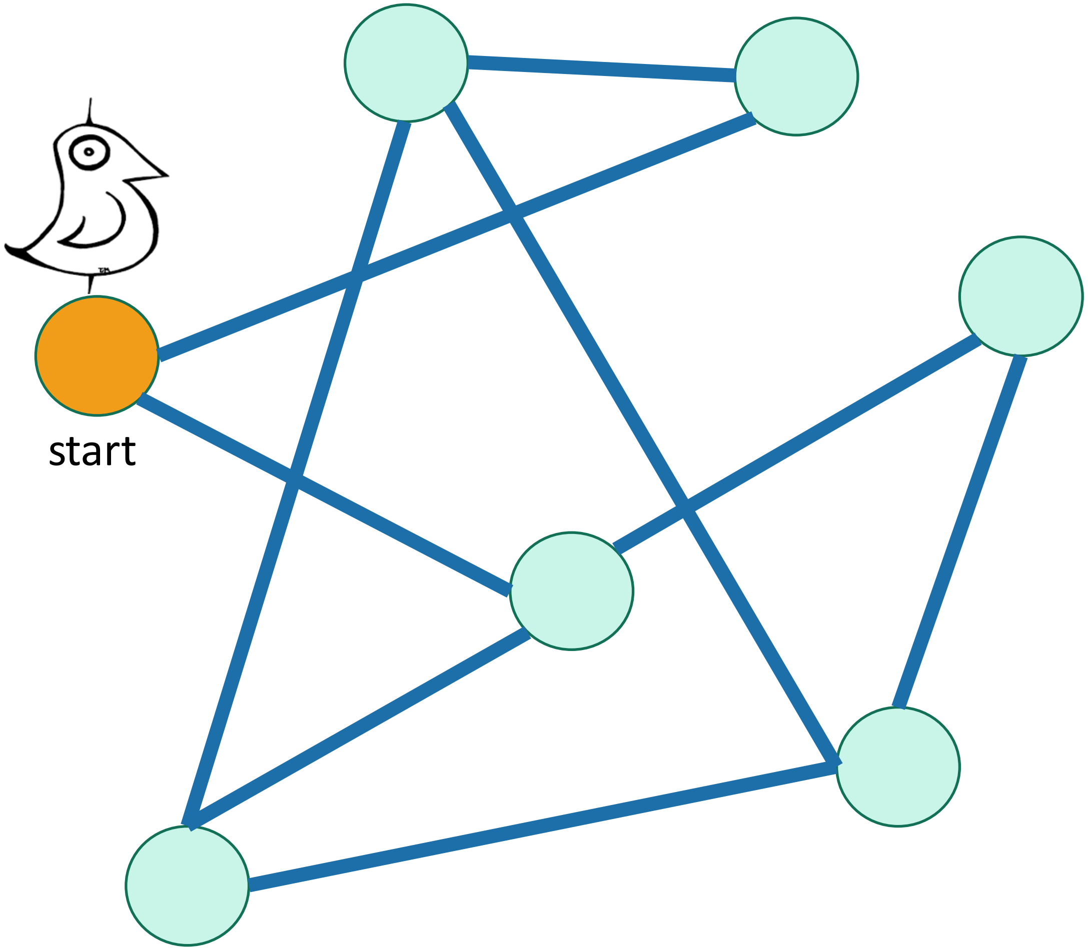

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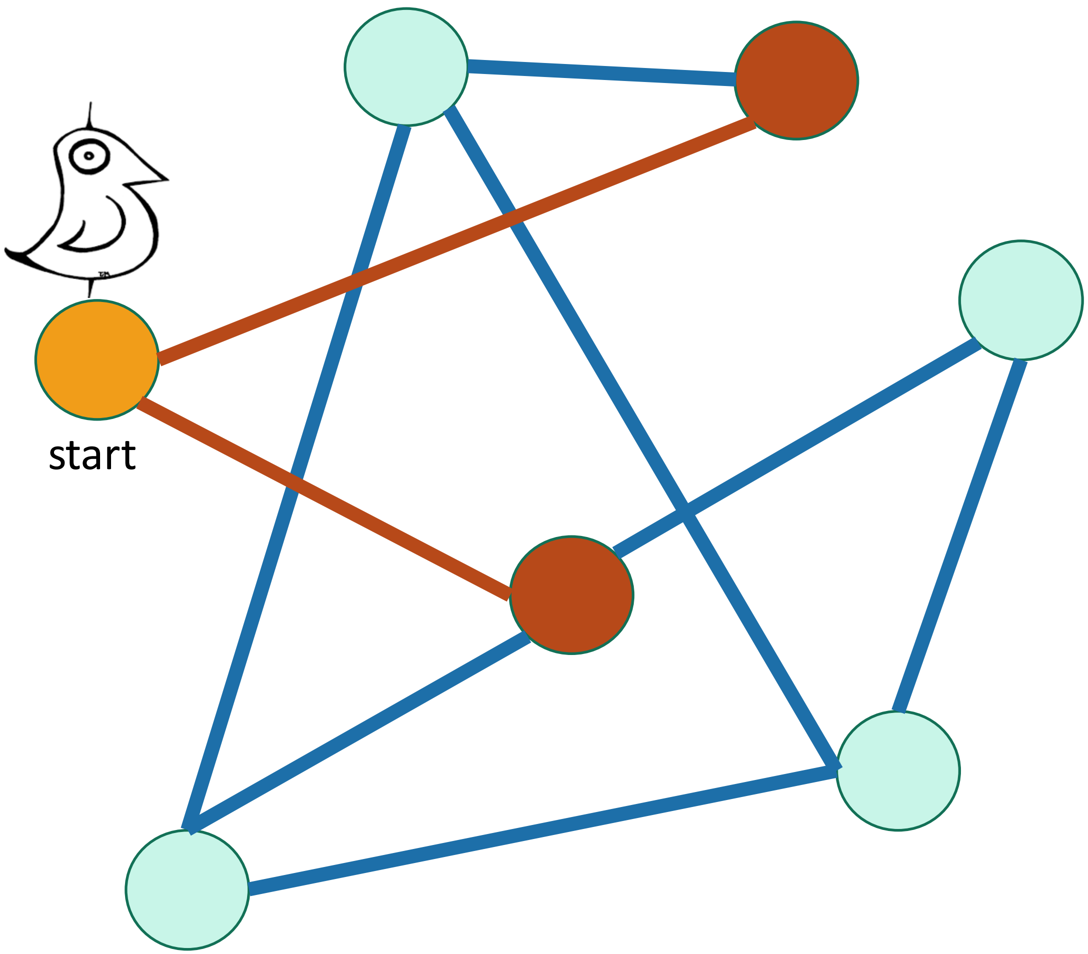

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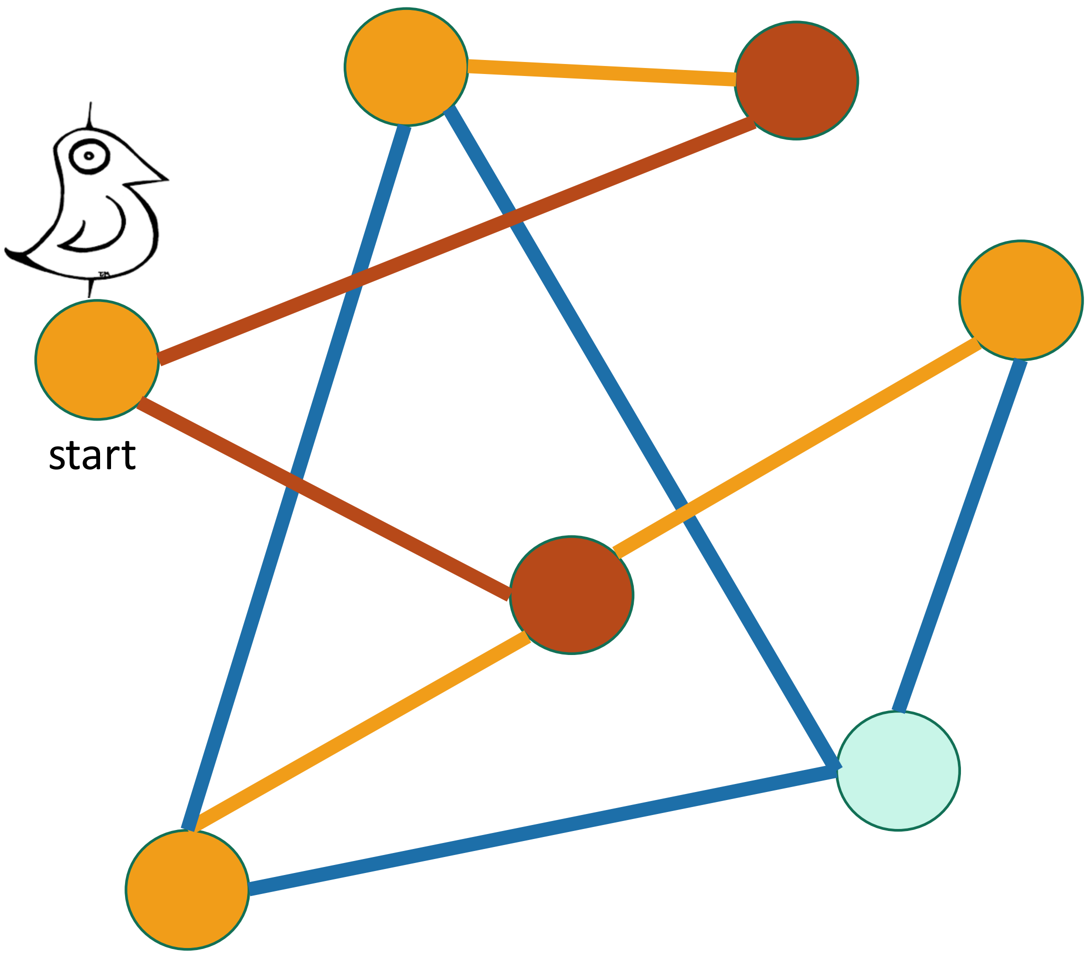

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Breadth-First Search

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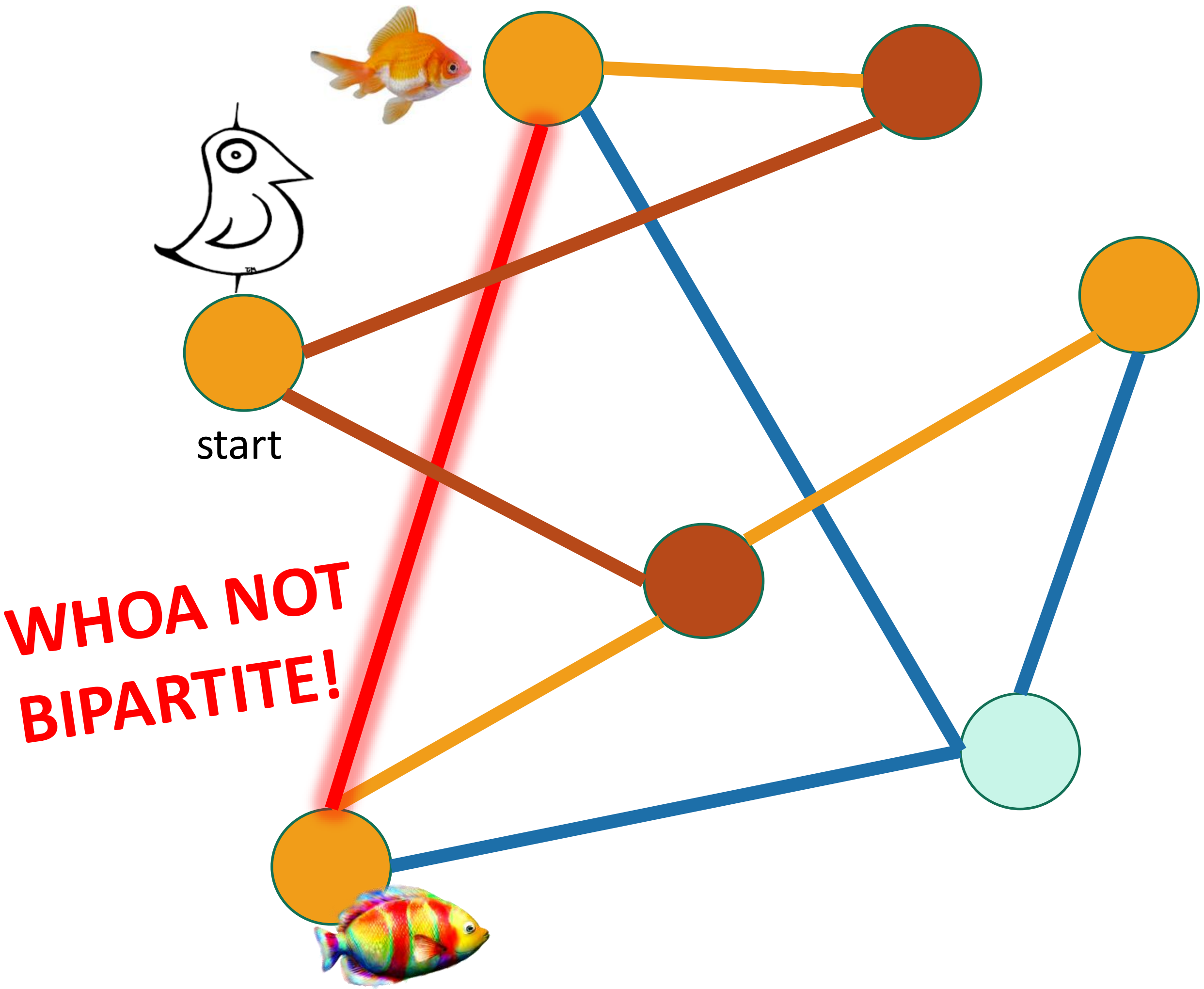


Not been there yet

Can reach there in zero steps

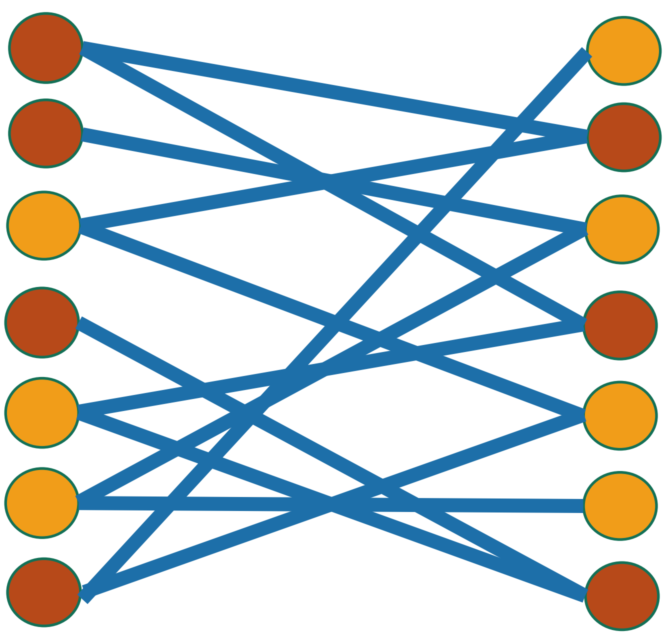
Can reach there in one step

Can reach there in two steps

Can reach there in three steps

Hang on now.

- Just because **this** coloring doesn't work, why does that mean that there is **no** coloring that works?

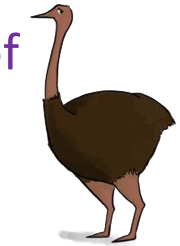


I can come up with plenty of bad colorings on this legitimately bipartite graph…

Plucky the pedantic penguin

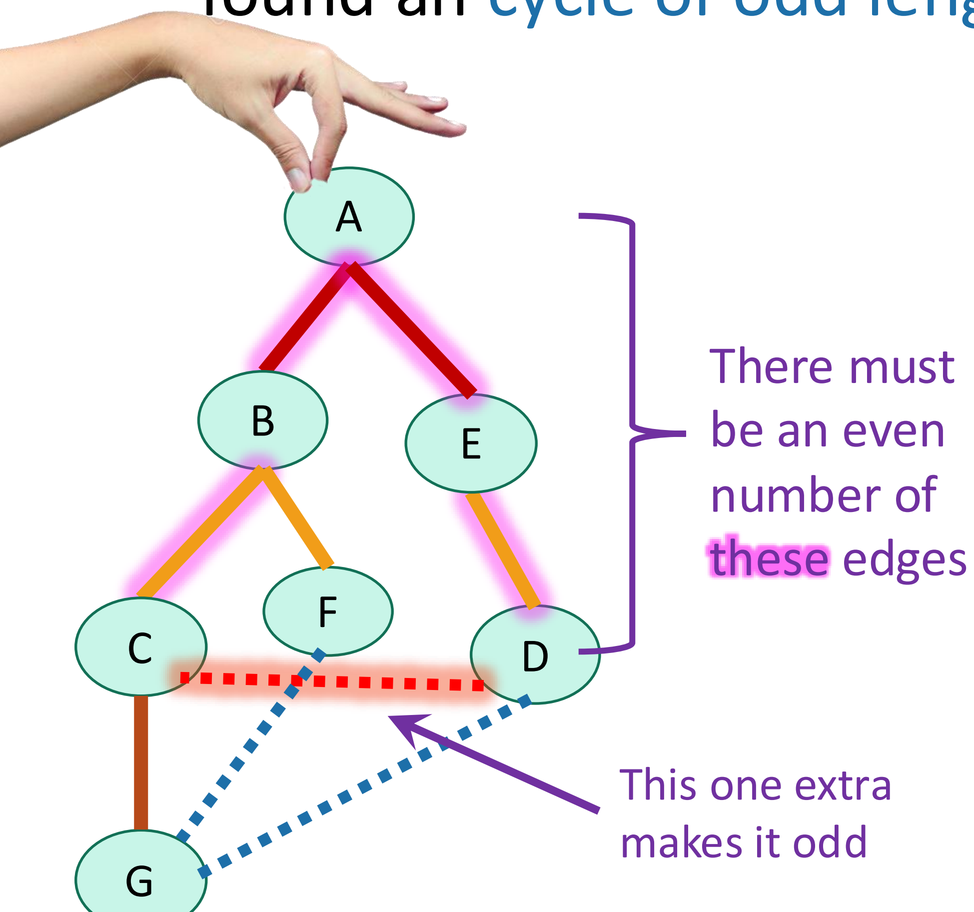
Some proof required

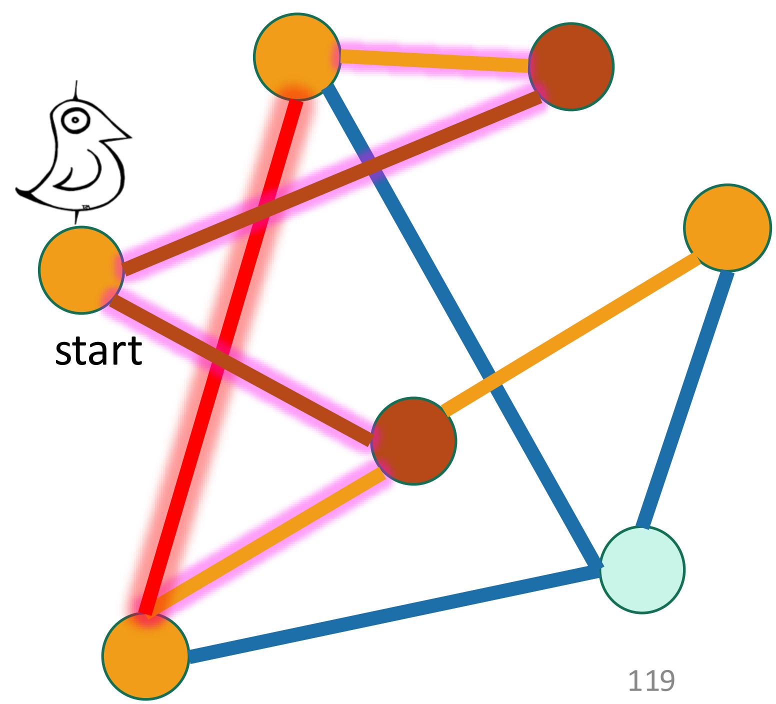
Make this proof sketch formal!



Ollie the over-achieving ostrich

- If BFS colors two neighbors the same color, then it's found an cycle of odd length in the graph.





Some proof required

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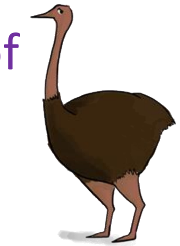
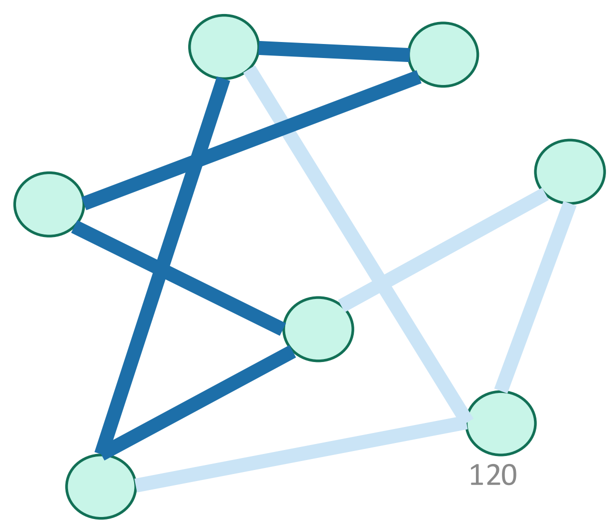


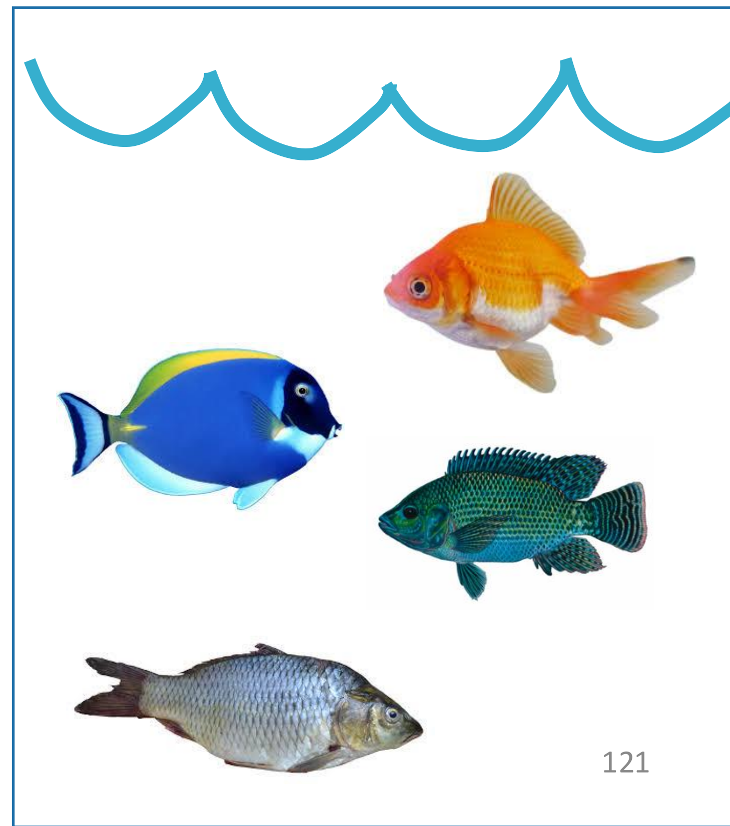
Ollie the over-achieving ostrich

- If BFS colors two neighbors the same color, then it's found an cycle of odd length in the graph.
- But you can **never** color an odd cycle with two colors so that no two neighbors have the same color.
  - [Fun exercise!]
- So you can't legitimately color the whole graph either.
- **Thus it's not bipartite.**



What have we learned?

BFS can be used to detect bipartite-ness in time O(n + m).



Outline

- Part 0: Graphs and terminology

- Part 1: Depth-first search
  - Application: topological sorting
  - Application: in-order traversal of BSTs
- Part 2: Breadth-first search
  - Application: shortest paths
  - Application (if time): is a graph bipartite?

Recap

Recap

- Depth-first search
  - Useful for topological sorting
  - Also in-order traversals of BSTs
- Breadth-first search
  - Useful for finding shortest paths
  - Also for testing bipartiteness
- Both DFS, BFS:
  - Useful for exploring graphs, finding connected components, etc

Still open (next few classes)

- We can now find components in undirected graphs…
  - What if we want to find strongly connected components in directed graphs?

- How can we find shortest paths in weighted graphs?