Lecture 5

Randomized algorithms and QuickSort

Announcements

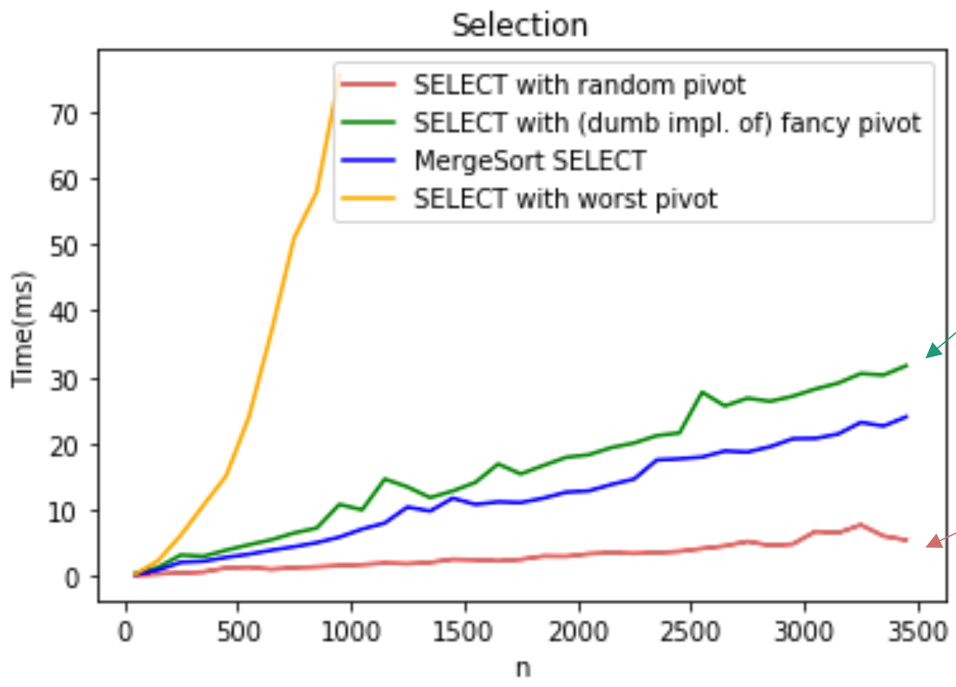
- Homework 2 is due today by midnight
- Homework 3 will be released today (still solo)

- This Friday we will have an EthiCS lecture taught by Justin Shin
  (same place/time as regular lecture)

  Friday Jan 23, 1:30-2:50pm (STLC 111)

Last time

- We saw a divide-and-conquer algorithm to solve the **`Select`** problem in time O(n) in the worst-case.
- It all came down to picking the pivot…



We choose a pivot **cleverly**

We choose a pivot **randomly.**

Randomized algorithms

- We make some random choices during the algorithm.
- We hope the algorithm works.
- We hope the algorithm is fast.

For today we will look at algorithms that always work and are probably fast. These are called "Las Vegas".

E.g., `Select` with a random pivot is a randomized algorithm.

- Always works (aka, is correct).
- Probably fast.

Today

- How do we analyze randomized algorithms?
- A few randomized algorithms for sorting.
  - BogoSort
  - QuickSort

- BogoSort is a pedagogical tool.
- QuickSort is important to know. (in contrast with BogoSort…)

How do we measure the runtime of a randomized algorithm?

Scenario 1

1. You publish your algorithm.
2. Bad guy picks the input.
3. You run your randomized algorithm.

Scenario 2

1. You publish your algorithm.
2. Bad guy picks the input.
3. Bad guy chooses the randomness (fixes the dice) and runs your algorithm.

- In **Scenario 1**, the running time is a **random variable.**
  - It makes sense to talk about **expected running time**.
- In **Scenario 2**, the running time is **not random**.
  - We call this the **worst-case running time** of the randomized algorithm.

Today

- How do we analyze randomized algorithms?
- A few randomized algorithms for sorting.
  - BogoSort
  - QuickSort

- BogoSort is a pedagogical tool.
- QuickSort is important to know. (in contrast with BogoSort…)

Assume A has distinct entries

From your pre-lecture exercise: BogoSort

Suppose that you can draw a random integer in {1,…,n} in time O(1). How would you randomly permute an array in-place in time O(n)?

Ollie the over-achieving ostrich

```
• **BogoSort**(A)
    • **While** true:
        • Randomly permute A.
        • Check if A is sorted.
        • **If** A is sorted, **return** A.
```

- Let $X_i = \begin{cases} 1 \text{ if A is sorted after iteration i} \\ 0 \text{ otherwise} \end{cases}$
- $E[X_i] = \frac{1}{n!}$
- $E[\text{number of iterations until A is sorted}] = n!$

Assume A has distinct entries

From your pre-lecture exercise: BogoSort

Suppose that you can draw a random integer in {1,…,n} in time O(1). How would you randomly permute an array in-place in time O(n)?

Ollie the over-achieving ostrich

```
• BogoSort(A)
    • While true:
        • Randomly permute A.
        • Check if A is sorted.
        • If A is sorted, return A.
```

- Let $X_i = \begin{cases} 1 & \text{if A is sorted after iteration i} \\ 0 & \text{otherwise} \end{cases}$
- $E[X_i] = \frac{1}{n!}$
- $E[\text{number of iterations until A is sorted}] = n!$

Expected Running time of BogoSort

This isn't random, so we can pull it out of the expectation.

E[running time on a list of length n]

= E[(number of iterations) * (time per iteration)]

= (time per iteration) * E[number of iterations]

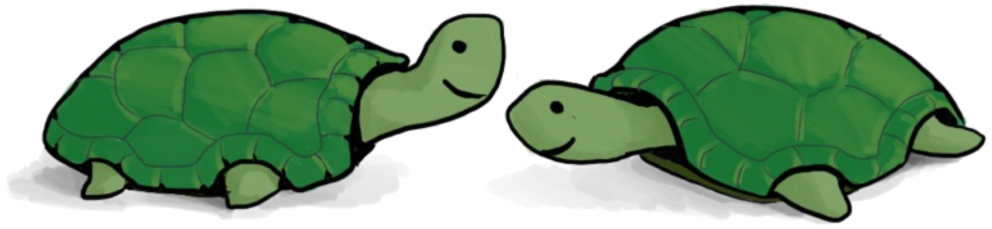
This is O(n) (to permute and then check if sorted)

We just computed this. It's n!.

= $O(n \cdot n!)$

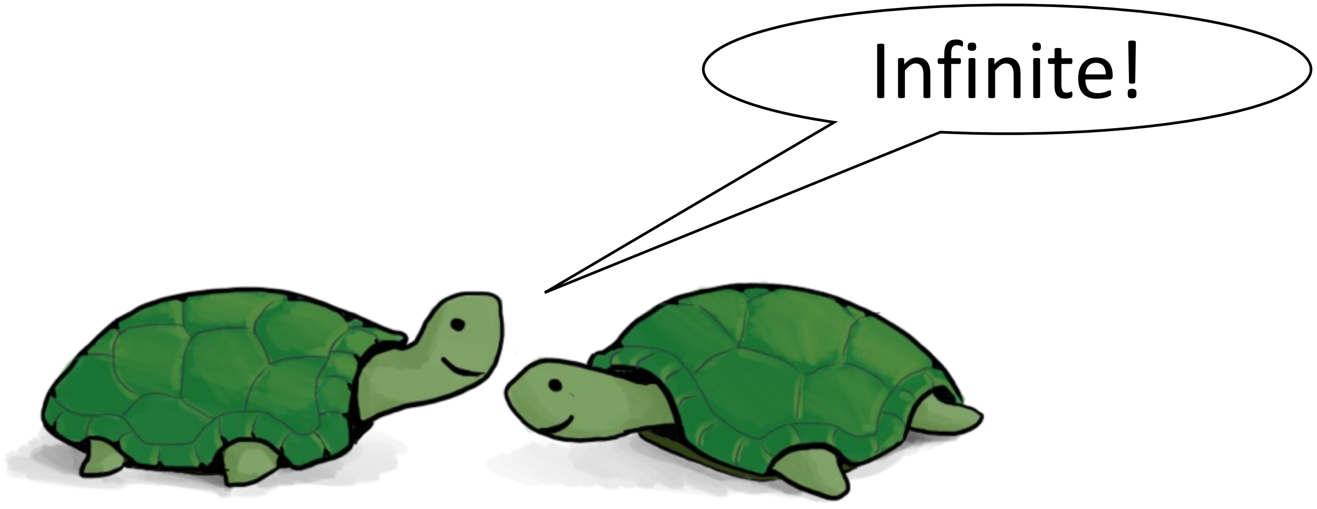
= REALLY REALLY BIG.

Worst-case running time of BogoSort?


Think-Share Terrapins!

```
• BogoSort(A)
    • While true:
        • Randomly permute A.
        • Check if A is sorted.
        • If A is sorted, return A.
```

Worst-case running time of BogoSort?



Think-Share Terrapins!

```
• BogoSort(A)
    • While true:
        • Randomly permute A.
        • Check if A is sorted.
        • If A is sorted, return A.
```

What have we learned?

- Expected running time:
  1. You publish your randomized algorithm.
  2. Bad guy picks an input.
  3. You get to roll the dice.

- Worst-case running time:
  1. You publish your randomized algorithm.
  2. Bad guy picks an input.
  3. Bad guy gets to “roll” the dice.

- Don’t use BogoSort.

Today

- How do we analyze randomized algorithms?
- A few randomized algorithms for sorting.
  - BogoSort
  - QuickSort

- BogoSort is a pedagogical tool.
- QuickSort is important to know. (in contrast with BogoSort…)

a better randomized algorithm: QuickSort

- Expected runtime O(nlog(n)).
- Worst-case runtime O(n^2).
- In practice works great!

QuickSort uses very similar methods to the Select algorithm we saw last time. Can you modify the QuickSort algorithm we'll learn today to make sure its worst-case runtime is O(nlog(n))?

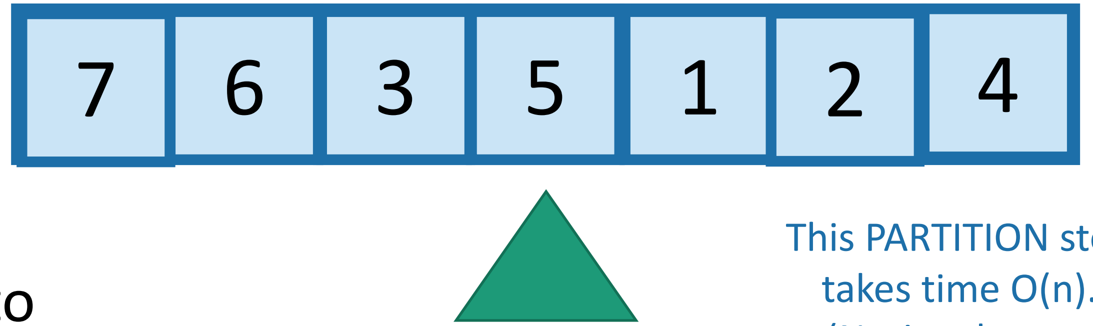
Siggi the Studious Stork

For the rest of the lecture, assume all elements of A are distinct.

Quicksort

We want to sort this array.

First, pick a "pivot."
Do it at random.



random pivot!

This PARTITION step takes time O(n). (Notice that we don't sort each half). [same as in SELECT]

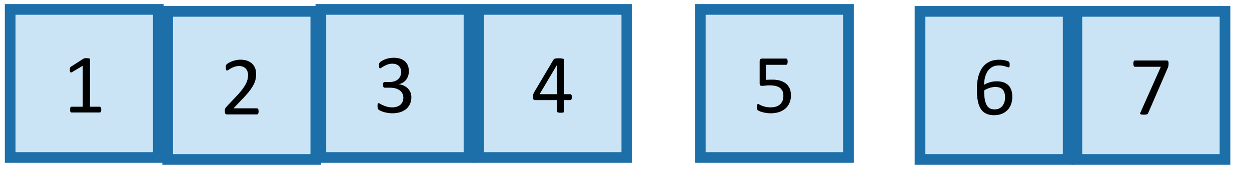
Next, partition the array into "bigger than 5" or "less than 5"

Arrange them like so:

L = array with things smaller than A[pivot]

R = array with things larger than A[pivot]

Recurse on L and R:



PseudoPseudoCode for what we just saw

Lecture 5 Python notebook for actual code.

- QuickSort(A):
    - **If** len(A) <= 1:
        - **return**
    - Pick some x = A[i] at random. Call this the **pivot**.
    - PARTITION the rest of A into:
        - L (less than x) and
        - R (greater than x)
    - Replace A with [L, x, R] (that is, rearrange A in this order)
    - QuickSort(L)
    - QuickSort(R)

Assume that all elements of A are distinct. How would you change this if that's not the case?

Running time?

- $T(n) = T(|L|) + T(|R|) + O(n)$

- In an ideal world…
  - if the pivot splits the array exactly in half…

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + O(n)$$

- We've seen that a bunch:

$$T(n) = O(n \log(n)).$$

The expected running time of QuickSort is O(nlog(n)).

Proof:*

- $E[|L|] = E[|R|] = \frac{n-1}{2}$.
  - The expected number of items on each side of the pivot is half of the things.

Remember, we are assuming all elements of A are distinct

Aside

why is $E[|L|] = \frac{n-1}{2}$?

- $E[|L|] = E[|R|]$
  - by symmetry
- $E[|L| + |R|] = n - 1$
  - because L and R make up everything except the pivot.
- $E[|L|] + E[|R|] = n - 1$
  - By linearity of expectation
- $2E[|L|] = n - 1$
  - Plugging in the first bullet point.
- $E[|L|] = \frac{n-1}{2}$
  - Solving for $E[|L|]$.

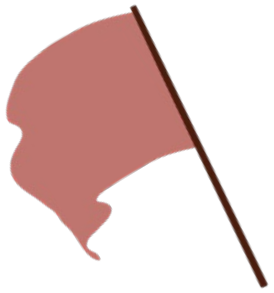
The expected running time of QuickSort is O(nlog(n)).

Proof:*

- $E[|L|] = E[|R|] = \frac{n-1}{2}$.
  - The expected number of items on each side of the pivot is half of the things.
- If that occurs, the running time is $T(n) = O(n\log(n))$.
  - Since the relevant recurrence relation is $T(n) = 2T\left(\frac{n-1}{2}\right) + O(n)$
- Therefore, the expected running time is $O(n\log(n))$.

***Disclaimer: this proof is WRONG.**

Red flag



We can use the same argument to prove something false.

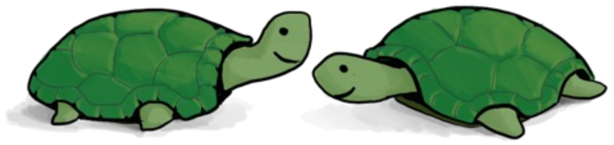
```
• Slow Sort(A):
    • If len(A) <= 1:
        • return
    • Pick the pivot x to be either max(A) or min(A), randomly
        • \\ We can find the max and min in O(n) time
    • PARTITION the rest of A into:
        • L (less than x) and
        • R (greater than x)
    • Replace A with  [L, x, R]  (that is, rearrange A in this order)
    • Slow Sort(L)
    • Slow Sort(R)
```

- Same recurrence relation:
  $$T(n) = T(|L|) + T(|R|) + O(n)$$
- We still have $E[|L|] = E[|R|] = \frac{n-1}{2}$
- But now, one of |L| or |R| is always n-1.
- You check: Running time is $\Theta(n^2)$, with probability 1.

The expected running time of SlowSort is O(nlog(n)).

Proof:[*]

What's wrong???



- $E[|L|] = E[|R|] = \frac{n-1}{2}$.
    - The expected number of items on each side of the pivot is half of the things.
- If that occurs, the running time is $T(n) = O(n\log(n))$.
    - Since the relevant recurrence relation is $T(n) = 2T\left(\frac{n-1}{2}\right) + O(n)$
- Therefore, the expected running time is $O(n\log(n))$.

*Disclaimer: this proof is WRONG.

What’s wrong?

- $E[|L|] = E[|R|] = \frac{n-1}{2}$.
  - The expected number of items on each side of the pivot is half of the things.
- If that occurs, the running time is $T(n) = O(n\log(n))$.
  - Since the relevant recurrence relation is $T(n) = 2T\left(\frac{n-1}{2}\right) + O(n)$
- Therefore, the expected running time is $O(n\log(n))$.

That’s not how expectations work!

Plucky the Pedantic Penguin

- The running time in the “expected” situation is not the same as the expected running time.
- Sort of like how $E[X^2]$ is not the same as $(E[X])^2$

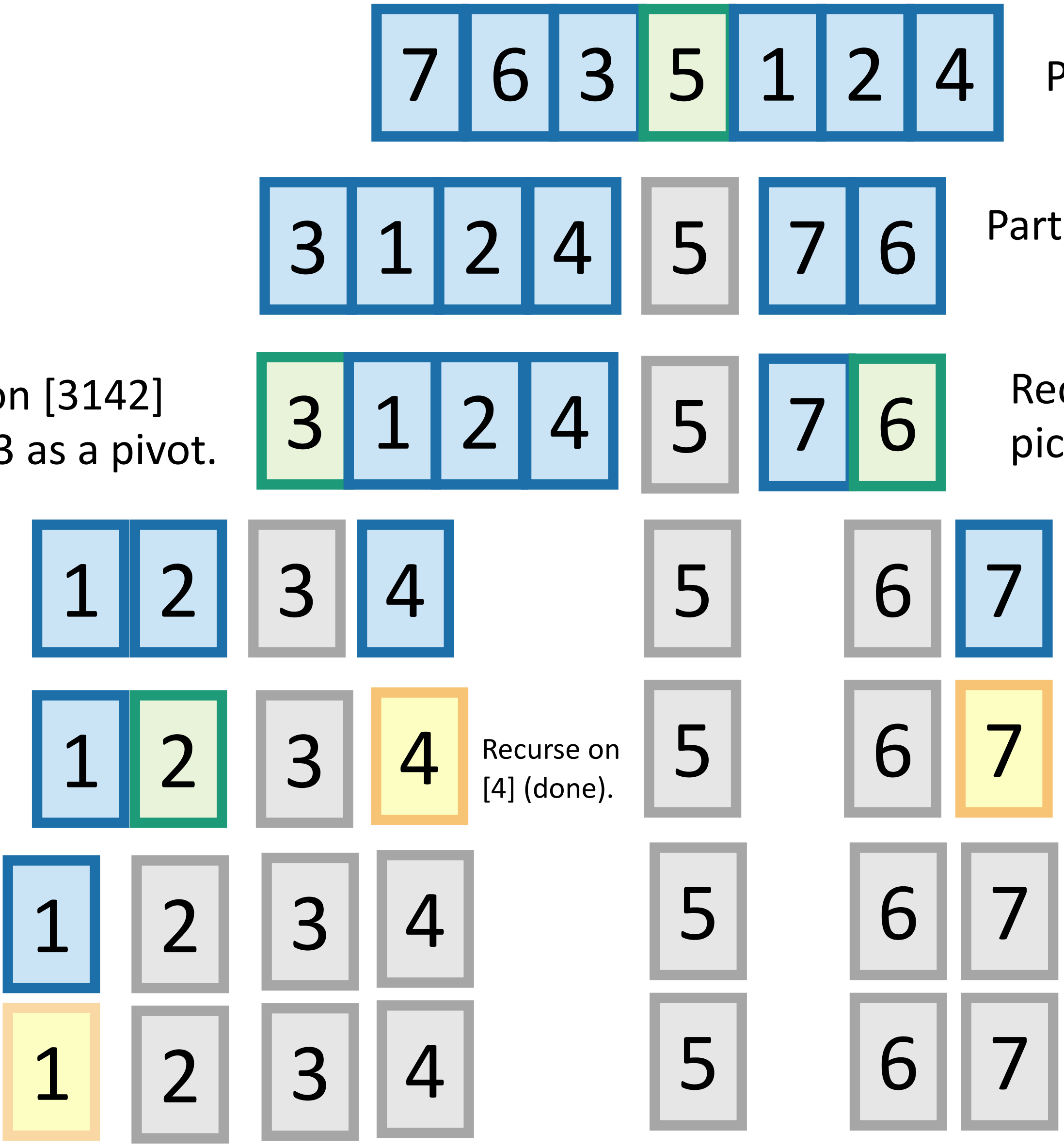
Instead

- We’ll have to think a little harder about how the algorithm works.

Next goal:

- Get the same conclusion, correctly!

Example of recursive calls



Pick 5 as a pivot

Partition on either side of 5

Recurse on [3142] and pick 3 as a pivot.

Recurse on [76] and pick 6 as a pivot.

Partition around 3.

Partition on either side of 6

Recurse on [12] and pick 2 as a pivot.

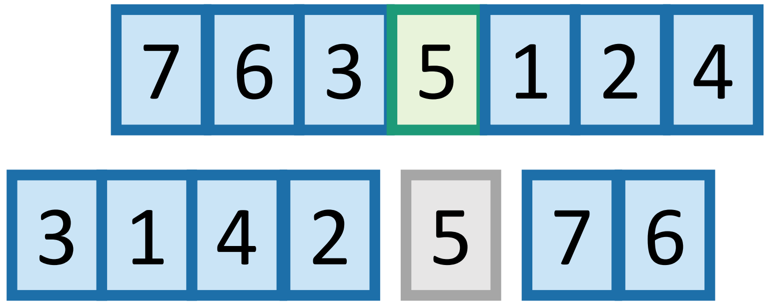
Recurse on [7], it has size 1 so we're done.

partition around 2.

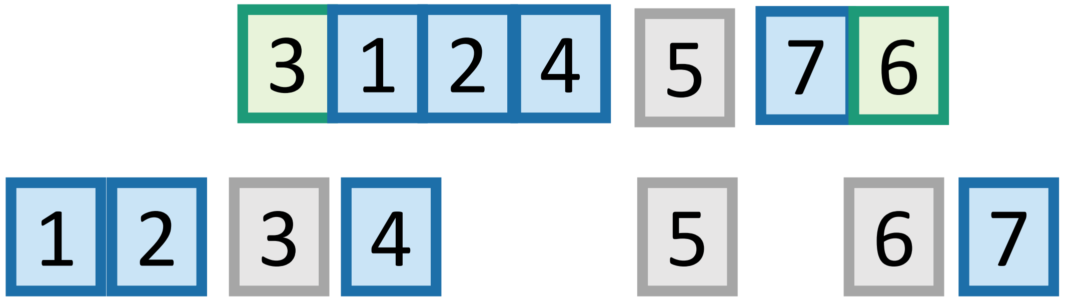
Recurse on [1] (done).

How long does this take to run?

- We will count the number of comparisons that the algorithm does.
    - This turns out to give us a good idea of the runtime. (Not obvious, but we can "charge" all operations to comparisons).
- How many times are any two items compared?

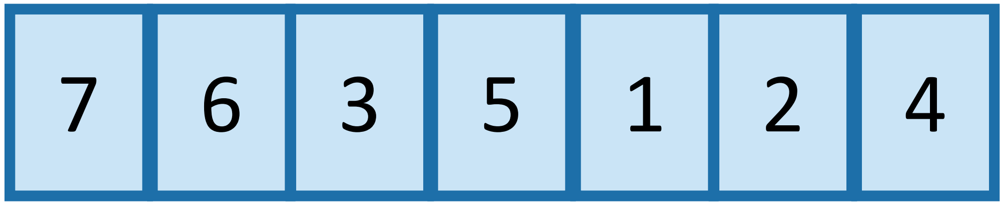


In the example before, everything was compared to 5 once in the first step….and never again.



But not everything was compared to 3.
5 was, and so were 1,2 and 4.
But not 6 or 7.

Each pair of items is compared either 0 or 1 times. Which is it?



Let's assume that the numbers in the array are actually the numbers 1,…,n

Of course this doesn't have to be the case! It's a good exercise to convince yourself that the analysis will still go through without this assumption.

- **Whether or not a, b are compared** is a random variable, that depends on the choice of pivots. Let's say

$$X_{a,b} = \begin{cases} 1 & \text{if } a \text{ and } b \text{ are ever compared} \\ 0 & \text{if } a \text{ and } b \text{ are never compared} \end{cases}$$

- In the previous example $X_{1,5} = 1$, because item 1 and item 5 were compared.
- But $X_{3,6} = 0$, because item 3 and item 6 were NOT compared.

Counting comparisons

- The number of comparisons total during the algorithm is

$$\sum_{a=1}^{n-1} \sum_{b=a+1}^{n} X_{a,b}$$

- The expected number of comparisons is

$$E\left[\sum_{a=1}^{n-1} \sum_{b=a+1}^{n} X_{a,b}\right] = \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} E[X_{a,b}]$$

by using linearity of expectations.

expected number of comparisons:

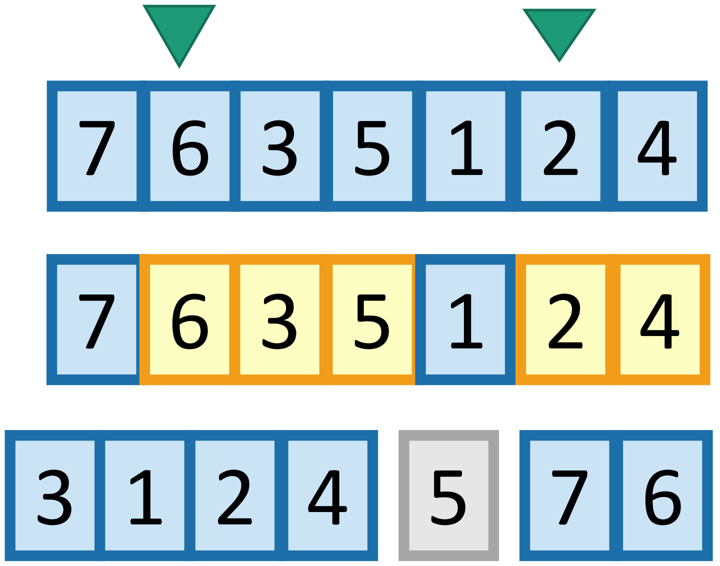
$$\sum_{a=1}^{n-1} \sum_{b=a+1}^{n} E[\, X_{a,b}]$$

Counting comparisons

- So we just need to figure out $E[\, X_{a,b} \,]$
- $E\left[X_{a,b}\right] = P(X_{a,b} = 1) \cdot 1 + P(X_{a,b} = 0) \cdot 0 = P(X_{a,b} = 1)$

  (by the definition of expectation)
- So we need to figure out:

$P(X_{a,b} = 1)$ = the probability that a and b are ever compared.



Say that a = 2 and b = 6. What is the probability that 2 and 6 are ever compared?

This is exactly the probability that either 2 or 6 is first picked to be a pivot out of the highlighted entries.

If, say, 5 were picked first, then 2 and 6 would be separated and never see each other again.

Counting comparisons

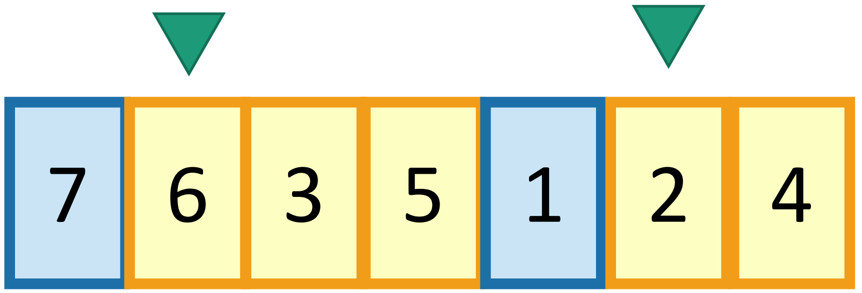
$P(\,X_{a,b} = 1\,)$

= probability a,b are ever compared

= probability that one of a,b are picked first out of all of the b – a +1 numbers between them.

2 choices out of b-a+1…

$$= \frac{2}{b-a+1}$$



All together now…

Expected number of comparisons

- $E\left[\sum_{a=1}^{n-1}\sum_{b=a+1}^{n} X_{a,b}\right]$ This is the expected number of comparisons throughout the algorithm
- $= \sum_{a=1}^{n-1}\sum_{b=a+1}^{n} E[\,X_{a,b}]$ linearity of expectation
- $= \sum_{a=1}^{n-1}\sum_{b=a+1}^{n} P(\,X_{a,b} = 1)$ definition of expectation
- $= \sum_{a=1}^{n-1}\sum_{b=a+1}^{n} \frac{2}{b-a+1}$ the reasoning we just did

- This is a big nasty sum, but we can do it.
- We get that this is less than 2n ln(n).
- (asymptotics on board if time …)

Do this sum!

Ollie the over-achieving ostrich

Almost done

- We saw that E[number of comparisons] = O(n log(n))
- Is that the same as E[running time]?

- In this case, **yes**.
- We need to argue that the running time is dominated by the time to do comparisons.
- See lecture notes.

```
• QuickSort(A):
    • If len(A) <= 1:
        • return
    • Pick some x = A[i] at random.  Call this the pivot.
    • PARTITION the rest of A into:
        • L (less than x) and
        • R (greater than x)
    • Replace A with  [L, x, R]  (that is, rearrange A in
      this order)
    • QuickSort(L)
    • QuickSort(R)
```

What have we learned?

- The expected running time of QuickSort is O(n log(n))

Worst-case running time

- Suppose that an adversary is choosing the "random" pivots for you.
- Then the running time might be $O(n^2)$
  - E.g., they'd choose to implement SlowSort
  - In practice, this doesn't usually happen.

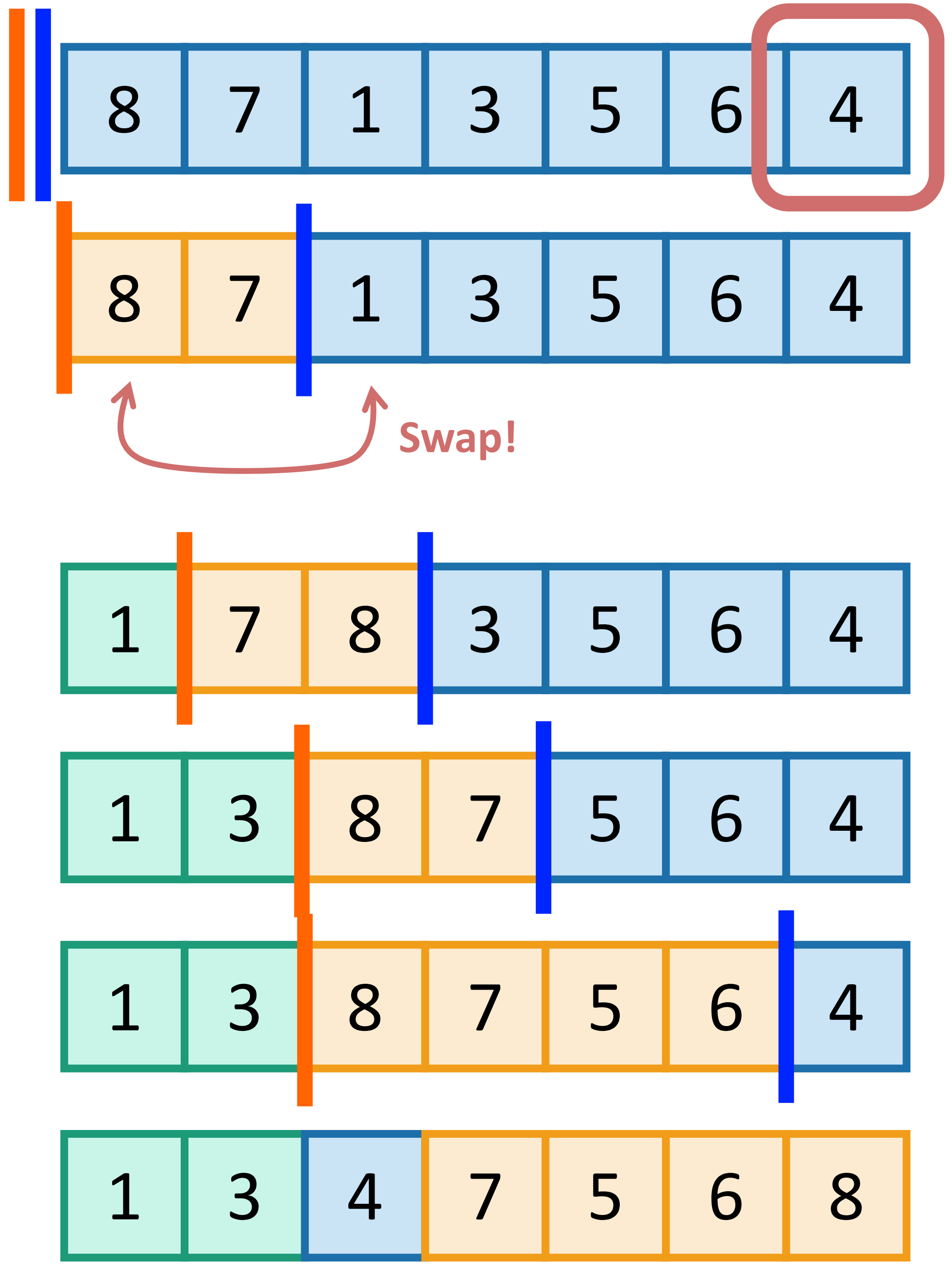
How should we implement this?

- Our pseudocode is easy to understand and analyze, but is not a good way to implement this algorithm.

```
• QuickSort(A):
    • If len(A) <= 1:
        • return
    • Pick some x = A[i] at random.  Call this the pivot.
    • PARTITION the rest of A into:
        • L (less than x) and
        • R (greater than x)
    • Replace A with  [L, x, R]  (that is, rearrange A in this order)
    • QuickSort(L)
    • QuickSort(R)
```

- Instead, implement it **in-place** (without separate L and R)
  - You may have seen this in CS 106b.
  - Here are some Hungarian Folk Dancers showing you how it’s done: https://www.youtube.com/watch?v=ywWBy6J5gz8
  - Check out Python notebook for Lecture 5 for two different ways.

A better way to do Partition



Pivot

Choose it randomly, then swap it with the last one, so it's at the end.

Initialize 🟧 and 🟦

Step 🟦 forward.

When 🟦 sees something smaller than the pivot, **swap** the things ahead of the bars and increment both bars.

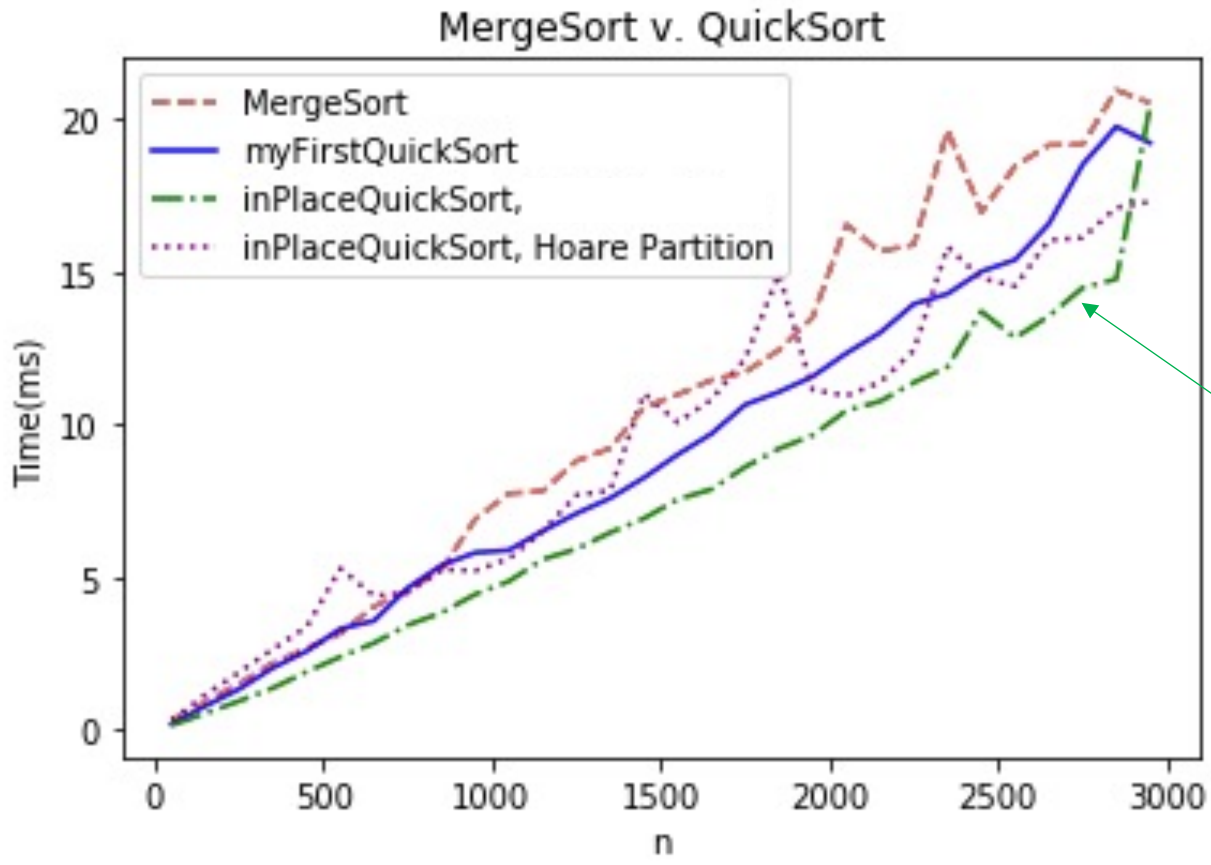
Repeat till the end, then put the pivot in the right place.

See lecture 5 Python notebook.

QuickSort vs. smarter QuickSort vs. Mergesort?

See Python notebook for Lecture 5

- All seem pretty comparable…



Hoare Partition is a different way of doing it (c.f. CLRS Problem 7-1), which you might have seen elsewhere. You are not responsible for knowing it for this class.

In-place partition function uses less space, and also is a smidge faster in this implementation.

QuickSort vs MergeSort

*What if you want O(n log(n)) worst-case runtime and stability? Check out "Block Sort" on Wikipedia!

	QuickSort (random pivot)	MergeSort (deterministic)
Running time	• Worst-case: $O(n^2)$ • Expected: $O(n \log(n))$	Worst-case: $O(n \log(n))$
Used by	• Java for primitive types • C qsort • Unix • g++	• Java for objects • Perl • Python (variant of it called Timsort)
In-Place? (With $O(\log(n))$ extra bits of memory)	Yes, pretty easily	Not easily* if you want to maintain both stability and runtime. (But pretty easily if you can sacrifice runtime).
Stable?	No	Yes
Other Pros	Good cache locality if implemented for arrays	Merge step is really efficient with linked lists

Understand this

These are just for fun. (Not on exam).

Today

- How do we analyze randomized algorithms?
- A few randomized algorithms for sorting.
    - BogoSort
    - QuickSort

- BogoSort is a pedagogical tool.
- QuickSort is important to know. (in contrast with BogoSort…)

Recap

Recap

- How do we measure the runtime of a randomized algorithm?
  - Expected runtime
  - Worst-case runtime

- QuickSort (with a random pivot) is a randomized sorting algorithm.
  - In many situations, QuickSort is nicer than MergeSort.
  - In many situations, MergeSort is nicer than QuickSort.

Code up QuickSort and MergeSort in a few different languages, with a few different implementations of lists A (array vs linked list, etc). What's faster?
(This is an exercise best done in C where you have a bit more control than in Python).

Ollie the over-achieving ostrich

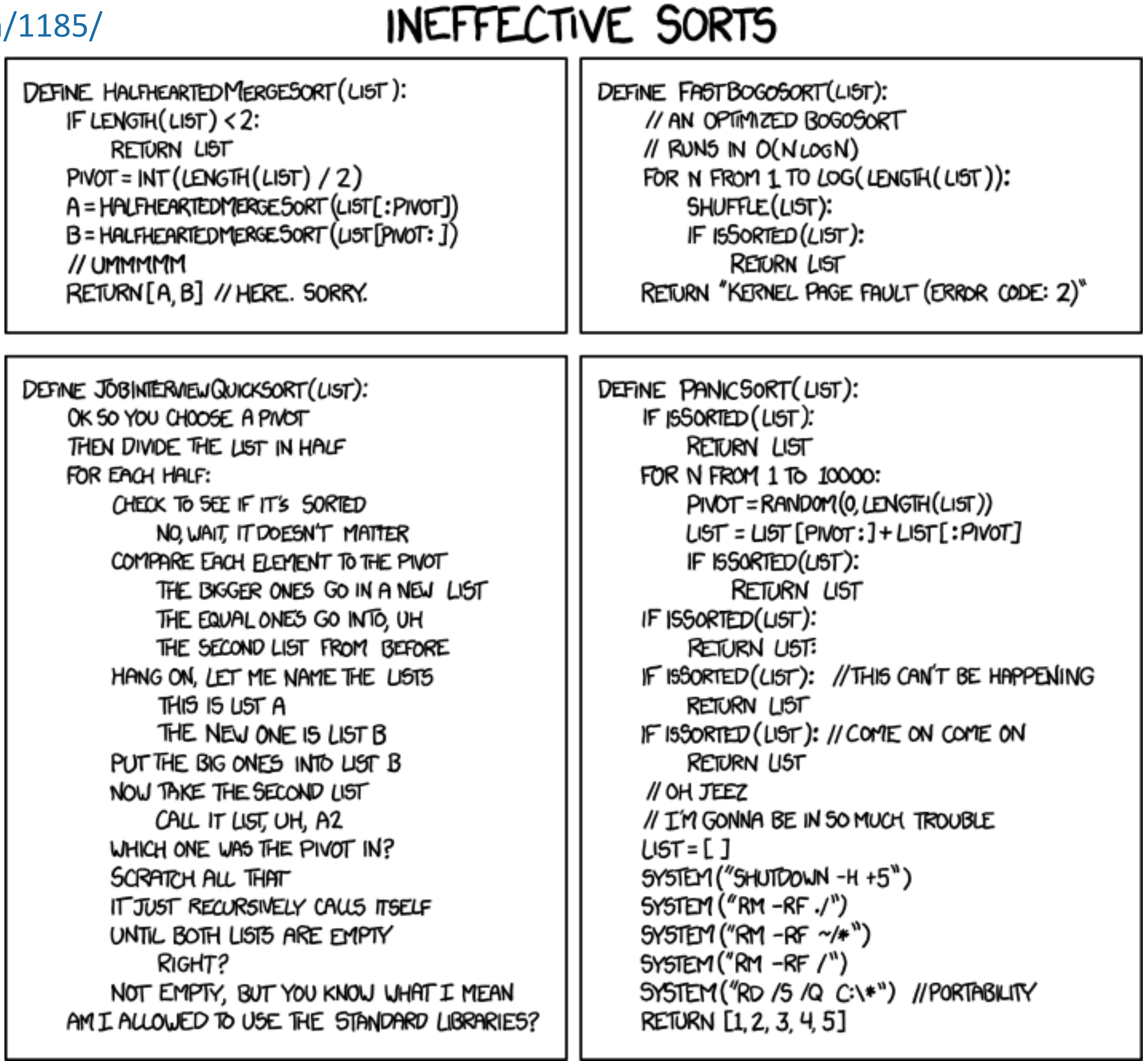
Next time

- Can we sort faster than $O(n \log(n))$??

Before next time

- ***Pre-lecture exercise*** for Lecture 6.
  - Can we sort even faster than QuickSort/MergeSort?

https://xkcd.com/1185/



INEFFECTIVE SORTS

```
DEFINE HALFHEARTEDMERGESORT(LIST):
    IF LENGTH(LIST) < 2:
        RETURN LIST
    PIVOT = INT(LENGTH(LIST) / 2)
    A = HALFHEARTEDMERGESORT(LIST[:PIVOT])
    B = HALFHEARTEDMERGESORT(LIST[PIVOT:])
    // UMMMMM
    RETURN [A, B] // HERE. SORRY.
```

```
DEFINE FASTBOGOSORT(LIST):
    // AN OPTIMIZED BOGOSORT
    // RUNS IN O(N LOG N)
    FOR N FROM 1 TO LOG(LENGTH(LIST)):
        SHUFFLE(LIST):
        IF ISSORTED(LIST):
            RETURN LIST
    RETURN "KERNEL PAGE FAULT (ERROR CODE: 2)"
```

```
DEFINE JOBINTERVIEWQUICKSORT(LIST):
    OK SO YOU CHOOSE A PIVOT
    THEN DIVIDE THE LIST IN HALF
    FOR EACH HALF:
        CHECK TO SEE IF IT'S SORTED
            NO, WAIT, IT DOESN'T MATTER
        COMPARE EACH ELEMENT TO THE PIVOT
            THE BIGGER ONES GO IN A NEW LIST
            THE EQUAL ONES GO INTO, UH
            THE SECOND LIST FROM BEFORE
        HANG ON, LET ME NAME THE LISTS
            THIS IS LIST A
            THE NEW ONE IS LIST B
        PUT THE BIG ONES INTO LIST B
        NOW TAKE THE SECOND LIST
            CALL IT LIST, UH, A2
        WHICH ONE WAS THE PIVOT IN?
        SCRATCH ALL THAT
        IT JUST RECURSIVELY CALLS ITSELF
        UNTIL BOTH LISTS ARE EMPTY
            RIGHT?
        NOT EMPTY, BUT YOU KNOW WHAT I MEAN
    AM I ALLOWED TO USE THE STANDARD LIBRARIES?
```

```
DEFINE PANICSORT(LIST):
    IF ISSORTED(LIST):
        RETURN LIST
    FOR N FROM 1 TO 10000:
        PIVOT = RANDOM(0, LENGTH(LIST))
        LIST = LIST[PIVOT:] + LIST[:PIVOT]
        IF ISSORTED(LIST):
            RETURN LIST
    IF ISSORTED(LIST):
        RETURN LIST:
    IF ISSORTED(LIST): //THIS CAN'T BE HAPPENING
        RETURN LIST
    IF ISSORTED(LIST): //COME ON COME ON
        RETURN LIST
    // OH JEEZ
    // I'M GONNA BE IN SO MUCH TROUBLE
    LIST = [ ]
    SYSTEM("SHUTDOWN -H +5")
    SYSTEM("RM -RF ./")
    SYSTEM("RM -RF ~/*")
    SYSTEM("RM -RF /")
    SYSTEM("RD /S /Q C:\*") //PORTABILITY
    RETURN [1, 2, 3, 4, 5]
```