CS 161 (Stanford, Winter 2026) Prereq Quiz

Exercises. The following questions cover material that we will use going forward in CS 161. Your work will not be graded and you are not required to complete all problems in detail, but you should make sure you are comfortable with all concepts used here. You are encouraged to ask for help on Ed or in office hours.
Note: many of these problems can be solved using more than one method. If your solution looks different than the official answer, it does not mean that you are wrong. If you aren't sure of your answer, feel free to post on Ed or ask during office hours.

1 Induction

1.1 Sums of squares

Show that for all $n \geq 1$,

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution

Base case: If $n = 1$, then $1^2 = \frac{1\cdot(1+1)(2+1)}{6} = 1$, so the claim holds.
Induction step: Assume that the claim holds for $n = k-1$, so $1^2+2^2+\ldots+(k-1)^2 = \frac{(k-1)k(2k-1)}{6} = \frac{2k^3-3k^2+k}{6}$. Then

$$\begin{aligned}
1^2 + 2^2 + \ldots + k^2 &= \frac{2k^3 - 3k^2 + k}{6} + k^2 && \text{Induction hypothesis} \\
&= \frac{2k^3 + 3k^2 + k}{6} && \text{Common denominator} \\
&= \frac{k(k+1)(2k+1)}{6}. && \text{Factor}
\end{aligned}$$

Therefore the claim holds for $n = k$ as well, which proves the claim for all n by induction.

1.2 Fibonacci parity

The Fibonacci numbers are defined by $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Show that every third Fibonacci number is even.

Solution

We'll work in groups of three numbers (a form of strong induction), since the claim we need to show refers to a pattern of length three.

Base case: The first three Fibonacci numbers are $F(0) = 0$, $F(1) = 1$, $F(2) = 0+1 = 1$, which have parity even, odd, odd, so the claim holds for $n = 0, 1, 2$.
Induction step: Assume that the Fibonacci numbers $F(3k-3)$, $F(3k-2)$, $F(3k-1)$ have parity even, odd, odd. Then $F(3k) = F(3k-2) + F(3k-1) = \text{odd} + \text{odd}$ is even. Similarly, $F(3k+1) = F(3k-1) + F(3k) = \text{odd} + \text{even}$ is odd, and $F(3k+2) = F(3k) + F(3k+1) = \text{even} + \text{odd}$ is odd. Therefore $F(3k)$, $F(3k+1)$, $F(3k+2)$ also have parity even, odd, odd, so the claim holds for all n by induction.

1.3 Sums of cubes

Show that for all $n \geq 1$, $1^3 + 2^3 + \ldots + n^3$ is a perfect square.

Solution

We will prove a stronger claim that is more amenable to induction. We'll show that not only is $\sum_{i=1}^{n} i^3$ a perfect square, but it is equal to $(n(n+1)/2)^2$.
Bonus fact: $n(n+1)/2 = \sum_{i=1}^{n} i$, so in fact we're proving that $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2$.
You can prove this bonus fact via induction or directly.
Base case: If $n = 1$, then $1^3 = (1 \cdot 2/2)^2 = 1$, so the claim holds.
Induction step: Assume that the claim holds for $n = k-1$, so $\sum_{i=1}^{k-1} i^3 = ((k-1)k/2)^2$.
Then

$$\begin{aligned}
\sum_{i=1}^{k} i^3 &= ((k-1)k/2)^2 + k^3 && \text{Induction hypothesis} \\
&= \frac{k^4 - 2k^3 + k^2}{4} + k^3 \\
&= \frac{k^4 + 2k^3 + k^2}{4} \\
&= \left(\frac{k(k+1)}{2}\right)^2.
\end{aligned}$$

Therefore the claim holds for $n = k$ as well, so we've proved the claim for all n by induction.

1.4 Dividing chocolate

Consider a chocolate bar made up of squares in an $n \times m$ grid pattern. Show that it takes $nm - 1$ breaks to break the bar completely into 1×1 squares.

Solution

Base case: A 1×1 chocolate bar takes $1 \cdot 1 - 1 = 0$ breaks to divide into 1×1 squares, so the claim holds.

Induction step: Assume the claim holds for all chocolate bars of size $m' \times n'$, where $m' < m$ or $n' < n$ (strong induction). We will prove that it also holds for a chocolate bar of size $m \times n$. Break the bar in any spot; assume without loss of generality that we break it along the first dimension, so we end up with two pieces of size $m_1 \times n$ and $m_2 \times n$, where $m_1 + m_2 = m$. We can apply our induction hypothesis to both of these pieces, so they both require $m_1 n - 1$ and $m_2 n - 1$ breaks to fully divide. Therefore the total number of breaks required, including our one initial break, is

$$(m_1 n - 1) + (m_2 n - 1) + 1 = (m_1 + m_2)n - 1 = mn - 1,$$

so the claim holds for a bar of size $m \times n$. By strong induction, the claim holds for a bar of any size.

1.5 Friendship parity

Consider a group of n people where some pairs of people are friends with each other. (For example, in a group of Alice, Bob, and Carol, perhaps Alice and Bob are friends, and Alice and Carol are friends, but Bob and Carol are not friends.) Show that there is an even number of people who have an odd number of friends.

Solution

Base case: in a group of one person, there are no friendships, so zero people have an odd number of friends and the claim holds.
Induction step: Assume that any group of $n = k - 1$ people has an even number of people with an odd number of friends. Consider a group of size $n = k$. Pick one arbitrary person, say Person A, and first consider the group without Person A, say Group B. By the induction hypothesis, Group B has an even number of people with an odd number of friends. Now consider Person A's friends. Each of these friends has their friend count parity flipped by the addition of Person A to the group. If Person A has an even number of friends, then an even number of people change friendship parity, so the total number of people with an odd number of friends remains even. If Person A has an odd number of friends, then by similar reasoning, an odd number of people from Group B now have an odd number of friends. Including Person A, this results overall in an even number of people with an odd number of friends. Therefore the claim holds for groups of size $n = k$, so by induction it holds for all n.

1.6 Coin values

Suppose a country only has coins of value 3 and 5. Show that it's possible to pay for any value that is at least 8.

Solution

In this problem, it's unclear how paying a value of k helps us at all to pay a value of $k+1$, which suggests that we should consider if strong induction might help.
Base case: We can create a value of 8 using one 3 coin and one 5 coin. We can create 9 as 3 coins of value 3, and we can create 10 as 2 coins of value 5.
Induction step: Assume we can create $n = k-3$ value out of coins. Then we can also create the value $n = k$ by adding one coin of value 3. Since any number greater than 8 can be obtained by adding multiples of 3 to either 8, 9, or 10, we have proved the result via strong induction.
Bonus challenge: Show that if the coins come in values a and b, where the greatest common divisor of a and b is 1, then it's possible to create any value that is at least $(a-1)(b-1)$.

1.7 Coin flip parity

Show that if a fair coin is flipped n times, the number of heads is equally likely to be even or odd.

Solution

Base case: If $n = 1$, then since the coin is fair, we are equally likely to get one heads or zero heads (aka one tails).
Induction step: Assume the claim holds for $n = k-1$. Then

$$\begin{aligned} P(\text{even heads after } k) &= P(\text{even heads after } k-1)P(k\text{th flip is tails}) \\ &\quad + P(\text{odd heads after } k-1)P(k\text{th flip is heads}) \\ &= \frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2} \\ &= \frac{1}{2}. \end{aligned}$$

Therefore the claim holds for $n = k$, so by induction it holds for all n.

1.8 Binary search

Suppose we are using binary search to find a given number in a sorted list of n numbers. Show that if $n \leq 2^k - 1$, then we will need to access at most k elements of the list.

Solution

We will do induction on k.
Base case: if $k = 1$, we need to search lists of size at most $2^1 - 1 = 1$, which requires one access to determine if our item is in the list or not.

Induction step: Assume that all sorted lists of size at most $2^{k-1} - 1$ require accessing at most $k - 1$ elements of the list. Given a sorted list of size at most $2^k - 1$, check the middle element (or either of the middle two if the list has even length). If this is the desired element, we are done with one access. If we find something larger than our desired element, search recursively on all elements to the left of the one we accessed. Since we looked at the middle element, the left sublist has at most $((2^k - 1) - 1)/2 = 2^{k-1} - 1$ elements. By our induction hypothesis, we need to access at most $k-1$ elements in this sublist, for a total of k elements overall. If we initially find something smaller than our desired element, repeat the same analysis with all elements in the right half of the list, with the same result. Therefore we have proved the claim for all k by induction.

2 Probability

2.1 Coin flips 1

Flip a fair coin until it lands on heads, and let T be the total number of times the coin was flipped. What is the expected value of T?

(a) 1 (b) 2 (c) e (d) Undefined

Solution

Answer: b.
T is a geometric random variable, and there are many ways to calculate its expectation. Here is one method that doesn't involve computing difficult infinite series. Let's condition on the result of the first flip. If we get heads, then we're done right away and $T = 1$. If we get tails, then we're right back where we started and the expected remaining number of flips is the same as the original expected value of T. In equation form:

$$\begin{aligned}\mathbb{E}[T] &= \mathbb{E}[T \mid \text{first flip heads}] \Pr[\text{first flip heads}] \\ &\quad + \mathbb{E}[T \mid \text{first flip tails}] \Pr[\text{first flip tails}] \\ &= 1 \cdot \frac{1}{2} + (1 + \mathbb{E}[T]) \cdot \frac{1}{2}.\end{aligned}$$

Solving for $\mathbb{E}[T]$, we find that $\mathbb{E}[T] = 2$.

2.2 Coin flips 2

Flip n fair coins. Gather any coins that landed on tails and flip them again. Repeat until every coin has landed on heads. What is the expected total number of coin flips you will complete?

(a) $2n$ (b) $n \log n$ (c) $\frac{1}{2}n^2$ (d) Undefined

Solution

Answer: a.
This question repeats the procedure from the previous question for n independent coins. Therefore, by linearity of expectation, the total expected number of flips is $n \cdot \mathbb{E}[T] = 2n$.

2.3 Coin flips 3

Flip a coin until it lands on heads. Starting with \$1 on the table, each time the coin lands on tails, double the amount of money on the table. When you flip heads for the first time, collect all the money on the table. If it costs \$2 to play this game, what is the expected amount of money you will earn?

(a) -1 (b) 0 (c) 2 (d) Undefined

Solution

Answer: d.
Let's calculate the expectation in a similar way to Question 2.1. Let X be the expected winnings, ignoring the \$2 cost for now. Since the amount won doubles each round, we have

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[X \mid \text{first flip heads}] \Pr[\text{first flip heads}] \\ &\quad + \mathbb{E}[X \mid \text{first flip tails}] \Pr[\text{first flip tails}] \\ &= 1 \cdot \frac{1}{2} + 2\mathbb{E}[X] \cdot \frac{1}{2}.\end{aligned}$$

This equation has no solution, so the expectation is undefined.
Another way to get this result is to consider the expected value won at the kth round of the game. In order to win money at the kth round, you must flip $k-1$ tails followed by one heads, which has a probability of 2^{-k}. If this occurs, you win $\$2^k$, so the expected winnings are 1. Using linearity of expectation to add up the expected winnings across all rounds, we find that the expected value of X is infinite.

2.4 Conditional expectation

Let X and Y be the results of rolling two standard six-sided dice. What is the expected value of X given that the sum of the dice is 9?

(a) 3.5 (b) 4 (c) 4.5 (d) 5

Solution

Answer: c.
Solution 1: There are four ordered pairs of dice rolls that sum to 9:

$(3,6)$, $(4,5)$, $(5,4)$, $(6,3)$. Each of these is equally likely, so we can calculate the expected value of the first die by taking the average of the first number in each pair across all four pairs: $(3+4+5+6)/4 = 4.5$.
Solution 2: We want to calculate $\mathbb{E}[X \mid X+Y=9]$. By linearity of expectation, $9 = \mathbb{E}[X+Y \mid X+Y=9] = \mathbb{E}[X \mid X+Y=9] + \mathbb{E}[Y \mid X+Y=9]$. But X and Y have the same distribution, so $\mathbb{E}[X \mid X+Y=9] = \mathbb{E}[Y \mid X+Y=9]$. Therefore $2\mathbb{E}[X \mid X+Y=9] = 9$, so $\mathbb{E}[X \mid X+Y=9] = 4.5$.

2.5 Counting 1

Consider putting three identical balls into 10 numbered bins. Out of all possible configurations, in approximately what fraction is each ball in a different bin?

(a) 0.55 (b) 0.67 (c) 0.72 (d) 0.93

Solution

Answer: a.
Let's count all possible configurations of balls. There are 10 ways to put all three balls into the same bin. There are $\binom{10}{2} \cdot 2 = 90$ ways to put two balls in one bin and one in another: first choose two bins out of 10, then choose one of those two to have two balls. There are $\binom{10}{3} = 120$ ways to put three balls into three different bins: just choose 3 bins out of 10. This gives a total of 220 possible configurations, 120 of which have each ball in a different bin, for a fraction of about 0.55.

2.6 Counting 2

Put each of three balls independently into one of 10 bins uniformly at random. What is the approximate probability that each ball is in a different bin?

(a) 0.55 (b) 0.67 (c) 0.72 (d) 0.93

Solution

Answer: c.
Let's calculate the probability by conditioning on each ball one at a time. The probability that all three balls have a valid arrangement is

$$\begin{aligned}\Pr[\text{all balls valid}] &= \Pr[\text{ball 1 valid}] \cdot \Pr[\text{ball 2 valid} \mid \text{ball 1 valid}] \\ &\quad \cdot \Pr[\text{ball 3 valid} \mid \text{balls 1, 2 valid}] \\ &= 1 \cdot \frac{9}{10} \cdot \frac{8}{10},\end{aligned}$$

since the first ball can go anywhere, the second can go anywhere except where the first went, and the third can go in any bin except for two. This gives a probability of 0.72.

Bonus question: What's the intuition for why this question and the previous one have different answers?

2.7 Conditional probability

There are 300 students enrolled in CS161, and on any given day, 5000 people are present on Stanford campus (including the 300 CS161 students). 10% of students enrolled in CS161 attend lectures in person, and a person on campus who isn't enrolled in CS161 has a 0.5% chance of wandering into a lecture anyway. If you see someone attending lecture in person, what is the probability they are enrolled, rounded to the nearest percent?

(a) 21% (b) 56% (c) 67% (d) 92%

Solution

Answer: b.
First, let's calculate Pr[attending], the probability that a randomly selected person on campus attends a CS161 lecture. Using the law of total probability,

$$\begin{aligned}\Pr[\text{attending}] &= \Pr[\text{attending} \mid \text{enrolled}] \cdot \Pr[\text{enrolled}] \\ &\quad + \Pr[\text{attending} \mid \text{not enrolled}] \cdot \Pr[\text{not enrolled}] \\ &= 0.1 \cdot \frac{300}{5000} + 0.005 \cdot \frac{5000-300}{5000}.\end{aligned}$$

Now we apply Bayes' Rule:

$$\begin{aligned}\Pr[\text{enrolled} \mid \text{attending}] &= \frac{\Pr[\text{attending} \mid \text{enrolled}] \cdot \Pr[\text{enrolled}]}{\Pr[\text{attending}]} \\ &= \frac{0.1 \cdot \frac{300}{5000}}{0.1 \cdot \frac{300}{5000} + 0.005 \cdot \frac{4700}{5000}} \\ &\approx 0.56.\end{aligned}$$

2.8 Random permutation

Pick a uniformly random permutation of the digits 1 to n. What is the expected number of adjacent pairs of digits such that the first digit in the pair is less than the second? For example, in the permutation (4, 1, 5, 2, 3), there are two such pairs: 1, 5 and 2, 3.

(a) $\log n$ (b) $2\sqrt{n}$ (c) $\frac{1}{2}(n-1)$ (d) $\frac{1}{2}n$

Solution

Answer: c.
We will use linearity of expectation. Define random variables $P_1, \ldots, P_{n-1}$, where $P_i = 1$

if the ith digit is less than the $(i+1)$th digit, and $P_i = 0$ otherwise. Then the total number P of digit pairs in ascending order is equal to $\sum_{i=1}^{n-1} P_i$, and $\mathbb{E}[P] = \sum_{i=1}^{n-1} \mathbb{E}[P_i]$. Now we will show that $\mathbb{E}[P_i] = \frac{1}{2}$ for all i. Intuitively, two digits in the permutation are equally likely to be in ascending or descending order. To make this rigorous, let's pair up all possible permutations of n digits by matching each permutation with the one obtained by reversing the digit order (i.e. replace 1 with n, 2 with $n-1$, and so on). For any adjacent pair of digits, each pair of permutations contains one where that pair is in ascending order and one where it is in descending order. Therefore exactly half of all the permutations have the ith digit less than the $(i+1)$th digit, so $\mathbb{E}[P_i] = \frac{1}{2}$. Putting this together, we find that $\mathbb{E}[P] = \sum_{i=1}^{n-1} \mathbb{E}[P_i] = \frac{1}{2}(n-1)$.

2.9 Independence

Answer true or false to the following questions about independence. Let A, B, C be events, and let X, Y be random variables.

1. If A and B are independent, then their complements A^c and B^c are also independent.

Solution

True. We use De Morgan's laws:

$$\begin{aligned} \Pr[A^c \text{ and } B^c] &= \Pr[(A \text{ or } B)^c] \\ &= 1 - \Pr[A \text{ or } B] \\ &= 1 - \Pr[A] - \Pr[B] + \Pr[A \text{ and } B] \\ &= 1 - \Pr[A] - \Pr[B] + \Pr[A]\Pr[B] && A \text{ and } B \text{ independent} \\ &= (1 - \Pr[A])(1 - \Pr[B]) \\ &= \Pr[A^c]\Pr[B^c]. \end{aligned}$$

2. If A and B are independent, then they are also independent conditioned on any C.

Solution

False. For example, let A and B be the events that two different coin flips are heads, and let C be the event that both coin flips have the same result. Then A and B are independent, but conditioning on C makes A completely determined by B.

3. If A and B are independent conditioned on any C, then they are independent.

Solution

True. Since A and B are independent conditioned on any C, take C to be the always-true event.
Bonus question: if A and B are independent conditioned on any C, what else can you say about A and B?

4. If A and B are independent, and B and C are independent, then A and C are independent.

Solution

False. Let A and B be the events that two independent coin flips are heads, and let C be equal to A. Then A and B are independent, and so are B and C by the definition of C, but C is completely determined by A.

5. If A and B are independent, and B and C are independent, and A and C are independent, then A, B, and C are independent.

Solution

False. Let A and B be the events that two independent coin flips are heads, and let C be the event that both coin flips have the same result. Then A and B are independent, and each is also independent of C:

$$\begin{aligned}\Pr[A \text{ and } C] &= \Pr[\text{coin 1} = \text{heads and coin 2} = \text{coin 1}] \\ &= \frac{1}{4} = \Pr[A]\Pr[C],\end{aligned}$$

and likewise for B. However, A, B, and C are not mutually independent because A and B completely determine C.

6. If X and Y are independent, then $P(X > Y) = P(Y > X)$.

Solution

False. Independent random variables do not need to have the same distribution as each other. For example, take X to be a Bernoulli(1/2) random variable, and let Y be a Bernoulli(1/2) random variable plus 10.

7. If X and Y are independent, then $P(X > 10, Y < 10) = P(X > 10)P(Y < 10)$.

Solution

True. If X and Y are independent, then all events of the form $X \in S_1$ and $Y \in S_2$, where S_1 and S_2 are sets[a], are independent.

[a] There are some technical constraints on S_1 and S_2, but those are well beyond the scope of

this course.

8. If X and Y are independent, then $\text{Var}(XY) = \text{Var}(X)\text{Var}(Y)$.

Solution

False. Let X and Y be two independent Bernoulli(1/2) random variables, so $\text{Var}(X) = \text{Var}(Y) = \frac{1}{4}$. Then XY is a Bernoulli(1/4) random variable, which has variance $\text{Var}(XY) = \frac{1}{4} \cdot (1 - \frac{1}{4}) = \frac{3}{16} \neq \text{Var}(X)\text{Var}(Y)$.

9. If X and Y are independent, then $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.

Solution

True. We'll use linearity of expectation, plus the fact that if X and Y are independent, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

$$
\begin{aligned}
\text{Var}(X+Y) &= \mathbb{E}[(X+Y)^2] - (\mathbb{E}[X+Y])^2 \\
&= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + E[Y^2] - \mathbb{E}[X]^2 - 2\mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[Y]^2 \\
&= \mathbb{E}[X^2] + 2\mathbb{E}[X]\mathbb{E}[Y] + E[Y^2] - \mathbb{E}[X]^2 - 2\mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[Y]^2 \\
&= \mathbb{E}[X^2] - \mathbb{E}[X]^2 + E[Y^2] - \mathbb{E}[Y]^2 \\
&= \text{Var}(X) + \text{Var}(Y).
\end{aligned}
$$

3 Asymptotic Analysis

3.1

Determine if each function is asymptotically equivalent to n^2. All logarithms are base 2.

1. $2n^2$

Solution

Yes. A general rule is that if two polynomials have the same degree (maximum power of n), then they are asymptotically equivalent.

2. $(n+1)(n-2)$

Solution

Yes. This is a degree 2 polynomial in n, hence asymptotically equivalent to n^2.

3. $2^{\sqrt{2n}}$

Solution

No. A useful fact when dealing with functions involving exponents is: if $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty$ and $\log f(n)$ is not $O(\log g(n))$, then $f(n)$ is not $O(g(n))$. In this case, $\log 2^{\sqrt{2n}} = \sqrt{2n}$, which is asymptotically larger than $\log n^2 = 2\log n$.

Proof of fact: Assume $\log f(n)$ is not $\log g(n)$. Then for all $c > 0$, there are infinitely many n such that $\log f(n) > c \cdot \log g(n)$. Rearranging, $f(n) > g(n)^c$. Given any constant $d > 0$, since $g(n) \to \infty$ we can choose a $c > 0$ such that $g(n)^c > d \cdot g(n)$ for all n past some point. Putting this all together, we've shown that for any $d > 0$, $f(n) > d \cdot g(n)$ for infinitely many n, which means that $f(n)$ is not $O(g(n))$.

4. $\log(1 + 2^{n^2})$

Solution

Yes. While this function is difficult to analyze, we can sandwich it between two functions that are easier to analyze and are both asymptotically equivalent to n^2:

$$n^2 = \log(0 + 2^{n^2}) < \log(1 + 2^{n^2}) < \log(2 \cdot 2^{n^2}) = n^2 + 1.$$

5. $n^2 \log\log n$

Solution

No. Even though $\log\log n$ grows very slowly, it still eventually grows to ∞ and makes this function asymptotically larger than n^2.

6. $\frac{(n+1)^2(n-3)^2}{n^2+4}$

Solution

Yes. A general rule is that if two rational functions have the same degree (degree of numerator minus degree of denominator), then they are asymptotically equivalent. We can prove this using L'Hospital's rule:

$$\begin{aligned}\lim_{n\to\infty} \frac{(n+1)^2(n-3)^3}{n^2(n^2+4)} &= \lim_{n\to\infty} \frac{\frac{d}{dn}(n+1)^2(n-3)^3}{\frac{d}{dn}n^2(n^2+4)} \\ &= [\ldots\text{repeat differentiation}\ldots] \\ &= \lim_{n\to\infty} \frac{24}{24} = 1.\end{aligned}$$

Therefore, for large enough n, $\frac{(n+1)^2(n-3)^2}{n^2+4} = (1 \pm \epsilon) \cdot n^2$ for a constant value of $\epsilon > 0$, which shows that this function is asymptotically equivalent to n^2. Bonus question: Convince yourself that the general rule is true, using a similar technique to above.

7. $\frac{n^3}{n+\log n}$

Solution

Yes. As in question 4, we can sandwich this function between two functions that are easier to analyze and both asymptotically equivalent to n^2:

$$n^2 = \frac{n^3}{n+0} > \frac{n^3}{n+\log n} > \frac{n^3}{n+n} = \frac{1}{2}n^2.$$

8. $\binom{2n+1}{2}$

Solution

Yes. $\binom{2n+1}{2} = \frac{1}{2}(2n+1)(2n)$, which is a degree 2 polynomial in n.

9. $\sum_{k=1}^{\infty} \frac{n^k}{2^k \cdot k!}$

Solution

No. The third term in the sum is $n^3/48$, which is already asymptotically larger than n^2.

3.2

For each pair of functions, is $f(n) = O(g(n))$?

1. $f(n) = n^2, \ g(n) = n \log n$

Solution

No, because n grows much faster than $\log n$. (We can prove this rigorously using calculus.) A general rule is that any polynomial in n is asymptotically larger than any polynomial in $\log n$.

2. $f(n) = 43n^2 + 228n + 91, \ g(n) = n^2$

Solution

Yes. As in the previous problem, these are two polynomials of the same degree, so $f(n) = O(g(n))$. (In fact, $f(n) = \Theta(g(n))$.)

3. $f(n) = n \log n, \ g(n) = n \log \log n$

Solution

No. Using L'Hospital's Rule,

$$\lim_{n\to\infty} \frac{n \log n}{n \log\log n} = \lim_{n\to\infty} \frac{\log n}{\log\log n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{\log n} \cdot \frac{1}{n}} = \lim_{n\to\infty} \log n = \infty.$$

Therefore $n \log n$ is not bounded by any constant multiple of $n \log\log n$.

4. $f(n) = \log(n^2 + 2n + 1)$, $g(n) = \log n$

Solution

Yes. $\log(n^2 + 2n + 1) = \log((n + 1)^2) = 2\log(n + 1)$, and then we can use L'Hospital's Rule:

$$\lim_{n\to\infty} \frac{2\log(n+1)}{\log n} = \lim_{n\to\infty} 2 \cdot \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} 2 \cdot \frac{n}{n+1} = 2.$$

Therefore, for large enough n, $\log(n^2+2n+1) \leq (2+\epsilon)\cdot\log n$, so $f(n) = O(g(n))$. Bonus problem: Show that if $p(n)$ is any polynomial in n, then $\log(p(n)) = O(\log n)$.

5. $f(n) = 3^n$, $g(n) = 2^n$

Solution

No. Let's prove this via contradiction. Assume $3^n = O(2^n)$. Then there is some $c > 0$ such that $3^n < c\dot{2}^n$ for all $n \geq n_0$. Rearranging, $(3/2)^n \leq c$ for all $n \geq n_0$. This cannot be true, since $(3/2)^n$ grows arbitrarily large as n increases.

6. $f(n) = 2^{n^{1/2}}$, $g(n) = n^3$

Solution

No. We'll use the fact from question 3.1, part 3: $\log f(n) = n^{1/2}$, and $\log g(n) = 3\log n$, so $\log f(n)$ is not $O(\log g(n))$ because polynomials are asymptotically larger than logarithms. Therefore $f(n)$ is not $O(g(n))$ as well.

7. $f(n) = n$, $g(n) = 2^{\sqrt{\log n}}$

Solution

No. Using the same logarithm trick again: $\log f(n) = \log n$, and $\log g(n) = \sqrt{\log n}$, and $\log n$ is not $O(\sqrt{\log n})$.

8. $f(n) = (\log n)^n$, $g(n) = n^{\log n}$

Solution

No. Using the logarithm trick: $\log f(n) = n \log \log n$, which is asymptotically larger than $\log g(n) = (\log n)^2$, so $f(n)$ is not $O(g(n))$.